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COMPACT OPERATOR RELATED WITH POISSON-SZEGö INTEGRAL

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Abstract. Suppose that μ is a finite positive Borel measure on the unit ball $B \subset C^n$. The boundary of B is the unit sphere $S = \{z : |z| = 1\}$. Let σ be the rotation-invariant measure on S such that $\sigma(S) = 1$. In this paper, we will show that if $\sup_{\zeta \in S} \int_B P(z,\zeta) d\mu(z) < \infty$ where $P(z,\zeta)$ is the Poission-Szegö kernel for B, then μ is a Carleson measure. We will also show that if $\sup_{\zeta \in S} \int_B P(z,\zeta) d\mu(z) < \infty$, then the operator T such that $T(f) = P[f]$ is compact as a mapping from $L^p(\sigma)$ into $L^p(B, d\mu)$.

1. Introduction

Throughout this paper, $C^n (n \geq 1)$ will be the Cartesian product of n copies of C. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in C^n, the inner product is defined by $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the norm by $|z|^2 = \langle z, z \rangle$. For $w \in C^n, r > 0$, let $B(w, r) = \{z \in C^n : |z - w| < r\}$. For simplicity, the unit ball $B(0, 1)$ will be denoted by B. The boundary of B is the unit sphere $S = \{z : |z| = 1\}$.

Let σ be the rotation-invariant measure on S such that $\sigma(S) = 1$. For $1 \leq p \leq \infty$, $L^p(\sigma)$ denote the Lebesgue space of S induced by σ.

For $f \in L^1(\sigma)$, $P[f]$ denotes the Poisson-Szegö integral defined for $z \in B$ by

$$P[f](z) = \int_S P(z, \zeta) f(\zeta) d\sigma(\zeta)$$

where

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$$P(z,\zeta) = \left(\frac{1-|z|^2}{|1-\langle z,\zeta\rangle|^2}\right)^n$$

is the Poisson-Szegö kernel for B.

A positive Borel measure μ on B is called a Carleson measure if

$$\sup\{\frac{\mu(B_\delta(\zeta))}{\delta^n} : \zeta \in S, \delta > 0\} < \infty$$

where $B_\delta(\zeta) = \{z \in B : |1-\langle z,\zeta\rangle| < \delta\}$(See [4] and [8]).

In this paper, we will let $\| \mu \|_P = \sup_{\zeta\in S}\int_B P(z,\zeta)d\mu(z)$ for positive Borel measure μ on B. In section 2, we will show that if $\| \mu \|_P < \infty$, then μ is a Carleson measure.

Let X and Y be Banach spaces. We will denote the set of all bounded operators form X to Y by $\mathfrak{L}(X,Y)$. An operator $T \in \mathfrak{L}(X,Y)$ is called compact if and only if for every bounded sequence $\{x_n\} \subset X$, $\{Tx_n\}$ has a subsequence convergent in Y.

In Section 3, we will also show that if $\| \mu \|_P < \infty$, then the operator T such that $T(f) = P[f]$ is compact as a mapping from $L^p(\sigma)$ into $L^p(B, d\mu)$ where $1 < p < \infty$.

2. Carleson measure and Poisson integral

Let $A(B)$ be the class of all $f : B \to C$ that are continuous on the closed ball $\bar{B}$ and that holomorphic in its interior B.

Theorem 2.1. *If $f \in A(B)$, then $f(z) = P[f](z)$ for all $z \in B$.*

Proof. See [10, Theorem 3.2.4]. □

Theorem 2.2. *If $0 \le r < 1$, $\zeta \in S$ and $\eta \in S$, then*

$$P(r\eta,\zeta) = P(r\zeta,\eta).$$

Also,

$$\int_S P(r\eta,\zeta)d\sigma(\zeta) = 1 = \int_S P(r\zeta,\eta)d\sigma(\eta).$$

