JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **20**, No. 3, September 2007

ASYMPTOTIC EQUIVALENCE OF VOLTERRA DIFFERENCE SYSTEMS

Sung Kyu Choi *, Jin Soon Kim **, and Namjip Koo ***

Abstract. We obtain a discrete analogue of Nohel's result in [5] about asymptotic equivalence between perturbed Volterra system and unperturbed system.

1. Introduction

Let $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$ and $\mathbb{R}^d$ be the d-dimensional real Euclidean space with norm

$$|x| = \sum_{i=1}^{d} |x_i|, \qquad x = (x_1, \cdots x_d) \in \mathbb{R}^d.$$

For a $d \times d$ matrix $A = [a_{ij}]$ on $\mathbb{Z}_+ \times \mathbb{Z}_+$, the norm of A is given by

$$|A| = \sum_{i=1}^{d} \sum_{j=1}^{d} |a_{ij}|.$$

Let BC be the space of all bounded sequences equipped with the norm

$$|\phi| = \sup_{n \geq 0} |\phi(n)| \ , \quad \phi \in BC.$$

We consider the perturbed system of Volterra difference equations

$$x(n) = f(n) + \sum_{s=0}^{n} B(n,s)[x(s) + g(s, x(s))], \quad n \geq 0 \tag{1.1}$$

Received August 6, 2007.
2000 Mathematics Subject Classification: Primary 39A12, 45E99.
Key words and phrases: Volterra difference system, resolvent matrix, asymptotic equivalence.
This work was supported by the Korea Research Foundation Grant founded by the Korea Government(MOEHRD)(KRF-2005-070-C00015).

and unperturbed system

$$y(n) = f(n) + \sum_{s=0}^{n} B(n,s)y(s). \tag{1.2}$$

Two systems (1.1) and (1.2) are *asymptotically equivalent* if for every bounded solution $x(n)$ of (1.1), there exists a bounded solution $y(n)$ of (1.2) such that

$$\lim_{n\to\infty} [x(n) - y(n)] = 0 \tag{1.3}$$

and conversely, for each bounded solution $v(n)$ of (1.2) there exists a bounded solution $u(n)$ of (1.1) such that

$$\lim_{n\to\infty} [u(n) - v(n)] = 0.$$

The purpose of this paper is to obtain the asymptotic equivalence between (1.1) and (1.2) as a discrete analogue of Nohel's result in [5].

For the asymptotic equivalence between perturbed Volterra difference system

$$x(n+1) = A(n)x(n) + \sum_{s=0}^{n} B(n,s)x(s) + f(n), \quad n \ge 0$$

and linear Volterra difference system

$$y(n+1) = A(n)y(n) + \sum_{s=0}^{n} B(n,s)y(s), \quad n \ge 0,$$

see [1] and [2].

2. Main Results

Consider the perturbed system of Volterra difference equations

$$x(n) = f(n) + \sum_{s=0}^{n} B(n,s)[x(s) + g(s, x(s))], \quad n \ge 0 \tag{2.1}$$

and unperturbed system

$$y(n) = f(n) + \sum_{s=0}^{n} B(n,s)y(s), \tag{2.2}$$

where $x, y, f : \mathbb{Z}_+ \to \mathbb{R}^d$, B is a $d \times d$ matrix on $\mathbb{Z}_+ \times \mathbb{Z}_+$, and $g(n,x)$ is defined for $n \ge 0$, $|x| < \infty$ and is continuous in x, $g(n,0) = 0$ with

$$g(n,x) = o(|x|) \text{ uniformly in } n, \text{ as } |x| \to 0. \tag{2.3}$$

Note that the solution $y(n)$ can be represented by

$$y(n) = f(n) - \sum_{k=0}^{n} R(n,k)f(k), \tag{2.4}$$

where the resolvent matrix $R(n,m)$ satisfies

$$R(n,m) = \sum_{j=m}^{n} B(n,j)B(j,m) - B(n,m), \quad 0 \leq m \leq n. \tag{2.5}$$

See[4].

Firstly, we show that the following representation is a solution of (2.1).

