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CONTINUOUS SHADOWING AND INVERSE SHADOWING FOR FLOWS

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Abstract. The notions of continuous shadowing and inverse shadowing for flows are introduced, and show that an expansive flow on a compact manifold with the shadowing property has the continuous shadowing property. Moreover it is proved that the continuous shadowing property implies the inverse shadowing property.

1. Introduction

Let M be a compact smooth manifold with a Riemannian metric d, and consider a C^1-vector field X on M and the system of differential equations

$$\dot{x} = X(x) \tag{1}$$

Let $\chi^1(M)$ be the set of all C^1-vector fields on M with the C^1-topology, and let $F : M \times \mathbb{R} \longrightarrow M$ be the flow induced by the system (1). We shall write xt instead of $F(x,t)$ for $x \in M$ and $t \in \mathbb{R}$ for simplicity. For $\delta, \tau > 0$ we say that a mapping

$$\phi : \mathbb{R} \longrightarrow M$$

is a (δ, τ)-*pseudo solution* of system (1) if there exists an increasing sequence $\{t_k \in \mathbb{R} : k \in \mathbb{Z}\}$ such that

(i) $t_0 = 0$,
(ii) $t_{k+1} - t_k \geq \tau$,
(iii) $\lim_{t \to t_k^+} \phi(t) = \phi(t_k)$,
(iv) $\dot{\phi}(t) = X(\phi(t))$ for $t \in (t_k, t_{k+1})$,
(v) $d(\phi(t_k), \phi_-(t_k)) < \delta$,

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where $\phi_-(t_k) = \lim_{t\to t_k^-} \phi(t)$ and $k \in \mathbb{Z}$.

For $\delta, \tau > 0$ we say that a mapping $\Phi : M\times\mathbb{R} \longrightarrow M$ is a (δ,τ)-*method* for F if, for any $x \in M$, the map $\Phi_x : \mathbb{R} \longrightarrow M$ defined by

$$\Phi_x(t) = \Phi(x,t),\ t \in \mathbb{R},$$

is a (δ,τ)-pseudo solution of system (1). A method Φ is said to be *complete* if $\Phi(x,0) = x$ for all $x \in M$. Note that a (δ,τ)-method for F can be considered as a family of (δ,τ)-pseudo solution of system (1). A method Φ of F is said to be *continuous* if the map

$$\tilde{\Phi} : M \longrightarrow M^{\mathbb{R}}$$

given by

$$\tilde{\Phi}(x)(t) = \Phi(x,t),\ x \in M,\ t \in \mathbb{R}$$

is continuous under the topology of compact convergence on $M^{\mathbb{R}}$, where $M^{\mathbb{R}}$ denotes the set of all functions from $\mathbb{R}$ into M. The set of all complete (δ,τ)-methods [resp. complete continuous (δ,τ)-methods] for F will be denote by $\mathcal{T}_a(\delta,\tau,F)$ [resp. $\mathcal{T}_c(\delta,\tau,F)$]. It is clear that if Y is another vector field on M which is sufficiently close to X then the system

$$\dot{x} = Y(x) \tag{2}$$

induces a complete continuous method for F.

Let $\mathcal{T}_h(\delta,\tau,F)$ be the set of all complete continuous (δ,τ)-methods for F which are induced by system (2) with $d_0(X,Y) < \delta$, where d_0 is a C^0-metric on $\chi^1(M)$.

Let $C(\mathbb{R})$ be the set of all continuous maps from $\mathbb{R}$ to itself, and we let

$$Rep = \{h \in C(\mathbb{R}) : h(t) < h(s) \text{ for } t < s,\ h(0) = 0\},$$
$$Rep^* = \{h \in Rep : h(\mathbb{R}) = \mathbb{R}\}$$

and

$$Rep(\varepsilon) = \{h \in Rep^* : \left| \frac{h(s)-h(t)}{s-t} \right| \leqslant \varepsilon,\ (t \neq s)\},\quad (\varepsilon > 0).$$

Each element of Rep [or $Rep^*, Rep(\varepsilon)$] is called a *reparametrization.* We say that a (δ,τ)-pseudo solution ϕ of (1) is *weakly ε-shadowed* [resp. *normally ε-shadowed, strongly ε-shadowed*]) by a point $x \in M$ if there is $h \in Rep$ [resp. $h \in Rep^*$, $h \in Rep(\varepsilon)$] such that

$$d(xh(t), \phi(t)) < \varepsilon$$

for all $t \in \mathbb{R}$.

We say that the flow F of system (1) has the *shadowing property* [or *pseudo orbit tracing property*] if for any $\varepsilon > 0$ and $\tau > 0$, there exists $\delta > 0$ such that any (δ, τ)-pseudo solution of system (1) is normally ε-shadowed by some point of M.

