JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **20**, No. 3, September 2007

BOUNDEDNESS OF THE SOLUTIONS OF VOLTERRA DIFFERENCE EQUATIONS

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Abstract. Using the representation of the solution by means of the resolvent, we study the boundedness of the solutions of some Volterra difference equations.

1. Introduction

Volterra difference equations arise mainly in the process of moldeling of some real phenomena or by applying a numerical method to a Volterra integral equation. Sometimes Volterra difference equations describe processes whose current state is determined by their entire prehistory. For a detailed applications of Volterra difference equations, see [5].

A property of crucial importance is the boundedness of the solutions of a Volterra difference equation. In fact, error between the true and the numerical solutions of a Volterra integral equation satisfies a discrete Volterra equation. Thus the boundedness of the solution of this Volterra discrete equation assumes the boundedness of the global error, that is, the stability of the considered numerical method [4].

In this paper, we represent the solution of Volterra difference equation

$$y(n+1) = \sum_{j=n_0}^{n} B(n,j)y(j) + f(n) \tag{1}$$

by means of the resolvent of the equation

$$x(n+1) = \sum_{j=n_0}^{n} B(n,j)x(j), \tag{2}$$

Received July 16, 2007.
2000 Mathematics Subject Classification: Primary 39A11.
Key words and phrases: boundedness, Volterra difference equation, representation of the resolvent .
This work was supported by the Korea Research Foundation Grant founded by the Korea Government(MOEHRD)(KRF-2005-070-C00015).

and then investigate the boundedness of the solutions of equations (1) and

$$y(n+1) = \sum_{j=n_0}^{n} [B(n,j) + C(n,j)]y(j). \tag{3}$$

The main reference is [4].

For asymptotic behaviors of Volterra difference equations, see [2] and [3].

2. Main results

We denote by $\mathbb{R}^d$ the d-dimensional real space, $x = \{x(n)\}_{n\in\mathbb{Z}_+}$ a sequence with $x(n) \in \mathbb{R}^d$, where $\mathbb{Z}_+$ is the set of all nonnegative integers. We consider the discrete linear Volterra equation

$$x(n+1) = \sum_{j=n_0}^{n} B(n,j)x(j),\ x(n_0) = x_0,\ n \geq n_0 \in \mathbb{Z}_+ \tag{4}$$

and the associated linear equation

$$y(n+1) = \sum_{j=n_0}^{n} B(n,j)y(j) + f(n),\ y(n_0) = y_0,\ n \geq n_0, \tag{5}$$

where the *kernel* of (4), $B(n,j)$, is a $d \times d$ matrix for each $j, n \in \mathbb{Z}_+$ with $j \leq n$ and $f : \mathbb{Z}_+ \to \mathbb{R}^d$ is a given sequence in $\mathbb{R}^d$.

The *resolvent* $R(n,s)$ associated with (4) satisfies

$$\begin{aligned} R(n+1,s) &= \sum_{j=s}^{n} B(n,j)R(j,s) \ \text{ if } s \leq n, \\ R(s,s) &= I, \text{ the identity matrix}, \\ R(n,s) &= 0 \ \text{ if } n < s. \end{aligned} \tag{6}$$

To examine the boundedness of the solution of (5) we need the following representation of the solution by the resolvent instead of the variation of constants formula

Lemma 2.1. *The unique solution $y(n)$ of (5) satisfying $y(n_0) = y_0$ is given by*

$$y(n) = R(n,n_0)y_0 + \sum_{j=n_0}^{n-1} R(n,j+1)f(j). \tag{7}$$

Proof. We show that $y(n)$ given by (7) satisfies the equation (5).