Proof. See [10, Proposition 3.3.3]. □

Theorem 2.3. *If $\| \mu \|_P < \infty$ and $f \in L^p(\sigma)$, then*

$$\int_B |P[f](z)|^p d\mu(z) \le \| \mu \|_P \int_S |f(\zeta)|^p d\sigma(\zeta).$$

Proof. By the Hölder inequaltiy,

$$\begin{aligned} |P[f](z)|^p &= |\int_S f(\zeta)P(z,\zeta)d\sigma(\zeta)|^p \\ &\leq \int_S |f(\zeta)|^p P(z,\zeta)d\sigma(\zeta) \int_S P(z,\zeta)d\sigma(\zeta) \\ &= \int_S |f(\zeta)|^p P(z,\zeta)d\sigma(\zeta) \end{aligned}$$

where last inequality follows from Theorem 2.2. This implies that

$$\begin{aligned} \int_B |P[f](z)|^p d\mu(z) &\leq \int_B \int_S |f(\zeta)|^p P(z,\zeta)d\sigma(\zeta)d\mu(z) \\ &= \int_S |f(\zeta)|^p \int_B P(z,\zeta)d\mu(z)d\sigma(\zeta) \\ &\leq \parallel \mu \parallel_P \int_S |f(\zeta)|^p d\sigma(\zeta). \end{aligned}$$

□

THEOREM 2.4. *If $f \in A(B)$ and $z \in B$, Then*

$$f(z) = \int_S \frac{f(\zeta)}{(1-\langle z,\zeta\rangle)^n} d\sigma(\zeta).$$

Proof. See [10, Theorem 3.2.4]. □

THEOREM 2.5. *If $\parallel \mu \parallel_P < \infty$, then*

$$\sup_{w\in B} \int_B \frac{(1-|w|^2)^n}{|1-<w,z>|^{2n}} d\mu(z) < \infty.$$

Proof. For the following function f such that

$$f(\zeta) = \left(\frac{(1-|w|^2)^n}{(1-\langle \zeta,w\rangle)^{2n}} \right)^{1/p},$$

$$\begin{aligned} P[f](z) &= \int_S P(z,\zeta) \left(\frac{(1-|w|^2)^n}{(1-\langle \zeta,w\rangle)^{2n}} \right)^{1/p} d\sigma(\zeta) \\ &= \left(\frac{(1-|w|^2)^n}{(1-\langle z,w\rangle)^{2n}} \right)^{1/p} \end{aligned}$$

by Theorem 2.4. Since

$$\int_S \frac{(1-|w|^2)^n}{|1-\langle \zeta, w\rangle|^{2n}} d\sigma(\zeta) = 1$$

by Theorem 2.4,

$$\begin{aligned}\int_B |P[f](z)|^p d\mu(z) &= \int_B \frac{(1-|w|^2)^n}{|1-\langle z, w\rangle|^{2n}} d\mu(z) \\ &\leq \| \mu \|_P \int_S \frac{(1-|w|^2)^n}{|1-\langle \zeta, w\rangle|^{2n}} d\sigma(\zeta) \\ &= \| \mu \|_P .\end{aligned}$$

This implies that

$$\sup_{w\in B} \int_B \frac{(1-|w|^2)^n}{|1- < w, z > |^{2n}} d\mu(z) < \infty.$$

□

THEOREM 2.6. *If $\| \mu \|_P < \infty$, then μ is a Carleson measure on B.*

Proof. For $z = 0$,

$$\int_B \frac{(1-|z|^2)^n}{|1-\langle w, z\rangle|^{2n}} d\mu(w) = \mu(B).$$

This implies that if $\delta \geq 1/4$, then

$$\begin{aligned}\mu(B_\delta(\zeta)) &\leq \mu(B) \\ &\leq \sup_{z\in B} \int_B \frac{(1-|z|^2)^n}{|1-\langle w, z\rangle|^{2n}} d\mu(w) \\ &\leq 4^n \delta^n \sup_{z\in B} \int_B \frac{(1-|z|^2)^n}{|1-\langle w, z\rangle|^{2n}} d\mu(w)\end{aligned}$$

for all $\zeta \in \partial B$. This implies that if $\delta \geq 1/4$, then $\frac{\mu(B_\delta(\zeta))}{\delta^n} \leq \| \mu \|_P$ for all $\zeta \in S$ by Theorem 2.5.