2. Continuous shadowing

In this section, we introduce the concept of continuous shadowing for flows. Let

$$\begin{aligned}\mathcal{P}_\alpha(\delta,\tau,F) &= \bigcup\{\Phi_x : x\in M, \Phi\in\mathcal{T}_\alpha(\delta,\tau,F)\}\subset M^{\mathbb{R}}\\ &= \textstyle\bigcup_{x\in M,\Phi\in\mathcal{T}_\alpha(\delta,\tau,F)}\Phi_x(\mathbb{R}),\end{aligned}$$

where $\alpha = a, c, h$. Clearly we have

$$\mathcal{P}_h(\delta,\tau,F)\subset\mathcal{P}_c(\delta,\tau,F)\subset\mathcal{P}_a(\delta,\tau,F).$$

Definition 2.1. *We say that the flow F has the shadowing property with respect to the class $\mathcal{T}_\alpha$ [or $\mathcal{T}_\alpha$-shadowing property], $\alpha = a, c, h$ if for any $\varepsilon > 0$ and $\tau > 0$ there exists $\delta > 0$ and a map $\gamma : \mathcal{P}(\delta,\tau,F)\longrightarrow M$ such that for any (δ,τ)-pseudo solution $\Phi_x\in\mathcal{P}_\alpha$, there exists $h\in Rep^*$ for which*

$$d(\gamma(\Phi_x)h(t),\Phi_x(t)) < \varepsilon$$

for all $t\in\mathbb{R}$. If γ is continuous, then we say that F has the continuous shadowing with respect to the class $\mathcal{T}_\alpha$.

It is easy to show that the flow F has the shadowing property with respect to the class $\mathcal{T}_a$ if and only if it has the shadowing property in the original sense. Clearly we see that the $\mathcal{T}_a$-shadowing property implies the $\mathcal{T}_c$-shadowing property.

Definition 2.2. *We say that the flow F has the inverse shadowing property with respect to the class $\mathcal{T}_\alpha$ [or $\mathcal{T}_\alpha$-inverse shadowing property], $\alpha = a, c, h$, if for any $\varepsilon > 0$, $\tau > 0$, there exists $\delta > 0$ such that for any (δ,τ)-method $\Phi\in\mathcal{T}_\alpha(\delta,\tau,F)$, there exists a map $s : M\longrightarrow M$ which has the following property: for any point $y\in M$ there exists $h\in Rep^*$ such that*

$$d(yh(t),\Phi_{s(y)}(t)) < \varepsilon$$

for all $t\in\mathbb{R}$. If s is continuous, then we say that F has the continuous inverse shadowing property.

DEFINITION 2.3. *We say that a flow F on a compact manifold M is expansive if for any $\varepsilon > 0$, there exists $\delta > 0$ with the property that if $d(xt, ys(t)) < \delta$ for all $t \in \mathbb{R}$, for a pair of points $x, y \in M$ and a continuous map $s : \mathbb{R} \longrightarrow \mathbb{R}$ with $s(0) = 0$, then $y = xt$, where $|t| < \varepsilon$. The constant $\delta > 0$ is said to be an expansive constant of F corresponding to ε.*

It is clear from the definition that there are only a finite number of fixed points for an expansive flow and each is an isolated point of M. This reduces the study of expansive flows to those without fixed points, and so we assume that all the expansive flows on M do not have fixed points throughout the section.

LEMMA 2.4. *([5]). A flow F on M is expansive if and only if for all $\varepsilon > 0$, there exists $r > 0$ such that if $t = (t_i)_{-\infty}^{\infty}$, $u = (u_i)_{-\infty}^{\infty}$ are doubly infinite sequences of real numbers with $u_0 = t_0 = 0$, $0 < t_{i+1} - t_i \leq r$, $|u_{i+1} - u_i| \leq r$, $t_i \longrightarrow \infty$, $t_{-i} \longrightarrow -\infty$, as $i \longrightarrow \infty$, and if $x, y \in M$ satisfy $d(xt_i, yu_i) \leq r$ for all $i \in \mathbb{Z}$, then there exists t such that $|t| < \varepsilon$ and $y = xt$.*

LEMMA 2.5. *([5]). Let F be an expansive flow. Then there is $T_0 > 0$ such that for every T satisfying $0 < T < T_0$, there exists $\gamma_0 > 0$ with $d(xT, x) \geq \gamma_0$ for every $x \in M$.*

THEOREM 2.6. *If a flow F on a compact manifold M is expansive and has the shadowing property then it has the continuous shadowing property with respect to the class $\mathcal{T}_\alpha$.*

Proof. Let $\tau > 0$ be arbitrary. Take T_0 as in lemma 2.5, choose $\varepsilon > 0$ with $\varepsilon < \frac{1}{2}T_0$ and select $\gamma_0 > 0$ as in lemma 2.4 for the ε. Then we can choose $\gamma_1 > 0$ with $d(y(\frac{1}{2}\gamma_0), y) \geq \gamma_1$ for all $y \in M$ by Lemma 2.5. Let $\varepsilon' > 0$ be an expansive constant corresponding to γ_0 with $\varepsilon' < \gamma_1$. Since F has the shadowing property, given $\varepsilon' > 0$ and $\tau > 0$, there is $\delta > 0$ such that any (δ, τ)-pseudo solution is $\frac{1}{12}\varepsilon'$-shadowed by some point of M. For any point $x \in M$, there are many other (δ, τ)-pseudo solutions Φ_x, Ψ_x .... Fix a (δ, τ)-pseudo solution $\Phi_x : \mathbb{R} \longrightarrow M$ with $\Phi_x(0) = x$. Then by expansiveness of F, any (δ, τ)-pseudo solution is $\frac{1}{6}\varepsilon'$-shadowed by unique real orbit of F, where $\varepsilon' < \varepsilon$.