Theorem 2.1. *The solution $x(n)$ of (2.1) is given by the form*

$$x(n) = f(n) - \sum_{k=0}^{n} R(n,k)[f(k) + g(k, x(k))]. \tag{2.6}$$

Proof. We have

$$\begin{aligned} x(n) &= \sum_{k=0}^{n} B(n,k)g(k,x(k)) + f(n) - \sum_{k=0}^{n} R(n,k)f(k) \\ &= \sum_{k=0}^{n} B(n,k)g(k,x(k)) + f(n) \\ &- \sum_{k=0}^{n} R(n,k)[\sum_{l=0}^{k} B(k,l)g(l,x(l)) + f(k)]. \end{aligned}$$

Thus

$$\begin{aligned} x(n) - f(n) &+ \sum_{k=0}^{n} R(n,k)f(k) - \sum_{l=0}^{n} B(n,l)g(l,x(l)) \\ &= -\sum_{k=0}^{n} R(n,k) \sum_{l=0}^{k} B(k,l)g(l,x(l)) \\ &= -\sum_{k=0}^{n} \sum_{l=0}^{k} R(n,k)B(k,l)g(l,x(l)). \end{aligned}$$

Hence we obtain

$$\begin{aligned} x(n) &= f(n) - \sum_{k=0}^{n} R(n,k)f(k) - \sum_{k=0}^{n} [R(n,k)B(k,l) - B(n,l)]g(l,x(l)) \\ &= f(n) - \sum_{k=0}^{n} R(n,k)[f(n) + g(k,x(k))]. \end{aligned}$$

by(2.5). □

We need the following fixed point theorem.

Schauder - Tychonoff Theorem [3] Let $C(J)$ denote the set of all functions which are continuous on the interval J, and let F be the subset formed by those functions $x(t)$ such that

$$|x(t)| \le \mu(t) \quad \text{for all} \;\; t \in J,$$

where $\mu(t)$ is a fixed positive continuous function.

Let T be a mapping of F into itself with the properties

(i) T is continuous, in the sense that if $x_n \in F$, $n = 1, 2, \cdots$, and $x_n \to x$ uniformly on every compact subinterval of J, then $Tx_n \to Tx$ uniformly on every compact subinterval of J,

(ii) The image set $T(F)$ is equicontinuous and bounded at every point of J.

Then T has at least one fixed point in F.

Theorem 2.2. *Let* $y(n)$ *be a bounded solution of (2.2). Suppose that*

(i) $\sum_{s=0}^{n} |R(n,s)| \le c$, *for some* $c > 0$,

(ii) *for any* $n_0 > 0$,

$$\lim_{n\to\infty} [\sum_{s=0}^{n_0} |R(n,s) - R(n_0,s)| + \sum_{s=n_0+1}^{n} |R(n,s)|] = 0, n > n_0 > 0,$$

(iii) *for any* λ *with* $0 < \lambda < 1$ *and* $\epsilon > 0$, $|y| \le \lambda\epsilon$ *for some* $\epsilon_0 > 0$ *with* $0 < \epsilon \le \epsilon_0$.

Then there exists at least one solution $x(n)$ *of (2.1) such that* $x \in BC$ *and* $|x| \le \epsilon$.

Proof. In view of the assumption (2.3), there exists an $\epsilon_0 > 0$ such that $|x| \le \epsilon_0$ implies $|g(n,x)| \le \beta|x|$ uniformly in n for some $\beta > 0$ with $\beta c < 1 - \lambda$. Let $0 < \epsilon \le \epsilon_0$. Consider the set

$$S_\epsilon = \{\phi \in BC : |\phi| \le \epsilon\}.$$

Define the operator $T : S_\epsilon \to BC$ by the relation

$$T\phi(n) = y(n) + \sum_{s=0}^{n} R(n,s)g(s,\phi(s)), \qquad n \geq 0.$$

We claim that $T(S_\epsilon) \subset S_\epsilon$. Using (i) and (iii), we have

$$\begin{aligned} |T\phi(n)| &\leq |y(n)| + \sum_{s=0}^{n} |R(n,s)||g(s,\phi(s))| \\ &\leq |y| + c\beta \sup_{n\geq 0} |\phi(n)| \\ &\leq |y| + c\beta\epsilon \\ &\leq \lambda\epsilon + (1-\lambda)\epsilon = \epsilon. \end{aligned}$$

For the proof of continuity of T, we let $\phi_m \in S_\epsilon$ and suppose $\phi_m \to \phi$ uniformly on every compact subset of $\mathbb{Z}_+$. Then

$$\begin{aligned} |T\phi(n) - T\phi_m(n)| &\leq \sum_{s=0}^{n} |R(n,s)||g(s,\phi(s)) - g(s,\phi_m(s))| \\ &\leq c \sup_{0\leq s\leq n} |g(s,\phi(s)) - g(s,\phi_m(s))| \end{aligned}$$

which tends to zero as $n \to \infty$, uniformly on compact subset of $\mathbb{Z}_+$ since $g(n,x)$ is continuous in x.

