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A CHARACTERIZATION OF SOLUTIONS OF AFFINE VARIATIONAL INEQUALITY DEFINED BY SECOND-ORDER CONE

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Abstract. A variational inequality defined by the second-order cone is considered, and it is shown that a necessary and sufficient condition for a solution of the variational inequality holds under some regularity condition.

1. Introduction and preliminaries

Recently, many authors have studied optimization problems over second-order cones and their applications to nonlinear optimization problems [1, 2, 6]. Such investigation has a strong motivation since many nonlinear optimization problems can be relaxed to optimization problems over second-order cones. It is well-known that variational inequality problems are closely related to optimization problems. Very recently, Auslender [3] studied algorithms for variational inequalities over the cone of semidefinite positive symmetric matrices and over the second-order cone.

The purpose of this paper is to give a complete characterization of solutions for a variational inequality problem defined by the second-order cone. We prove that a necessary and sufficient condition for a solution of the variational inequality holds under some regularity condition. We recall in this section some notations and basic results which will be used in next section.

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For a subset $D \subset \mathbb{R}^n$, the closure of D will be denoted by cl$D$ and the convex hull of D by co$D$. Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous convex function. The conjugate function of h, $h^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) := \sup\{v(x) - h(x) \mid x \in \text{dom } h\},$$

where dom $h := \{x \in \mathbb{R}^n \mid h(x) < +\infty\}$ is the effective domain of h. The epigraph of h is defined by

$$\text{epi } h := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \text{dom } h, h(x) \le r\}.$$

The set (possibly empty)

$$\partial h(a) := \{v \in \mathbb{R}^n \mid h(x) - h(a) \ge v(x - a), \forall x \in \text{dom } h\}$$

is the subdifferential of the convex function h at $a \in$ dom h. For a closed convex subset D of $\mathbb{R}^n$, the indicator function δ_D is defined as $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$. Then $\partial\delta_D(x) = N_D(x)$, which is known as the normal cone of D of x.

For a proper lower semicontinuous convex functions $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the lower semicontinuous convex hull of g is denoted by cl co$g$. That is, epi(cl co$g$) = cl co(epi g). For details, see [7]. Let g_i, $i \in I$ (where I is an arbitrary index set) be proper lower semicontinuous convex functions. It is well known (see [7]) that if $\sup_{i\in I} g_i$ is proper, then

$$\left(\sup_{i\in I} g_i\right)^* = \text{cl co}\left(\inf_{i\in I} g_i^*\right).$$

Thus we can check that

$$\text{epi}\left(\sup_{i\in I} g_i\right)^* = \text{cl co}\left(\bigcup_{i\in I} \text{epi} g_i^*\right). \tag{1.1}$$

2. A characterization of solutions

In this section, we introduce a variational inequality defined by the second-order cone, and then we prove that a necessary and sufficient condition for a point to be a solution of the inequality holds under a regularity condition. The condition gives a complete characterization of solutions of Aff(VI).

Let $Q := \{x \in \mathbb{R}^n \mid \|Hx + b\| \le c^t x + d\ \}$, where H is an $(m-1) \times n$ matrix, $b \in \mathbb{R}^{m-1}$, $c \in \mathbb{R}^n$, $d \in \mathbb{R}$ and $\|z\| = \sqrt{z^t z}$, $z \in \mathbb{R}^{m-1}$. Suppose that $Q \neq \emptyset$. Let $K = \{(y, t)^t \in \mathbb{R}^{m-1} \times \mathbb{R} \mid \|y\| \le t\ \}$, that is, K is a second-order cone in $\mathbb{R}^m$. Then K is self-dual, that is, $K = \{z \in \mathbb{R}^m \mid k^t z \geq 0 \ \forall z \in K\}$. Consider the following affine variational inequality defined by the second-order cone K.

$$\text{Aff(VI)}: \qquad \text{Find } \bar{x} \in Q \text{ such that } \langle M\bar{x}, x - \bar{x}\rangle \geq 0 \quad \forall x \in Q,$$

where $M \in \mathbb{R}^{n\times n}$ is a given matrix. Note that

$$x \in Q \ \Leftrightarrow \begin{pmatrix} Hx + b \\ c^t x + d \end{pmatrix} \in K. \tag{2.1}$$

Thus Aff(VI) is a variational inequality defined by the second-order cone K. We begin with the following lemma.

Lemma 2.1. *Let $\bar{x} \in Q$. Then we have the following:*

$$u \in N_Q(\bar{x}) \iff (u, u^t\bar{x}) \in \operatorname{cl}\Big[\bigcup_{\lambda\in K}\left\{\left(-\begin{pmatrix} H \\ c^t \end{pmatrix}^t \lambda, \begin{pmatrix} b \\ d \end{pmatrix}^t \lambda\right)\right\} + \{0\}\times\mathbb{R}^+\Big].$$