Define a set A_y^Φ by

$$\begin{aligned} A_y^\Phi = \ & \{x \in M |\ \text{for any}\ \ \eta, T > 0 \text{ there is a homeomorphism} \\ & \alpha : \mathbb{R} \longrightarrow \mathbb{R} \text{ with } \alpha(0) = 0 \text{ such that} \\ & d(x\alpha(t), \Phi_y(t)) < \tfrac{1}{6}\varepsilon' + \eta \ \text{ for all } \ t \in [-T, T]\}. \end{aligned}$$

Then it is clear that $A_y \subset O(F, z)$ for some $z \in M$, and have the following two properties;

(1) the length of the interval $\{t \in \mathbb{R} : F(z,t) \in A_y^\Phi\}$ is less than ε,
(2) the set A_y is closed in M.

Define $\gamma : \mathcal{P}_c(\delta, \tau, F) \longrightarrow M$ by

$$\gamma(\Phi_x) = L.L.A_x^\Phi,$$

where $L.L.A_x^\Phi$ is the largest limit point of A_x^Φ. Define a set $A_{y.\eta.T}^\Phi$ by

$$A_{y.\eta.T}^\Phi = \{ \quad x \in M| \text{ there exists a homeomorphism } \alpha : \mathbb{R} \longrightarrow \mathbb{R} \text{ such that } \alpha(0) = 0, \; d(x\alpha(t), \Phi_y(t)) < \tfrac{1}{6}\varepsilon' + \eta \text{ for all } t \in [-T, T]\}.$$

Then we can easily check that

$$A_{y.\eta.T}^\Phi = \bigcap_i A_{y.\eta_i.T_i}^\Phi,$$

where $\eta_i \longrightarrow 0$, $T_i \longrightarrow \infty$ as $i \longrightarrow 0$.

Now we will show that the map $\gamma : \mathcal{P}_c(\delta, \tau, F) \longrightarrow M$ is continuous. Let $\{\Psi_{y_n}^{(n)}\}$ be a sequence in $\mathcal{P}_c(\delta, \tau, F)$ such that $\Psi_{y_n}^{(n)} \longrightarrow \Phi_y$. Assume that if $n \neq k$ then $\Psi^{(n)} \neq \Psi^{(k)}$. Let

$$A_{y_n.\eta.T}^{\Psi^{(n)}} = \{ \quad x \in M| \text{ there exists a homeomorphism } \beta : \mathbb{R} \longrightarrow \mathbb{R} \text{ such that } \beta(0) = 0, \; d(x\beta(t), \Psi_{y_n}^{(n)}(t)) < \tfrac{1}{6}\varepsilon' + \eta \text{ for all } t \in [-T, T]\}.$$

Then we have

$$A_{y_n}^{\Psi^{(n)}} = \bigcap_i A_{y_n.\eta_i.T_i}^{\Psi^{(n)}},$$

where $\eta_i \longrightarrow 0$, $T_i \longrightarrow \infty$ as $i \longrightarrow 0$. It is clear that A_y^Φ and $A_{y_n}^{\Psi^{(n)}}$ are closed subsets of M.

Let M^* be the set of closed subsets of M with the Hausdorff metric R. Without loss of generality, we may assume that $A_{y_n}^{\Psi^{(n)}} \longrightarrow A_z \in M^*$ as $n \longrightarrow \infty$.

First of all we prove the following claim:

Claim $A_y^\Phi = A_z$.

To show the claim, we need following two lemmas 2.7 and 2.8. □

Lemma 2.7. *For every η_i, T_i, there are η_i', T_i' with $\eta_i' < \eta_i$, $T_i < T_i'$ and n_0 such that for all $n \geq n_0$,*

$$A^{\Psi^{(n)}}_{y_n.\eta_i'.T_i'} \subset A^{\Phi}_{y.\eta_i.T_i}.$$

Proof. Let $\eta_i' = \frac{1}{2}\eta_i$ and $T_i \leqslant T_i'$. If $\Psi^{(n)}_{y_n}$ is sufficiently close to Φ_y, then for given $\eta_i' = \frac{1}{2}\eta_i$, $T_i \leqslant T_i'$, there is a homeomorphism h_n with $h_n(0) = 0$ and

$$d(\Psi^{(n)}_{y_n}(h_n(t)), \Phi_y(t)) < \frac{\eta_i}{2}$$

for all $t \in [-T_i', T_i']$ $n \geq n_0$. Let $x \in A^{\Psi^{(n)}}_{y_n.\eta_i'.T_i'}$, then there is a homeomorphism $\beta : \mathbb{R} \longrightarrow \mathbb{R}$ with $\beta(0) = 0$ such that

$$d(x\beta(t), \Psi^{(n)}_{y_n}(t)) < \frac{1}{6}\varepsilon' + \frac{\eta_i}{2}$$

for all $t \in [-T_i', T_i'\ ]$. Then we have

$$\begin{aligned} d(x\beta \circ h_n(t), \Phi_y(t)) &\leqslant (x\beta(h_n(t)), \Psi^{(n)}_{y_n}(h_n(t))) + d(\Psi^{(n)}_{y_n}(h_n(t)), \Phi_y(t)) \\ &< \frac{1}{6}\varepsilon' + \eta_i \end{aligned}$$

for all $t \in [-T_i', T_i']$. □

Lemma 2.8. *For every η_i, T_i, there is a η_i', T_i' with $\eta_i' < \eta_i$, $T_i < T_i'$ and n_0 such that for all $n \geq n_0$, $A^{\Phi}_{y.\eta_i'.T_i'} \subset A^{\Psi^{(n)}}_{y_n.\eta_i.T_i}$.*