We consider the case for $\delta < \frac{1}{4}$. For $\zeta \in S$, put $z_0 = (1-\frac{\delta}{2})\zeta$. For $w \in B_\delta(\zeta) = \{z \in B : |1-\langle z, \zeta\rangle| < \delta\}$,

$$\begin{aligned}|1-\langle z_0,w\rangle| &= |1-\langle z_0, w-z_0+z_0\rangle| \\ &= |1-\langle z_0,z_0\rangle - \langle z_0, w-z_0\rangle| \\ &\le (1-|z_0|^2) + |\langle z_0, w-z_0\rangle| \\ &= (1-|z_0|^2) + |\langle z_0,w\rangle - \langle z_0,z_0\rangle| \\ &= (1-|z_0|^2) + \left|\langle \frac{z_0}{|z_0|}, w\rangle |z_0| - |z_0|^2\right|.\end{aligned}$$

Since $\zeta = \frac{z_0}{|z_0|}$ and $\delta = 2(1-|z_0|)$,

$$\begin{aligned}&(1-|z_0|^2) + \left|\langle \frac{z_0}{|z_0|}, w\rangle |z_0| - |z_0|^2\right| \\ &= (1-|z_0|^2) + |z_0||\langle \zeta, w\rangle - |z_0|| \\ &\le (1-|z_0|^2) + |z_0|2\delta \\ &\le (1-|z_0|^2) + |z_0|4(1-|z_0|) \\ &\le (1-|z_0|^2) + (1+|z_0|)4(1-|z_0|) \\ &\le 5(1-|z_0|^2).\end{aligned}$$

This implies that

$$|1-\langle z_0,w\rangle|^{2n} \le 5^{2n}(1-|z_0|^2)^{2n}.$$

Since

$$\begin{aligned}\frac{(1-|z_0|^2)^n}{|1-\langle z_0,w\rangle|^{2n}} &\ge \frac{1}{5^{2n}}\frac{1}{(1-|z_0|^2)^n} \\ &\ge \frac{1}{5^{2n}}\frac{1}{2^n(1-|z_0|)^n} \\ &\ge \frac{1}{25^n}\frac{1}{\delta^n},\end{aligned}$$

$$\begin{aligned}\sup_{z\in B}\int_B \frac{(1-|z|^2)^n}{|1-\langle z,w\rangle|^{2n}}d\mu(w) &\ge \int_B \frac{(1-|z_0|^2)^n}{|1-\langle z_0,w\rangle|^{2n}}d\mu(w) \\ &\ge \frac{1}{25^n}\int_{B_\delta(\zeta)} \frac{1}{\delta^n}d\mu(w) \\ &= \frac{1}{25^n}\frac{\mu(B_\delta(\zeta))}{\delta^n}.\end{aligned}$$

This implies that if $\delta < \frac{1}{4}$, then $\frac{\mu(B_\delta(\zeta))}{\delta^n} \le 25^n \parallel \mu \parallel_P$ for all $\zeta \in S$ by Theorem 2.5. Thus we can conclude that

$$\sup\left\{\frac{\mu(B_\delta(\zeta))}{\delta^n} : \zeta \in S, \delta > 0\right\} < \infty.$$

□

3. Compact operator

Suppose that $1 < p < \infty$. In this section, we will let

$$\| f \|_p = \left\{\int_S |f(\zeta)|^p d\sigma(\zeta)\right\}^{1/p}$$

for $f \in L^p(\sigma)$.