Now we show that $T(S_\epsilon)$ is equicontinuous. To do this we let $n_o \in \mathbb{Z}_+$ and $n > n_0$ (the same argument applies to $n < n_0$). Let $\epsilon > 0$ be given. We show that there exists a $\delta > 0$ such that $|n - n_0| \leq \delta$ implies $|T\phi(n) - T\phi(n_0)| < \epsilon$. From the assumptions we have

$$\begin{aligned} |T\phi(n) \quad &- \quad T\phi(n_0)| \\ &\leq |y(n) - y(n_0)| + |\sum_{s=0}^{n} R(n,s) - \sum_{s=0}^{n_0} R(n_0,s)||g(s,\phi(s))| \\ &= |y(n) - y(n_0)| + [\sum_{s=0}^{n_0} |R(n,s) - R(n_0,s)| \\ &\qquad + \sum_{s=n_0+1}^{n} |R(n,s)|]|g(s,\phi(s))| \end{aligned}$$

$$\leq \quad |y(n) - y(n_0)| + \sup_{0 \leq s \leq n} |g(s, \phi(s))| [\sum_{s=0}^{n_0} |R(n,s) - R(n_0, s)| + \sum_{s=n_0+1}^{n} |R(n,s)|].$$

Let $\eta > 0$ be given. Choose a $\delta_1 > 0$ such that

$$|y(n) - y(n_0)| \leq \eta/2 \qquad \text{when } |n - n_0| \leq \delta_1.$$

By (ii), choose a $\delta_2 > 0$ such that

$$\sum_{s=0}^{n_0} |R(n,s) - R(n_0,s)| + \sum_{s=n_0+1}^{n} |R(n,s)| \leq \frac{\eta}{2(1+\beta\epsilon_0)}$$

when $|n - n_0| \leq \delta_2$. Putting $\delta = \min\{\delta_1, \delta_2\}$, we obtain

$$\begin{aligned} |T\phi(n) - T\phi(n_0)| &\leq \frac{\eta}{2} + \beta\epsilon_0 \frac{\eta}{2(1+\beta\epsilon_0)} \\ &\leq \frac{\eta}{2} + \frac{\eta}{2} = \eta. \end{aligned}$$

This shows that the pointwise equicontinuity of the functions in $T(S_\epsilon)$. Therefore, by the Schauder - Tychonoff Theorem, there exists a function $x \in S_\epsilon$ such that $Tx = x$ or

$$x(n) = y(n) + \sum_{s=0}^{n} R(n,s) g(s, x(s)).$$

This completes the proof. □

Under the assumptions in Theorem 2.2 we can obtain one asymptotic stability theorem as a corollary.

Corollary 2.3. *Let $y(n)$ be a solution of (2.2). Suppose that*

(i) $\sum_{s=0}^{n} |R(n,s)| \leq c$ *for some* $c > 0$,
(ii) *for any* $n_0 > 0$,

$$\lim_{n\to\infty} [\sum_{s=0}^{n_0} |R(n,s) - R(n_0,s)| + \sum_{s=n_0+1}^{n} |R(n,s)|] = 0, n > n_0 > 0,$$

(iii) *for any λ with $0 < \lambda < 1$ and $\epsilon > 0$,*
$|y| \leq \lambda\epsilon$ *for some* $\epsilon_0 > 0$ *with* $0 < \epsilon \leq \epsilon_0$.
(iv) *for any* $N > 0$, $\lim_{n\to\infty} \sum_{s=0}^{N} |R(n,s)| = 0$.