Proof. Let $x \in A^{\Phi}_{y.\eta_i'.T_i'}$, where ${\eta_i}' = \frac{\eta_i}{2}$ and $T_i \leqslant T_i'$. Then there is a homeomorphism $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ with $\alpha(0) = 0$ such that

$$d(x\alpha(t), \Phi_y(t)) < \frac{1}{6}\varepsilon' + \frac{\eta_i}{2}$$

for all $t \in [-T_i', T_i']$. If $\Psi^{(n)}_{y_n}$ is sufficiently close to Φ_y, then there is n_0 such that

$$d(\Psi^{(n)}_{y_n}(h_n(t)), \Phi_y(t)) < \frac{\eta_i}{2}$$

for $n \geq n_0$, all $t \in [-T_i', T_i']$ and for a homeomorphism $h_n : \mathbb{R} \longrightarrow \mathbb{R}$, with $h_n(0) = 0$. Then we get

$$\begin{aligned} d(x\alpha \circ h_n(t), \Psi^{(n)}_{y_n}(t)) &\leqslant d(x\alpha \circ h_n(t), \Phi_y(h_n(t))) + d(\Phi_y(h_n(t)), \Psi^{(n)}_{y_n}(t)) \\ &< \frac{1}{6}\varepsilon' + \eta_i \end{aligned}$$

for all $t \in [-T_i', T_i']$.

Now we prove the above claim by the following two steps.

Step1. If $p \notin A_y^\Phi$, then $p \notin A_y^\Phi = \bigcap_i A_{y.\eta_i.T_i}^\Phi$. Hence there are η_i, T_i such that $p \notin A_y^\Phi = \bigcap_i A_{y.\eta_i.T_i}^\Phi$. By lemma 2.7, there are η_i', T_i' and n_0 such that

$$A_{y_n.\eta_i'.T_i'}^{\Psi^{(n)}} \subset A_{y.\eta_i.T_i}^\Phi$$

for all $n \geq n_0$. This implies that $p \notin A_z$.

Step 2. Let $p \notin A_z$. Then there is $\eta > 0$ such that $d(p, A_z) > \eta$. (A_z is closed subset of M). Then there is η_0 such that for all $n \geq n_0$, $R(A_{y_n}^{\Psi^{(n)}}, A_z) < \eta_0 < \eta$, where R is Hausdorff metric in M^*. This implies that $p \notin A_{y_n}^{\Psi^{(n)}}$ for all $n \geq n_0$. Since $p \notin A_{y_n}^{\Psi^{(n)}}$, there are $\eta_n > 0$, $T_n > 0$ such that $p \notin A_{y_n.\eta_n.T_n}^{\Psi^{(n)}}$ for all $n \geq n_0$. By lemma 2.8, there are $\eta_n' > 0$, $T_n' > 0$ such that

$$A_y^\Phi \subset A_{y.\eta_n'.T_n'}^\Phi \subset A_{y_n.\eta_n.T_n}^{\Psi^{(n)}}$$

for all $n \geq n_0$. This means that $p \notin A_y^\Phi$, and so completes our proof. □

Lemma 2.9. *([5]). For all $\lambda > 0$, there are $\eta > 0$, $T > 0$ such that $d(x, A_y^\Phi) < \lambda$, for every $y \in M$ and all $x \in A_{y.\eta.T}^\Phi$.*

Now we are going to show that the map γ is continuous. For our purpose, we assume that $\{y_n\}$ and $\{z_n\}$ are sequences of point in M so that $z_n = L.L.A_{y_n}^{\Psi^{(n)}}$ for all n. (i.e. $\gamma(\Psi_{y_n}^{(n)}) = z_n$). And assume that $\Psi_{y_n}^{(n)} \longrightarrow \Phi_y$, $z = L.L.A_y^\Phi$, where $A_y^\Phi = A_z$. From the compactness of M, $\{z_n\}$ has a convergent subsequence, so without loss of generality, we may assume that $z_n \longrightarrow z'$. It is obvious from the step 1 of the claim that $z \in A_y^\Phi = A_z$.

Let x be any point in A_z and let $\{\lambda_i\}_{i\in\mathbb{Z}}$ be a convergent subsequence of positive real numbers with 0 as the only limit point. For the convenience of notation, we denote $k_i \equiv k(i)$. If $\Psi_{y_n}^{(n)} \longrightarrow \Phi_y$ then there is a sequence $\{y_{k(i)}\}$ such that

$$d(\Psi_{y_{k(i)}}^{(k(i))}(h_{k(i)}(t)), \Phi_y(t)) < \frac{1}{2}\eta_i$$

for all $t \in [-T_i, T_i]$, $i \in \mathbb{N}$ and for a homeomorphism $h_{k(i)} \in Rep^*$ with $h_{k(i)}(0) = 0$.