THEOREM 3.1. *A linear operator T on Hilbert space H is compact if and only if $\| Tx_n \| \to 0$ whenever $x_n \to 0$ weakly in H.*

Proof. See [9, Theorem VI.11]. □

THEOREM 3.2. *Let X be a Banach space. Let Γ be a family of bounded linear transformations from X to some normed linear space Y. Suppose that for each $x \in X$, $\{\| Tx \|_Y : T \in \Gamma\}$ is bounded. Then $\{\| T \| : T \in \Gamma\}$ is bounded.*

Proof. See [5, Theorem 14.1]. □

Recall that the boundary of B is the unit sphere $S = \{z : |z| = 1\}$ and σ is the measure on S such that $\sigma(S) = 1$.

THEOREM 3.3. *If $f_n \to f$ in $L^p(\sigma)$, then f_n converges to f weakly in $L^p(\sigma)$.*

Proof. Let q be the integer such that $\frac{1}{p} + \frac{1}{q} = 1$ and $g \in L^q(\sigma)$. Then

$$\left|\int_S (f_n - f)(\zeta)g(\zeta)d\sigma(\zeta)\right| \le \| f_n - f \|_p \| g \|_q \to 0$$

as $n \to \infty$. □

THEOREM 3.4. *If $f_n \to f$ weakly in $L^p(\sigma)$ as $n \to \infty$, then $\{\| f_n \|_p : n \in N\}$ is bounded.*

Proof. Let q be the integer such that $\frac{1}{p} + \frac{1}{q} = 1$. Let us define $T_{f_n} : L^q(\sigma) \to R$ such that $T_{f_n}(g) = \int_S f_n(\zeta)g(\zeta)d\sigma(\zeta)$. Since $f_n \to f$ weakly in $L^p(\sigma)$, $\int_S f_n(\zeta)g(\zeta)d\sigma(\zeta) \to \int_S f(\zeta)g(\zeta)d\sigma(\zeta)$. This implies that there exists n_0 such that if $n > n_0$, $|\int_S f_n(\zeta)g(\zeta) - \int_S f(\zeta)g(\zeta)d\sigma(\zeta)| < 1$. If $n > n_0$,

$$\left| \int_S f_n(\zeta)g(\zeta)d\sigma(\zeta) \right| \le 1 + \| f \|_p \| g \|_q,$$

$$\left| \int_S f_n(\zeta)g(\zeta)d\sigma(\zeta) \right| \le max\{M, 1 + \| f \|_p \| g \|_q\}$$

where

$$M = max\left\{ \left| \int_S f_1(\zeta)g(\zeta)d\sigma(\zeta) \right|, \cdots, \left| \int_S f_{n_0}(\zeta)g(\zeta)d\sigma(\zeta) \right| \right\}.$$

Since $\{\| T_{f_n} g \| : n \in N\}$ is bounded, $\{\| T_{f_n} \| = \| f_n \|_p : n \in N\}$ is bounded by Theorem 3.2. □

THEOREM 3.5. *Suppose that* $1 < p < \infty$. *Let* μ *be a positive Borel measure on* B *and* $\mu_r = \mu|_{B_r}$ *where* $B_r = \{z \in B : |z| < r\}$. *If sequence* f_n *converges to* 0 *weakly in* $L^p(\sigma)$, *then* $P[f_n]$ *converges to* 0 *in* $L^p(B, d\mu_r)$*-norm.*

Proof. Since the sequence f_n converges to 0 weakly in $L^p(\sigma)$, $f_n(\zeta)$ converges to 0 pointwise and $\sup\{\| f_n \|_p : n \in N\}$ is bounded by Theorem 3.4.

If $z \in B_r$, then

$$P(z, \zeta) = \frac{(1 - |z|^2)^n}{|1 - \langle z, \zeta \rangle|^{2n}} < \frac{1}{(1 - r)^{2n}}.$$