If $\lim_{n\to\infty} y(n) = 0$, *then* $\lim_{n\to\infty} x(n) = 0$

Proof. Let $S_0 = \{\phi \in S_\epsilon : \lim_{n\to\infty} \phi(n) = 0\}$. Then S_0 is a closed subset of S_ϵ under the uniform norm. Thus it suffices to show that $R(S_0) \subset S_0$. Suppose that $\lim_{n\to\infty} x(n) \neq 0$. Then

$$\mu = \lim_{n\to\infty} \sup |x(n)| > 0 \ .$$

For a fixed number γ , let $1 - \lambda < \gamma < 1$. Choose $N > 0$ so large that $|x(n)| \le \mu/\gamma$ when $n \ge N$. Then

$$\begin{aligned} |x(n)| &\le |y(n)| + \sum_{s=0}^{n} |R(n,s)||g(s,x(s))| \\ &\le |y(n)| + \sum_{s=0}^{N} |R(n,s)||g(s,x(s))| + \sum_{s=N}^{n} |R(n,s)||g(s,x(s))| \\ &\le |y(n)| + \beta\epsilon \sum_{s=0}^{N} |R(n,s)| + \beta\frac{\mu}{\gamma} \sum_{s=N}^{n} |R(n,s)|. \end{aligned}$$

Taking the limit sup, we obtain

$$\begin{aligned} \mu &\le 0 + 0 + \beta\frac{\mu}{\gamma}c \\ &< \frac{\mu}{\gamma}(1-\lambda) \\ &< \mu, \end{aligned}$$

a contradiction. Therefore we have $\lim_{n\to\infty} x(n) = 0$ □

THEOREM 2.4. *Assume that*

(i) $\sum_{s=0}^{n} |R(n,s)| \le c$ *for some* $c > 0$,
(ii) *for any* $n_0 > 0$,

$$\lim_{n\to\infty} [\sum_{s=0}^{n_0} |R(n,s) - R(n_0,s)| + \sum_{s=n_0+1}^{n} |R(n,s)|] = 0, n > n_0 > 0,$$

(iii) *for any* $N > 0$, $lim_{n\to\infty} \sum_{s=0}^{N} |R(n,s)| = 0$,
(iv) $|g(n,x)| \le \lambda(n)|x|$.

where $\lambda(n) > 0$, *is bounded on* $\mathbb{Z}_+$ *with* $\lim_{n\to\infty} \lambda(n) = 0$ *and* $|\lambda|c \le \frac{1}{2}$. *Then (2.1) and (2.2) are asymptotically equivalent.*

Proof. Let $y(n) \in BC$ be a solution of (2.2) with $|y| \le k$ for some $k > 0$. Consider the set

$$S_k = \{\phi \in BS : |\phi| \le 2k\}.$$

Define the operator $T : S_k \to BC$ by

$$T\phi(n) = y(n) - \sum_{s=0}^{n} R(n,s)g(s,\phi(s)).$$

Then T is continuous as in the proof of Theorem 2.2.

Also, $T(S_k) \subset Sk$ since

$$\begin{aligned} |T\phi(n)| &\le |y(n)| + \sum_{s=0}^{n} |R(n,s)||g(s,\phi(s))| \\ &\le |y| + \sum_{s=0}^{n} |R(n,s)|\lambda(s)|\phi(s)| \\ &\le k + C|\lambda|2k \\ &= 2k. \end{aligned}$$

To show that $T(S_k)$ is equicontinuous, let $n_0 \in \mathbb{Z}_+$ and $n > n_0$. We have

$$\begin{aligned} | \quad T\phi(n) \quad & -T\phi(n_0)| \le |y(n) - y(n_0)| \\ + \quad & \sup_{0\le s\le n} |g(s,\phi(s))| [\sum_{s=0}^{n_0} |R(n,s) - R(n_0,s)| + \sum_{s=n_0+1}^{n} |R(n,s)|] \\ \le \quad & |y(n) - y(n_0)| + 2|\lambda|k[\sum_{s=0}^{n_0} |R(n,s) - R(n_0,s)| \\ & + \sum_{s=n_0+1}^{n} |R(n,s)|]. \end{aligned}$$

For any $\epsilon > 0$, there exists a $\delta_1 > 0$ such that

$$|y(n) - y(n_0)| \le \frac{\epsilon}{2}, \text{ when } |n - n_0| \le \delta_1.$$