Since $x \in A_z = A_y^\Phi$, there is a homeomorphism $\alpha_i : \mathbb{R} \longrightarrow \mathbb{R}$ with $\alpha_i(0) = 0$ such that

$$d(x\alpha_i(t), \Phi_y(t)) < \frac{1}{6}\varepsilon' + \frac{1}{2}\eta_i$$

for $t \in [-T_i, T_i]$. Therefore we get

$$\begin{aligned} d(x\alpha_i(t), \Psi^{(k(i))}_{y_{k(i)}}(h_{k(i)}(t))) &\leqslant d(x\alpha_i(t), \Phi_y(t)) + d(\Phi_y(t), \Psi^{(k(i))}_{y_{k(i)}}(h_{k(i)}(t))) \\ &< \frac{1}{6}\varepsilon' + \eta_i \end{aligned}$$

for all $t \in [-T_i, T_i]$ and $i \in \mathbb{N}$. From this, we have $x \in A^{\Psi^{(k(i))}}_{y_{k(i)} \cdot \eta_i \cdot T_i}$. If we choose η_i, T_i for all λ_i as in Lemma 2.9, then we get $d(x, A^{\Psi^{(k(i))}}_{y_{k(i)}}) < \lambda_i$ for all i. Choose $x_{k(i)} \in A^{\Psi^{(k(i))}}_{y_{k(i)}}$ such that

$$d(x, x_{k(i)}) = d(x, A^{\Psi^{(k(i))}}_{y_{K(i)}}) = \lambda_i$$

(This can be done because $A^{\Psi^{(k(i))}}_{y_{k(i)}}$ is closed). Obviously $x_{k(i)} \longrightarrow x$ as $i \longrightarrow \infty$. Since $z_{k(i)} = L.L.A^{\Psi^{(k(i))}}_{y_{k(i)}}$, there are $w_{k(i)} \geq 0$ such that $z_{k(i)} = x_{k(i)} w_{k(i)}$. Also $z_{k(i)} \longrightarrow z'$, and hence $z' = xw$ with $w \geq 0$ for every $x \in A_z = A^{\Phi}_y$. This means that z' is the largest limit point of A^{Φ}_y, i.e. $L.L.A^{\Phi}_y$. Since the largest limit point of A^{Φ}_y is unique, we have $z = z'$. By now we have proved that every convergent subsequence of $\{z_n\}$ has a limit point (z), and this means that $z_n \longrightarrow z$. Consequently we proved that γ is continuous, and so completes the proof of our theorem.

3. $\mathcal{T}_a$-Shadowing

DEFINITION 3.1. *A flow F is said to be have a finite shadowing property if for every ε, there is $\delta > 0$ such that every finite $(\delta, 1)$-pseudo solution is ε-traced by an orbit of F.*

LEMMA 3.2. *([5]). Suppose F is a flow with no fixed points. Then there is a $T_0 > 0$ such that if $0 < T < T_0$, there exists $\lambda_T > 0$ such that $d(x, y) < \lambda_T$ implies that $d(xT, y) > \lambda_T$ for all $x, y \in M$.*

PROPOSITION 3.3. *([5]). Every fixed point free flow with the finite shadowing property has the shadowing property.*

PROPOSITION 3.4. *([5]). Let F be a flow with the following properties: if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every finite $(\delta, 1)$-pseudo solution $\phi : [-T_i, T_i] \longrightarrow M$, $t_i \in [-T_i, T_i] \subset \mathbb{R}$ and $x_i = \phi(t_i)$, $-k \leqslant i \leqslant k$, $1 \leqslant t_{i+1} - t_i \leqslant 2$, is ε-traced by an orbit of F. Then F has the finite shadowing property.*

Let F be a C^1-flow on a compact manifold M generated by $\dot{x} = X(x)$, and let $L(F) = \{x | X(x) = 0\}$. (i.e. $L(F)$ is a set of fixed points)

Now given a non-zero vector $Y \in T_x M$, where $x \notin L(F)$ define the inclination of Y relative to F to be the length of the normalized difference, that is

$$\sigma(Y) = \| \frac{1}{\|Y\|} Y - \frac{1}{\|X(x)\|} X(x) \| \ .$$

LEMMA 3.5. *Given $\varepsilon > 0$ and a flow F on M, suppose γ is a C^1-curve in M(an embedded closed interval or circle) such that at each point x in the image of γ one of the following conditions hold;*

(i) $\|\dot{F}(x) = X(x)\| < \frac{1}{2}\varepsilon$, *or*

(ii) $x \notin L(F)$, *and γ has inclination $\sigma < \frac{\varepsilon}{\|\dot{F}\|}$ at x.*

Then, for any neighborhood U of the image of γ, there exists a flow ψ on M satisfying

(a) $\dot{\psi} = \dot{F}$ *off U*

(b) $\|\dot{\psi} - \dot{F}\| < \varepsilon$ *on M*

(c) *γ is an (segment of an) integral curve of ψ.*

THEOREM 3.6. *Let F be a fixed point free flow. Then F has the shadowing property if and only if it has the $\mathcal{T}_h$-shadowing property.*