Proof. Let $g(x) = -\begin{pmatrix} Hx + b \\ c^t x + d \end{pmatrix}$. For any $\lambda \in K$, we get

$$\begin{aligned}
(\lambda^t g)^*(v) &= \sup\{v^t x - \lambda^t g(x) \mid x \in \mathbb{R}^n\} \\
&= \sup\left\{\left[v + \begin{pmatrix} H \\ c^t \end{pmatrix}^t \lambda\right]^t x \;\Big|\; x \in \mathbb{R}^n\right\} + \begin{pmatrix} b \\ d \end{pmatrix}^t \lambda \\
&= \begin{cases} \begin{pmatrix} b \\ d \end{pmatrix}^t \lambda & \text{if } v = -\begin{pmatrix} H \\ c^t \end{pmatrix}^t \lambda \\ +\infty & \text{if } v \neq -\begin{pmatrix} H \\ c^t \end{pmatrix}^t \lambda, \end{cases}
\end{aligned}$$

and hence

$$\begin{aligned}
\operatorname{epi}(\lambda^t g)^* &= \left\{\left(-\begin{pmatrix} H \\ c^t \end{pmatrix}^t \lambda,\ \begin{pmatrix} b \\ d \end{pmatrix}^t \lambda + r\right) \;\Big|\; r \geq 0\right\} \\
&= \left(-\begin{pmatrix} H \\ c^t \end{pmatrix}^t \lambda,\ \begin{pmatrix} b \\ d \end{pmatrix}^t \lambda\right) + \{0\} \times \mathbb{R}^+.
\end{aligned}$$

Thus we have

$$\bigcup_{\lambda\in K} \operatorname{epi}(\lambda^t g)^* = \bigcup_{\lambda\in K}\left\{\left(-\begin{pmatrix} H \\ c^t \end{pmatrix}^t \lambda,\ \begin{pmatrix} b \\ d \end{pmatrix}^t \lambda\right)\right\} + \{0\} \times \mathbb{R}^+.$$

Since $\bar{x} \in Q$, we see

$$\begin{aligned} u \in N_Q(\bar{x}) \quad &\iff \quad u^t(x-\bar{x}) \le 0 \ \forall x \in Q \\ &\iff \quad \sup_{x\in Q} u^t x = u^t \bar{x} \\ &\iff \quad \delta_Q^*(u) = u^t\bar{x} \\ &\iff \quad (u,\ u^t\bar{x}) \in \text{epi } \delta_Q^*. \end{aligned}$$

As $\delta_Q(x) = \sup_{\lambda\in K} \lambda^t g(x)$ and $\bigcup_{\lambda\in K} \text{epi}(\lambda^t g)^*$ is convex [4], it follows from (1.1) that

$$\text{epi } \delta_Q^* = \text{cl co}\Big[\bigcup_{\lambda\in K} \text{epi}(\lambda^t g)^*\Big] = \text{cl}\Big[\bigcup_{\lambda\in K} \text{epi}(\lambda^t g)^*\Big].$$

This completes the proof. □

Now we present the main result of this note.

THEOREM 2.2. *Assume that* $\bigcup_{\lambda\in K}\left\{\left(-\begin{pmatrix} H \\ c^t\end{pmatrix}^t \lambda, \begin{pmatrix} b \\ d\end{pmatrix}^t \lambda\right)\right\} + \{0\} \times \mathbb{R}^+$ *is closed. Then* $\bar{x} \in \mathbb{R}^n$ *is a solution of Aff(VI) if and only if* $\begin{pmatrix} H\bar{x}+b \\ c^t\bar{x}+d\end{pmatrix} \in K$*, and there exists* $\lambda \in K$ *such that*

$$M\bar{x} = \begin{pmatrix} H \\ c^t\end{pmatrix}^t \lambda \quad \text{and} \quad \lambda^t \begin{pmatrix} H\bar{x}+b \\ c^t\bar{x}+d\end{pmatrix} = 0.$$

Proof. Let us denote by sol(Aff(VI)) the solution set of Aff(VI). Then we have the following equivalences:

$$\bar{x} \in \text{sol} \quad (\text{Aff(VI)}) \quad \iff \bar{x} \in Q \text{ and } -M\bar{x} \in N_Q(\bar{x})$$

$$\begin{aligned}
\Longleftrightarrow \quad & \bar{x} \in Q \text{ and } \Big[\exists \lambda \in K \text{ and } \exists \alpha \geq 0 \text{ satisfying} \\
& (-M\bar{x}, -\bar{x}^t M^t \bar{x}) = \left(-\begin{pmatrix} H \\ c^t \end{pmatrix}^t \lambda, \begin{pmatrix} b \\ d \end{pmatrix}^t \lambda + \alpha \right) \\
& \text{i.e., } -M\bar{x} = -\begin{pmatrix} H \\ c^t \end{pmatrix}^t \lambda, \ -\bar{x}^t M^t \bar{x} = \begin{pmatrix} b \\ d \end{pmatrix}^t \lambda + \alpha \Big] \\
\Longleftrightarrow \quad & \begin{pmatrix} H\bar{x} + b \\ c^t \bar{x} + d \end{pmatrix} \in K \text{ and } \Big[\exists \lambda \in K \text{ and } \exists \alpha \geq 0 \text{ such that} \\
& M\bar{x} = \begin{pmatrix} H \\ c^t \end{pmatrix}^t \lambda \text{ and } -\lambda^t \begin{pmatrix} H\bar{x} + b \\ c^t \bar{x} + d \end{pmatrix} = \alpha \Big] \\
\Longleftrightarrow \quad & \begin{pmatrix} H\bar{x} + b \\ c^t \bar{x} + d \end{pmatrix} \in K \text{ and } \Big[\exists \lambda \in K \text{ such that} \\
& M\bar{x} = \begin{pmatrix} H \\ c^t \end{pmatrix}^t \lambda \text{ and } \lambda^t \begin{pmatrix} H\bar{x} + b \\ c^t \bar{x} + d \end{pmatrix} = 0. \Big]
\end{aligned}$$

The second and third equivalences come from Lemma 2.1 and (2.1), respectively. This completes the proof. □

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