Also, there exists a $\delta_2 > 0$ such that

$$\sum_{s=0}^{n_0} |R(n,s) - R(n_0,s)| + \sum_{s=n_0+1}^{n} |R(n,s)| \le \frac{\epsilon}{4k|\lambda|}, \quad \text{when } |n - n_0| \le \delta_2. \tag{2.7}$$

Hence, by putting $\delta = \min\{\delta_1, \delta_2\}$, we obtain

$$\begin{aligned} |T\phi(n) - T\phi(n_0)| &\le \frac{\epsilon}{2} + 2|\lambda|k\frac{\epsilon}{4|\lambda|k} \\ &= \epsilon \end{aligned}$$

whenever $|n - n_0| \leq \delta$. Therefore there exists solution $x(n) \in BS$ of (2.1) by the Schauder - Tychonoff Theorem.

We show that $\lim_{n\to\infty}[x(n) - y(n)] = 0$. Let $\epsilon > 0$ be given. From (iv), there exists $N > 0$ such that

$$|\lambda(n)| \leq \frac{\epsilon}{4kc}\ , \quad n \geq N. \tag{2.8}$$

We can choose $N_1 \geq N$ such that

$$\sum_{s=0}^{N} |R(n,s)| \leq \frac{\epsilon}{4|\lambda|k}\ , \quad n \geq N_1$$

from (iii). Now we have

$$\begin{aligned} |x(n) - y(n)| &\leq \sum_{s=0}^{N} |R(n,s)|\lambda(s)|x(s)| + \sum_{s=N+1}^{n} |R(n,s)|\lambda(s)|x(s)| \\ &\leq 2k|\lambda| \sum_{s=0}^{N} |R(n,s)| + 2k \sup_{N\leq n<\infty} \lambda(n) \sum_{s=0}^{n} |R(n,s)| \\ &\leq 2k|\lambda| \frac{\epsilon}{4k|\lambda|} + 2kc\frac{\epsilon}{4kc} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\ , \qquad n \geq N_1. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary this shows the assertion.

For the converse, let $x \in BC$ be a solution of (2.1). Define

$$y(n) = x(n) + \sum_{s=0}^{n} R(n,s)g(s,x(s)).$$

Then it is easy to show that $y(n)$ is a solution of (2.2) and $y \in BC$ since

$$\begin{aligned} |y(n)| &\leq |x(n)| + c|\lambda||x| \\ &< \infty\ , \qquad 0 \leq n < \infty\ . \end{aligned}$$

To show that $\lim_{n\to\infty}[y(n) - x(n)] = 0$, let $\epsilon > 0$ and $|x| \leq k$. Then

$$\begin{aligned} |y(n) - x(n)| &\leq \sum_{s=0}^{N} |R(n,s)|\lambda(s)|x(s)| + \sum_{s=N}^{n} |R(n,s)|\lambda(s)|x(s)| \\ &\leq |\lambda|k\frac{\epsilon}{4k|\lambda|} + \frac{\epsilon}{4kc}kc \end{aligned}$$

by (2.7) and (2.8). Hence $|y(n) - x(n)| \leq \frac{\epsilon}{2}$. Since $\epsilon > 0$ is arbitrary we show that the asymptotic relationship holds, and the proof is complete. □

References

[1] S. K. Choi and N. J. Koo, *Asymptotic equivalence between two linear Volterra difference systems*, Computers Math. Applic. 47 (2004), 461-471.

[2] S. K. Choi and N. J. Koo, *Asymptotic property of linear Volterra difference systems,* J. Math. Anal. Appl. **321** (2006), 260-272.

[3] W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, D. C. Heath & Co., Boston, 1965.

[4] V. B. Kolmanovskii, *On asymptotic equivalence of the solutions of some Volterra difference equations*, Auto. Remote Control. **62** (2001), 548-556

[5] J. A. Nohel, *Asymptotic relationship between systems of Volterra equations*, Annali de Matematica Pura ed Applicata **90** (1971), 149-165.

*

Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: skchoi@math.cnu.ac.kr

**

Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea

***

Department of Mathematics
Chungnam National University
Daejeon 305-764, Republic of Korea
E-mail: njkoo@math.cnu.ac.kr