Proof. We need only necessary condition. Given $\varepsilon > 0$, without loss of generality take $T_0 > 0$ as in Lemma 2.5, and assume $0 < \varepsilon < T_0$. Choose $0 < \varepsilon' < \frac{1}{3}\varepsilon$ such that if $x = yt$ with $|t| < \varepsilon'$ then $d(x, y) < \frac{1}{3}\varepsilon$. Take $0 < \xi < \varepsilon''$ such that $d(x, y) < \xi$ implies that

$$d(xt, yt) < \varepsilon''$$

for all $t \in [0, 2]$. By assumption, there exists $\delta > 0$ satisfying $\mathcal{T}_h$-shadowing property with respect to $\frac{1}{2}\xi$. Take $0 < \delta' < \delta$ such that every C^1-flow η on M, $\|\dot{\eta} - \dot{F}\| < \delta'$ implies that

$$d(\eta(\cdot, t), F(\cdot, t)) < \delta$$

for all $t \in [0, 2]$. Take $0 < \lambda < \delta'$ (Later we are going to fix the value λ). Now let $\phi : [0, T] \longrightarrow M$ ($\phi(0) = x_0$) be a finite $(\frac{1}{3}\lambda)$-pseudo solution of F such that there is a finite increasing sequence $\{t_i\}_{i=0}^k$ with $\phi(t_i) = x_i$, $0 \leqslant i \leqslant k$, and $t_k = T$, $t_0 = 0$ and $1 \leqslant t_{i+1} - t_i \leqslant 2$. Then $(\{x_i\}_{i=0}^k, (\{s_i\}_{i=0}^k,))$ be a pair of sequence with $1 \leqslant s_i \leqslant 2$, where $s_i = t_{i+1} - t_i$, $0 \leqslant i \leqslant k-1$. Without loss of generality, we can choose a sequence of distinct points $(\{x_i{}'\}_{i=0}^k$ in M with the following properties $d(x_i' s, x_i s) < \frac{1}{3}\lambda$, for all $0 \leqslant s \leqslant 2$. For positive s_i with $1 \leqslant i \leqslant k$, we have

$$d({x_i}' s_i, {x_{i+1}}') \leqslant d({x_i}' s_i, x_i s_i) + d(x_i s_i, x_{i+1}) + \tag{3}$$

$$d(x_{i+1}, {x_{i+1}}') \leqslant \frac{1}{3}\lambda + \frac{1}{3}\lambda + \frac{1}{3}\lambda = \lambda \tag{4}$$

for all $0 \leqslant i \leqslant k-1$. Let $\phi' : [0, \sum_{j=0}^{k-1} s_j] \longrightarrow M$ such that $\phi'(t) = x'_i(t - \sum_{j=0}^{i-1} s_j)$ if $t \in [\sum_{j=0}^{i-1} s_j, \sum_{j=0}^{i} s_j]$. Then ϕ' is a finite $(\lambda, 1)$-pseudo solution of F. Now take $0 < \lambda < \delta'$, choose λ small enough for one to take a C^1-curve

$$\gamma : [0, \sum_{j=0}^{k-1} s_j] \longrightarrow M$$

with the following properties;

(a) γ is a closed curve in M
(b) $\gamma(t_n) = {x_n}'$ for $0 \leqslant n \leqslant k$
(c) γ has an inclination less than $\frac{\delta'}{\|\dot{F}\|}$ at every point x in the image of γ.

Using Lemma 3.5, we see that there exists a C^1-flow ψ on M such that

(i) γ is an integral curve of ψ
(ii) $\|\dot{\psi} - \dot{F}\| < \delta'$.

So we have

(i) $\psi_{{x_0}'}(t_n) = x'_n$
(ii) $d(\psi(\cdot, t), F(\cdot, t)) < \delta$, for all $t \in [0, 2]$.

Then $\psi|_{[0,t_k]} : [0, t_k] \longrightarrow M$ is a δ-pseudo solution of F. By assumption, there is a point $z \in M$ and $\alpha \in Rep^*$ such that

$$d(z\alpha(t), \psi_{x_0}(t)) < \frac{\xi}{2}$$

for all $t \in [0, t_k]$. Then

$$\begin{aligned} d(\phi_{x_0}(t), z\alpha(t)) \leqslant \quad & d(\phi_{x_i}(s), x_i s) + d(x_i s, x'_i s) + d(x'_i s, \psi_{x_i}(s)) + \\ & d(\psi_{x_i}(s), z\alpha(s)) \leqslant \tfrac{1}{3}\lambda + \tfrac{1}{3}\lambda + \delta + \tfrac{\xi}{2} < \lambda + \delta + \tfrac{\xi}{2} \end{aligned}$$

where $t \in [\sum_{j=0}^{i-1} s_j, \sum_{j=0}^{i} s_j]$, $s = t - \sum_{j=0}^{i-1} s_j$. If $\max\{\lambda, \delta, \frac{\xi}{2}\} < \frac{\varepsilon}{3}$, then

$$d(\phi_{x_0}(t), z\alpha(t)) < \varepsilon$$

for all $t \in [0, t_k]$. Therefore F has the finite shadowing property. By Proposition 3.3, F has the shadowing property. □