This implies that

$$\begin{aligned} |P[f_n](z)| &= \left| \int_S f_n(\zeta)P(z, \zeta)d\sigma(\zeta) \right| \\ &\le \left\{ \int_S |f_n(\zeta)|^p P(z, \zeta)d\sigma(\zeta) \right\}^{1/p} \\ &\le \left(\frac{1}{(1 - r)^{2n}} \right)^{1/p} \| f_n \|_p \\ &\le M \sup_n \| f_n \|_p \end{aligned}$$

for some constant M. Let us define linear functional l_z on $L^p(\sigma)$ by $l_z(f_n) = P[f_n](z)$. Since l_z is bounded linear functional on $L^p(\sigma)$ and f_n converges to 0 weakly in $L^p(\sigma)$,

$$\lim_{n\to\infty} P[f_n](z) = 0.$$

By the dominated convergence theorem,

$$\lim_{n\to\infty} \int_{B_r} |P[f_n](z)|^p d\mu_r = \int_{B_r} \lim_{n\to\infty} |P[f_n](z)|^p d\mu_r = 0.$$

□

THEOREM 3.6. *If μ and μ_r are the same measures in Theorem 3.5 and $\| \mu \|_P < \infty$, then*

$$\lim_{r\to 1} \int_B P(z,\zeta) d\mu_r(z) = \int_B P(z,\zeta) d\mu(z).$$

Proof. Since

$$\int_B P(z,\zeta) d\mu_r(z) = \int_B P(z,\zeta)\chi_{B_r}(z) d\mu(z)$$

and

$$P(z,\zeta)\chi_{B_r}(z) \leq P(z,\zeta),$$

$$\begin{aligned} \lim_{r\to 1} \int_{B_r} P(z,\zeta) d\mu(z) &= \lim_{r\to 1} \int_B P(z,\zeta)\chi_{B_r}(z) d\mu(z) \\ &= \int_B \lim_{r\to 1} P(z,\zeta)\chi_{B_r}(z) d\mu(z) \\ &= \int_B P(z,\zeta) d\mu(z). \end{aligned}$$

□

THEOREM 3.7. *If $\| \mu \|_P < \infty$, then the operator T such that $T(f) = P[f]$ is compact as a mapping from $L^p(\sigma)$ into $L^p(B, d\mu)$ where $1 < p < \infty$.*

Proof. Let $\{f_n\}$ be a sequence which converges to 0 weakly in $L^p(\sigma)$. Since

$$\begin{aligned} &\int_B |P[f_n](z)|^p d\mu(z) \\ &= \int_B |P[f_n](z)|^p d\mu_r(z) + \int_B |P[f_n](z)|^p d(\mu - \mu_r)(z) \end{aligned}$$

and

$$\lim_{n\to\infty}\int_B |P[f_n](z)|^p d\mu_r(z) = 0$$

by Theorem 3.5, we will only prove that

$$\lim_{r\to 1}\int_B |P[f_n](z)|^p d(\mu-\mu_r)(z) = 0.$$

Since

$$\begin{aligned}
&|f_n(\zeta)|^p \int_B P(z,\zeta) d(\mu-\mu_r)(z)\\
&\le |f_n(\zeta)|^p \int_B P(z,\zeta) d\mu(z)\\
&\le |f_n(\zeta)|^p \parallel \mu \parallel_P,
\end{aligned}$$

$$\begin{aligned}
&\lim_{r\to 1}\int_B |P[f_n](z)|^p d(\mu-\mu_r)(z)\\
&\le \lim_{r\to 1}\int_B\int_S |f_n(\zeta)|^p P(z,\zeta) d\sigma(\zeta) d(\mu-\mu_r)(z)\\
&= \lim_{r\to 1}\int_S |f_n(\zeta)|^p \int_B P(z,\zeta) d(\mu-\mu_r)(z) d\sigma(\zeta)\\
&= \int_S |f_n(\zeta)|^p \lim_{r\to 1}\int_B P(z,\zeta) d(\mu-\mu_r)(z) d\sigma(\zeta)\\
&= \int_S |f_n(\zeta)|^p \int_B \lim_{r\to 1} P(z,\zeta) d(\mu-\mu_r)(z) d\sigma(\zeta)\\
&= 0.
\end{aligned}$$

By Theorem 3.1, the operator T such that $T(f) = P[f]$ is compact as a mapping from $L^p(\sigma)$ into $L^p(B, d\mu)$.

□

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