$$y\quad(n+1) = R(n+1,n_0)y_0 + \sum_{r=n_0}^{n} R(n+1,r+1)f(r)$$

$$= \sum_{j=n_0}^{n} B(n,j)R(j,n_0)y_0 + \sum_{r=n_0}^{n}\sum_{j=r+1}^{n} B(n,j)R(j,r+1)f(r)$$

$$= \sum_{j=n_0}^{n} B(n,j)R(j,n_0)y_0 + f(n) + \sum_{r=n_0}^{n-1}\sum_{j=r+1}^{n} B(n,j)R(j,r+1)f(r)$$

$$= \sum_{j=n_0}^{n} B(n,j)R(j,n_0)y_0 + f(n) + \sum_{r=n_0}^{n-1} B(n,n)R(n,r+1)f(r)$$

$$+ \sum_{r=n_0}^{n-1}\sum_{j=r+1}^{n-1} B(n,j)R(j,r+1)f(r)$$

$$= \sum_{j=n_0}^{n} B(n,j)R(j,n_0)y_0 + f(n) + \sum_{r=n_0}^{n-1}\sum_{j=n_0}^{r-1} B(n,r)R(r,j+1)f(j)$$

$$+ \sum_{j=n_0}^{n-1} B(n,n)R(n,j+1)f(j)$$

$$= \sum_{r=n_0}^{n} B(n,r)[R(r,n_0)y_0 + \sum_{j=n_0}^{r-1} R(r,j+1)f(j)] + f(n)$$

$$= \sum_{r=n_0} B(n,r)y(r) + f(n).$$

This completes the proof. □

A difference equation $x(n+1) = f(n,x(n)),\ x(n_0) = x_0$, is called

(i) *bounded* if for any $n_0 \in \mathbb{Z}_+$ and a number $r > 0$ there exists a number $\alpha(n_0,r)$ depending on n_0 and r such that

$$|x(n)| = |x(n,n_0,x_0)| < \alpha(n_0,r)$$

for all $n \geq n_0$ and x_0 with $|x_0| \leq r$:

(ii) *uniformly bounded* with respect to the initial moment n_0 if $\alpha(n_0,r)$ $= \alpha(r)$, i.e., the constant bounding the solution doses not depend on the initial moment n_0.

Example 2.1 in [4] shows that a bounded equation is not necessarily uniformly bounded. However the representation of the resolvent

$R(n, n_0)$ in that example is incorrect, and so we examine Example 2.1 [4] in detail :

EXAMPLE 2.2. *Consider the two-dimensional difference equation*

$$x(n+1) = \begin{bmatrix} 1 & \frac{(n+2)^2}{(n+3)} \\ 0 & \frac{(n+2)^3}{(n+3)^3} \end{bmatrix} x(n) + \begin{bmatrix} 0 & \frac{1}{(n+1)^2} \\ 0 & 0 \end{bmatrix} x_0, \tag{8}$$

where $x_0 = x(n_0) = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \in \mathbb{R}^2$. *The solution of (8) can be represented in the form*

$$\begin{aligned} x(n) &= R(n, n_0)x_0 + \sum_{j=n_0}^{n-1} R(n, j+1)f(j) \\ &= R(n, n_0)x_0 + \sum_{l=n_0+1}^{n} R(n, l)f(l-1). \end{aligned} \tag{9}$$

We compute $R(n, n_0)$ as follows : If we write $x(n+1) = \begin{bmatrix} x_{n+1}^1 \\ x_{n+1}^2 \end{bmatrix}$, where

$$\begin{cases} x_{n+1}^1 &= x_n^1 + \frac{(n+2)^2}{n+3} x_n^2 \\ x_{n+1}^2 &= \frac{(n+2)^3}{(n+3)^3} x_n^2, \end{cases}$$

then we have

$$\begin{aligned} x_n^2 &= \prod_{l=n_0}^{n-1} \frac{(l+2)^3}{(l+3)^3} x_{02} \\ &= \frac{(n_0+2)^3}{(n_0+3)^3} \cdot \frac{(n_0+3)^3}{(n_0+4)^3} \cdots \frac{(n+1)^3}{(n+2)^3} = \frac{(n_0+2)^3}{(n+2)^3} x_{02}, \\ x_{n+1}^1 &= x_n^1 + \frac{(n+2)^2}{(n+3)} \cdot \frac{(n_0+2)^3}{(n+2)^3} x_{02} \\ &= x_n^1 + \frac{(n_0+2)^3}{(n+3)(n+2)} x_{02}, \\ x_{01}^1 &= x_{01}. \end{aligned}$$