Remark 3.7. *If F has fixed points then the finite shadowing property does not imply the shadowing property in general (see [2]).*

Let $M \subset \mathbb{R}^n$ be a compact metric space($n \geq 1$) with metric is ρ. Assume that $diam(M) \leqslant 1$. Let $f : M \longrightarrow M$ be a homeomorphism, and K be a suspension space of f under 1. i.e. $K = \{(x,t) \in M \times \mathbb{R} : 0 \leqslant t \leqslant 1\}/(x,1) \sim (f(x),0)$. Let d be the suspension metric on K induced by ρ. We identify $K = M \times [0,1)$, $u = (x,t) \in M \times [0,1)$ for all $u \in K$, $M = M \times \{0\} \subset K$. Fix a point $a \in M$ and set $e = (a, \frac{1}{2}) \in M \times \mathbb{R} \subset \mathbb{R}^{n+1}$. Define a flow on K which has a unique fixed point e.

$$U = U(e) = \{x \in \mathbb{R}^{n+1} : |e - x| < \frac{1}{4}\} \ .$$

Take a C^∞-function $C : \mathbb{R}^{n+1} \longrightarrow [0,1]$ such that

(i) $C(x) = 0$ if $x = e$

(ii) $0 \leqslant C(x) < 1$ if $x \in U$

(iii) $C(x) = 1$ if $x \notin U$

Let φ be a flow on R^{n+1} defined by a vector field

$$\begin{cases} \dot{x_1} = 0 \\ \dot{x_2} = C(x) \quad \text{for} \quad x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} \end{cases} \tag{5}$$

Consider the flow of K induced by the restriction of φ to $M \times [0,1]$ and denote by the same symbol φ. We say that (K, φ) is the *singular suspension* of f with a fixed point $e = (a, \frac{1}{2})$.

Theorem 3.8. *([2]). Let $f : M \longrightarrow M$ be a homeomorphism and (K, φ) be a singular suspension of (M, f) with fixed point $(a, \frac{1}{2})$. Assume that $M_0 = M - \{a\}$ is dense in M. Then the following two properties are pairwise equivalent:*

(a) *(M, f) has shadowing property;*

(b) *(K, φ) has finite shadowing property.*

Example 3.9. *Let $I = [0,1] \subset R$, $I_0 = I - \{0\}$. $f : I \longrightarrow I$ be a homeomorphism defined by*

$$f(x) = \begin{cases} \frac{1}{2}x, & x \in [0, \frac{2}{3}] \\ 2x - 1, & x \in [\frac{2}{3}, 1] \end{cases} \tag{6}$$

Let (K, φ) be the singular suspension of (I, f) with fixed point $e = (0, \frac{1}{2})$. Komuro [2] showed that (K, φ) has the finite shadowing property but it does not have the shadowing property and f has the shadowing property. We will show that (K, φ) has the finite shadowing but has not $\mathcal{T}_h$-shadowing property. Let $C : \mathbb{R}^{n+1} \longrightarrow [0,1]$ be the C^∞-function which generate (K, φ). If there is a (K, ψ) flow with $d_0(\varphi, \psi) < \delta$, then ψ has a fixed point in K. If not, then ψ is conjugate to a singular suspension of (I, f). Since f has the shadowing property, ψ has the shadowing property. But $O(e, \varphi)$ is not ε-shadowed by any orbit of ψ. This is a contradiction. By the continuity of C and $C(e) = 0$, there is a closed neighborhood $\bar{W}_e$ of e such that for every $u \in \bar{W}_e$, $d(ut, e) < \frac{\delta}{2}$, for $0 \leqslant t \leqslant 1$. Assume that $\bar{W}_e = \{(x,t), 0 \leqslant x \leqslant \beta, (\beta \neq 0), \frac{1}{2} - r \leqslant t \leqslant \frac{1}{2} + r, r \neq 0\}$. Let $C' : \mathbb{R}^{n+1} \longrightarrow [0,1]$ be a C^∞-function such that

(i) $C'(x) = 0$ if $x \in \frac{1}{2}\bar{W}_e$
(ii) $C'(x) = C(x)$ if $x \notin \bar{W}_e$
(iii) and $\bar{W}_e - \frac{1}{2}\bar{W}_e$ linear extension from $C'(x)$ to $C(x)$.

Let ψ be a vector field generated by C', then $d_0(\psi, \varphi) < \delta$. This means that every orbits of ψ is a $(\delta, 1)$-pseudo solution of φ. Let $y = max\{x \in I | (x,t) \in \frac{1}{2}\bar{W}_e\}$. Then $z = (y,t) \in \frac{1}{2}\bar{W}_e$, $z' = (y,s) \in \bar{W}_e - \frac{1}{2}\bar{W}_e$,where $s < t$. We can easily show that $O(\psi, z)$ is not ε-shadowed by any orbit of φ. This shows that (K, φ) does not have the $\mathcal{T}_h$-inverse shadowing property.

4. Continuous shadowing and inverse shadowing

Theorem 4.1. *If a flow F has the $\mathcal{T}_\alpha$-continuous shadowing property on a compact manifold M then it has the $\mathcal{T}_\alpha$-inverse shadowing property, where $\alpha = a, c, h$.*

Proof. We only prove the theorem in the case of $\alpha = c$. Let $\varepsilon > 0$, $\tau > 0$ be given. Take $\delta > 0$ by the $\mathcal{T}_\alpha$-continuous shadowing property corresponding to ε. Let $\Phi : M \times \mathbb{R} \longrightarrow M$ be an arbitrary continuous (δ, τ)-method of F. Let

$$\mathcal{P}_\Phi = \bigcup \{\Phi_x : x \in M\} \subset \mathcal{P}_c(\delta, \tau, F) \subset M^{\mathbb{R}}.$$

Since F has the $\mathcal{T}_c$-continuous shadowing property, there is a continuous map $\gamma : \mathcal{P}_c(\delta, \tau, F) \longrightarrow M$ such that for every $\Phi_x \in \mathcal{P}_c(\delta, \tau, F)$, there exists a homeomorphism $h \in Rep^*$ with $h(0) = 0$ satisfying

$$d(\gamma(\Phi_x)h(t), \Phi_x(t)) < \varepsilon$$

for all $t \in \mathbb{R}$. Let $\gamma' \equiv \gamma|_{\mathcal{P}_\Phi}$. By definition of a continuous method, the map $s = \tilde{\Phi} : M \longrightarrow \mathcal{P}_\Phi \subset M^{\mathbb{R}}$ by $s(x) = \tilde{\Phi}(x) = \Phi_x$ is continuous. Let $H = \gamma' \circ s : M \longrightarrow M$. Then H is a continuous map and $d_0(H, id_M) < \varepsilon$.

If ε is sufficiently small, H is surjective. For every $x \in M$, there are $y \in M$ and homeomorphism $h \in Rep^*$ with $h(0) = 0$ such that $H(y) = x$ and

$$d(\gamma(\Phi_y)h(t), \Phi_y(t)) < \varepsilon$$

for all $t \in \mathbb{R}$. Then

$$\begin{aligned} d(xh(t), \Phi_y(t)) &= d(H(y)h(t), \Phi_y(t)) = d(\gamma' \circ s(y)h(t), \Phi_y(t)) \\ &= d(\gamma'(\Phi_y)h(t), \Phi_y(t)) = d(\gamma(\Phi_y)h(t), \Phi_y(t)) < \varepsilon \end{aligned}$$

for all $t \in \mathbb{R}$. This completes the proof. □

DEFINITION 4.2. *We say that a flow F has the $\mathcal{T}_\alpha$-continuous inverse shadowing property ($\alpha = a, c, h$) if for any $\varepsilon > 0$ and $\tau > 0$, there exists $\delta > 0$ such that for each (δ, τ)-method $\Phi \in \mathcal{T}_\alpha(\delta, \tau, F)$ there is a continuous map $s : M \longrightarrow M$ such that for every point $y \in M$ there is a homeomorphism $h \in Rep^*$ with $h(0) = 0$ such that*

$$d(yh(t), \Phi_{s(y)}(t)) < \varepsilon$$

for all $t \in \mathbb{R}$.

THEOREM 4.3. *If F has the $\mathcal{T}_\alpha$-continuous inverse shadowing property then it has the $\mathcal{T}_\alpha$-shadowing property.*

Proof. We only prove the theorem in the case of $\alpha = c$. Let $\varepsilon > 0$, $\tau > 0$ be given. Choose $\delta > 0$ by the $\mathcal{T}_c$-continuous inverse shadowing property corresponding to ε. Then there exists a continuous map $s : M \longrightarrow M$ such that for every $y \in M$ there is a homeomorphism $h \in Rep^*$ with $h(0) = 0$ such that

$$d(yh(t), \Phi_{s(y)}(t)) < \varepsilon$$

for all $t \in \mathbb{R}$. If $t = 0$, then $d(y, \Phi_{s(y)}(0)) = d(y, s(y)) < \varepsilon$. Since the map s is continuous, it is surjective for small $\varepsilon > 0$. We claim that F has the $\mathcal{T}_c$-shadowing. Define $\gamma : \mathcal{P}_c(\delta, \tau, F) \longrightarrow M$ as following; for each $\Phi_x \in \mathcal{P}_c(\delta, \tau, F)$, there are $x \in M$ and $\Phi \in \mathcal{T}_c(\delta, \tau, F)$ and a continuous surjective map $s : M \longrightarrow M$ such that $\Phi_x(0) = x$. Choose $y \in M$ with $s(y) = x$ and define $\gamma(\Phi_x) = y$. Then γ is a desired map. For every $\Phi_x \in \mathcal{P}_c(\delta, \tau, F)$ there is a $y \in M$ such that $\gamma(\Phi_x) = y$. Then

$$d(\gamma(\Phi_x)h(t), \Phi_x(t)) = d(yh(t), \Phi_{s(y)}(t)) < \varepsilon$$

for all $t \in \mathbb{R}$. □

COROLLARY 4.4. *If F is a fixed point free flow with the $\mathcal{T}_h$-continuous inverse shadowing property then it has the $\mathcal{T}_h$-shadowing property.*

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