In view of Lemma 2.1,

$$\begin{aligned} x_n^1 &= R(n,n_0)x_{01} + \sum_{j=n_0}^{n-1} R(n,j+1)f(j) \\ &= x_{01} + \sum_{j=n_0}^{n-1} \frac{(n_0+2)^3}{(j+2)(j+3)} x_{02} \\ x_n^1 &= x_{01} + (n_0+2)^3 x_{02} \left(\frac{1}{n_0+2} - \frac{1}{n+2} \right) \\ &= x_{01} + \left[(n_0+2)^2 - \frac{(n_0+2)^3}{n+2} \right] x_{02}. \end{aligned}$$

Thus we obtain

$$x(n) = \begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} = \begin{bmatrix} 1 & (n_0+2)^2 - \frac{(n_0+2)^3}{n+2} \\ 0 & \frac{(n_0+2)^3}{(n+2)^3} \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.$$

It follows that

$$R(n,n_0) = \begin{bmatrix} 1 & (n_0+2)^2 - \frac{(n_0+2)^3}{n+2} \\ 0 & \frac{(n_0+2)^3}{(n+2)^3} \end{bmatrix}$$

and it is bounded. Moreover, we get

$$\sum_{l=n_0+1}^{n} R(n,l)f(l-1) = x_{02} \sum_{n=n_0+1}^{n} \begin{bmatrix} \frac{1}{l^2} \\ 0 \end{bmatrix} \le \pi^2 x_{02} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

This implies that the second addend at the right hand side of equality (9) is uniformly bounded with respect to n_0. At the same time the first addend at right hand side of (9) is unbounded in n_0 since for any $k \in \mathbb{Z}_+$, the component

$$\begin{aligned} r_{12}(kn_0, n_0) &= (n_0+2)^2 - \frac{(n_0+2)^3}{kn_0+2} \\ &= \left(1 - \frac{n_0+2}{kn_0+2}\right)(n_0+2)^2 \to \infty,\ n_0 \to \infty. \end{aligned}$$

Consequently any solution of (8) is bounded with respect to n for arbitrary fixed n_0. But (8) is bounded nonuniformly with respect to the initial moment n_0.

For the boundedness of the solution of (4), Crisci et al. [4] imposed the condition

$$\sum_{n=n_0}^{\infty}\sum_{j=n_0}^{n}|B(n,j)|<\infty.$$

Also, they showed that for the scalar equation

$$x(n+1)=\sum_{j=n_0}^{n}a_{n,j}x(j),\ a_{n,j}\in\mathbb{R}, n\geq n_0,$$

the solution $x(n)$ satisfies $\sum_{n=n_0}^{\infty}|x(n)|<\infty$ if $\sum_{l=n_0}^{\infty}|a_{l+n+1,n+1}|<1$.

Some important sequence spaces are the following :

$$\begin{aligned} l_p &= \{x:\mathbb{Z}_+\to\mathbb{R}^d \mid \sum_{n=1}^{\infty}|x(n)|^p<\infty\},\ 1\leq p<\infty, \\ l_\infty &= \{x:\mathbb{Z}_+\to\mathbb{R}^d \mid \sup_n|x(n)|<\infty\}. \end{aligned}$$

They are equipped with the norms

$$\begin{aligned} |x|_p &= \left(\sum_{n=1}^{\infty}|x(n)|^p\right)^{\frac{1}{p}},\ 1\leq p<\infty, \\ |x|_\infty &= \sup_n|x(n)|, \end{aligned}$$

respectively, and are Banach spaces.

THEOREM 2.3. *For the equation (5), assume that the following :*

(i) $\sum_{n=n_0}^{\infty}\sum_{j=n_0}^{n}|B(n,j)|<\infty$,
(ii) $f\in l_1$,
(iii) $\sum_{n=n_0}^{\infty}\sum_{j=n_0}^{n}|R(n,j)|\leq M$.

Then the solution $x(n)$ of (5) belongs to l_∞.

Proof. Using the formula (7) and the assumptions, we have

$$\begin{aligned} |x(n)| &\leq |R(n,n_0)||x_0|+\sum_{j=n_0}^{n-1}|R(n,j+1)||f(j)| \\ &\leq M|x_0|+M|f|_1 \\ &= M(|x_0|+|f|_1). \end{aligned}$$

□

This implies that $|x|_\infty=\sup_{n\geq n_0}|x(n)|<\infty$.

We prove the next result which is Theorem 4.1 in [4], without using the discrete version of the Gronwall-Bellman lemma [1].

THEOREM 2.4. *For the equation*

$$(10)\qquad y(n+1) = \sum_{j=n_0}^{n} [B(n,j) + C(n,j)]y(j),\ y(n_0) = y_0,\ n \geq n_0,$$

where $C(n,j)$ *is a* $d \times d$ *matrix, suppose that*

(i) *(4) is uniformly bounded,*
(ii) $\sum_{j=n_0}^{\infty} \sum_{n=j}^{\infty} |C(n,j)| < \infty$.

Then (10) is uniformly bounded.

Proof. The solution $y(n)$ of (10) is given by

$$y(n) = R(n, n_0)y_0 + \sum_{r=n_0}^{n-1} R(n, r+1) \sum_{j=n_0}^{r} C(r,j)y(j)$$

by the same manner as the proof in Lemma 2.1. From (i), $|R(n, n_0)| \leq K$ for some constant $K > 0$. Thus

$$\begin{aligned} u(n) \equiv |y(n)| &= K \left(|y_0| + \sum_{r=n_0}^{n-1} R(n, r+1) \sum_{j=n_0}^{r} |C(r,j)||y(j)| \right) \\ &\leq K \left[|y_0| + \sum_{j=n_0}^{n-1} |y(j)| \sum_{r=j}^{n-1} |C(r,j)| \right] \\ &\equiv v(n). \end{aligned}$$

Note that $v(n_0) = K|y_0|$. If follows that

$$\begin{aligned} \Delta v(n) &= v(n+1) - v(n) \\ &= \sum_{r=n_0}^{n} \sum_{j=n_0}^{r} |C(r,j)||y(j)| - \sum_{r=n_0}^{n-1} \sum_{j=n_0}^{r} |C(r,j)||y(j)| \\ &= \sum_{j=n_0}^{n} |C(n,j)||y(j)| \\ &= \sum_{j=n_0}^{n} |C(n,j)|u(j) \leq \sum_{j=n_0}^{n} |C(n,j)|v(j). \end{aligned}$$

Thus

$$v(n+1) = \left[1 + \sum_{j=n_0}^{n} |C(n,j)| \right] v(n),\ n \geq n_0.$$

Consequently, we have

$$\begin{aligned}|y(n)| \le v(n) &= \prod_{r=n_0}^{n-1}\left[1+\sum_{j=n_0}^{r}|C(r,j)|\right]\\ &\le \exp\left(\sum_{r=n_0}^{n-1}\sum_{j=n_0}^{r}|C(r,j)\right)\\ &= \exp\left(\sum_{j=n_0}^{n-1}\sum_{r=j}^{n-1}|C(r,j)|\right).\end{aligned}$$

It follows that (10) is uniformly bounded by (ii). □

If we assume that Eq. (4) is uniformly asymptotically stable and

$$|B(n,j)| \le M\nu^{n-j} \text{ for some } M>0, 0<\nu<1,$$

then there exist constants λ and $\gamma \in (0,1)$ such that

$$|R(n,j)| \le \lambda\gamma^{n-j},\ n \ge j$$

[4]. Using this fact we obtain the weaker version of Theorem 4.2 in [4]. For the definitions of the various stability notions, see [1].

THEOREM 2.5. *Assume that*

(i) *(4) is uniformly asymptotically stable,*
(ii) $|B(n,j)| \le M\nu^{n-j},\ M>0,\ 0<\nu<1,$
(iii) $\sum_{j=n_0}^{\infty}\sum_{n=j}^{\infty}|C(n,j)|\gamma^{j-n} < \infty.$

Then (10) is bounded.

Proof. We obtain

$$|x(n)| \le \lambda\gamma^{n-n_0}|x_0| + \lambda\sum_{r=n_0}^{n-1}\gamma^{n-r-1}\sum_{j=n_0}^{r}|C(r,j)||x(j)|.$$

Then

$$\begin{aligned}q(n) &\equiv \frac{|x(n)|}{\lambda^n} \le \lambda\frac{|x_0|}{\gamma^{n_0}} + \lambda\sum_{r=n_0}^{n-1}\gamma^{-r-1}\sum_{j=n_0}^{r}|C(r,j)|\frac{|x(j)|}{\gamma^j}\gamma^j,\\ q(n) &\le \lambda q(n_0) + \sum_{r=n_0}^{n-1}\frac{\lambda}{\gamma}\sum_{j=n_0}^{r}|C(r,j)|\gamma^{j-r}q(j).\end{aligned}$$

Thus, by the discrete version of the Gronwall-Bellman lemma,

$$q(n) \le \lambda q(n_0) \exp\left(\sum_{r=n_0}^{n-1}\sum_{j=n_0}^{r} \lambda\gamma^{j-r-1}|C(r,j)|\right).$$

In view of the discrete version of the Fubini's theorem [1], we have

$$|x(n)| \le \lambda\gamma^{n-n_0}|x_0| \exp\left(\frac{\lambda}{\gamma}\sum_{r=n_0}^{n-1}\sum_{j=r}^{n-1} |C(j,r)\gamma^{r-j}\right).$$

Letting

$$K \equiv \sum_{j=n_0}^{\infty}\sum_{r=j}^{\infty} |C(r,j)\gamma^{j-r},$$

we get

$$\begin{aligned} |x(n)| &\le \lambda\gamma^{n-n_0}|x_0| \exp(\frac{\lambda}{\gamma}K) \\ &= M\gamma^{n-n_0}|x_0|,\ M \equiv \lambda \exp(\frac{\lambda K}{\gamma}). \end{aligned}$$

This completes the proof. □

The following result concerns with the boundedness of Eq. (5) and appeared in [4, Theorem 4.3] without the proof.

THEOREM 2.6. *Suppose that*

(i) *(4) is uniformly bounded,*
(ii) $\sum_{n=n_0}^{\infty} |f(n)| < \infty$.

Then (5) is bounded.

Proof. By Lemma 2.1, the solution $y(n)$ of (5) is given by

$$y(n) = R(n,n_0)y_0 + \sum_{r=n_0}^{n-1} R(n,r+1)f(r).$$

Thus we have

$$\begin{aligned} |y(n)| &\le |R(n,n_0)||y_0| + \sum_{r=n_0}^{n-1} |R(n,r+1)||f(r)| \\ &\le M|y_0| + M\sum_{r=n_0}^{n-1} |f(r)| \end{aligned}$$

for some $M > 0$ by (ii). Hence we obtain

$$|y(n)| \leq M|y_0| + M\sum_{r=n_0}^{\infty}|f(r)| \equiv \tilde{M}(y_0),\ n \geq n_0.$$

□

REMARK 2.7. *Theorem 2.6 was improved in* [4, Theorem 4.4] *under the conditions*

(i) *(4) is uniformly asymptotically stable,*
(ii) $|B(n,j)| \leq M\nu^{n-j},\ M > 0,\ 0 < \nu < 1,\ n \geq j,$
(iii) $|f(j)| \leq C,\ C > 0.$

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