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INTUITIONISTIC (T, S)-NORMED FUZZY SUBALGEBRAS OF BCK-ALGEBRAS

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Abstract. Using t-norm T and s-norm S, we introduce the notion of intuitionistic (T, S)-normed fuzzy subalgebra in BCK/BCI-algebra, and some related properties are investigated.

1. Introduction

The notion of BCK-algebras was proposed by Imai and Iséki in 1966. In the same year, Iséki [5] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. After the introduction of the concept of fuzzy sets by Zadeh [10], several researches were conducted on the generalization of the notion of fuzzy sets. The idea of "intuitionistic fuzzy set" was first published by Atanassov [2, 3], as a generalization of the notion of fuzzy set. In the present paper, Using t-norm T and s-norm S, we introduce the notion of intuitionistic (T, S)-normed fuzzy subalgebra in BCK/BCI-algebra, and some related properties are investigated.

2. Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

Recall that a *BCI-algebra* is an algebra $(X, *, 0)$ of type (2, 0) satisfying the following axioms:

(I) $((x * y) * (x * z)) * (z * y) = 0$,

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(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$, and
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$
for every $x, y, z \in X$. A BCI-algebra X satisfying the condition
(V) $0 * x = 0$ for all $x \in X$
is called a *BCK-algebra*. In any BCK/BCI-algebra X one can define a partial order "$\leq$" by putting $x \leq y$ if and only if $x * y = 0$.

A BCK/BCI-algebra X has the following properties:
(2.1) $x * 0 = x$,
(2.2) $(x * y) * z = (x * z) * y$,
(2.3) $x \leq y$ implies that $x * z \leq y * z$ and $z * y \leq z * x$,
(2.4) $(x * z) * (y * z) \leq x * y$
for all $x, y, z \in X$.

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A mapping $f : X \to Y$ of BCK/BCI-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Let X be a BCK/BCI-algebra. A fuzzy set f in X, i.e., a mapping $f : X \to [0, 1]$, is called a *fuzzy subalgebra* of X if $f(x * y) \geq f(x) \wedge f(y)$ for all $x, y \in X$. Note that if f is a fuzzy subalgebra of a BCK/BCI-algebra X, then $f(0) \geq f(x)$ for all $x \in X$.

Definition 2.1. [1] By a *t-norm* T, we mean a function $T : [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following conditions:
(T1) $T(x, 1) = x$,
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$,
(T3) $T(x, y) = T(y, x)$,
(T4) $T(x, T(y, z)) = T(T(x, y), z)$,
for all $x, y, z \in [0, 1]$.

A function $\mu : X \to [0, 1]$ is called a *fuzzy subalgebra* of X with respect to a t-norm T (briefly, a *T-fuzzy subalgebra* of X) if

$$\mu(x * y) \geq T(\mu(x), \mu(y))$$

for all $x, y \in X$.

Every t-norm T has a useful property:

$$T(\alpha, \beta) \leq \min(\alpha, \beta) \text{ for all } \alpha, \beta \in [0, 1].$$

Definition 2.2. [9] By a *s-norm* S, we mean a function $S : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:
(S1) $S(x, 0) = x$,
(S2) $S(x, y) \leq S(x, z)$ if $y \leq z$,
(S3) $S(x, y) = S(y, x)$,
(S4) $S(x, S(y, z)) = S(S(x, y), z)$,
for all $x, y, z \in [0, 1]$.

Every s-norm S has a useful property:

$$\max(\alpha, \beta) \leq S(\alpha, \beta) \text{ for all } \alpha, \beta \in [0, 1].$$

For a t-norm (or s-norm) P on $[0, 1]$, denote by Δ_P the set of element $\alpha \in [0, 1]$ such that $P(\alpha, \alpha) = \alpha$, i.e., $\Delta_P := \{\alpha \in [0, 1] \mid P(\alpha, \alpha) = \alpha\}$.

Definition 2.3. Let P be a t-norm (or s-norm). A fuzzy set μ in X is said to satisfy *imaginable property with respect to* P if $\mathrm{Im}(\mu) \subseteq \Delta_P$.

A function $\mu : X \to [0, 1]$ is called a *fuzzy subalgebra* of X with respect to a s-norm S (briefly, a *S-fuzzy subalgebra* of X) if

$$\mu(x * y) \leq S(\mu(x), \mu(y))$$

for all $x, y \in X$.

Let X denote a BCK/BCI-algebra. An *intuitionistic fuzzy set* (IFS for short) A is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$$

where the functions $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$.

3. Intuitionistic (T, S)-normed fuzzy subalgebras

In what follows, let X denote a BCK/BCI-algebra unless otherwise specified.

DEFINITION 3.1. Let T be a t-norm and S be a s-norm. An IFS $A = (\mu_A, \gamma_A)$ in X is called an *intuitionistic* (T, S)*-normed fuzzy subalgebras of* X if

$$\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)) \text{ and } \gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y))$$

for all $x, y \in X$.

EXAMPLE 3.2. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $T : [0,1] \times [0,1] \to [0,1]$ be a function defined by

$$T(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$$

for all $\alpha, \beta \in [0,1]$ and $S : [0,1] \times [0,1] \to [0,1]$ be a function defined by

$$S(\alpha, \beta) = \min(\alpha + \beta, 1)$$

for all $\alpha, \beta \in [0,1]$. Then T is a t-norm and S is a s-norm. Define an intuitionistic fuzzy set IFS $A = (\mu_A, \gamma_A)$ by $\mu_A(0) = \mu_A(b) = \mu_A(c) = 1, \mu_A(a) = 0$ and $\gamma_A(0) = \gamma_A(b) = \gamma_A(c) = 0, \gamma_A(a) = 1$. Then IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-normed fuzzy subalgebra of X.

DEFINITION 3.3. Let T be a t-norm and S be a s-norm on $[0, 1]$. An intuitionistic (T, S)-normed fuzzy subalgebra $A = (\mu_A, \gamma_A)$ is called an *intuitionistic imaginable* (T, S)*-fuzzy subalgebra* of X if μ_A and γ_A satisfy the imaginable property with respect to T and S respectively.

EXAMPLE 3.4. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	c	b	0

Define an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ by

$$\mu_A(x) := \begin{cases} 1 & \text{if } x \in \{0, a\} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \gamma_A(x) := \begin{cases} 0 & \text{if } x \in \{0, a\} \\ 1 & \text{otherwise.} \end{cases}$$

and let $T : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by

$$T(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$$

and and $S : [0, 1] \times [0, 1] \to [0, 1]$ be a function defined by

$$S(\alpha, \beta) = \min(\alpha + \beta, 1)$$

for all $\alpha, \beta \in [0, 1]$. Then T is a t-norm and S is a s-norm. By routine calculations, we know that $\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y))$ and $\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y))$for all $x, y \in X$, and $\mathrm{Im}(\mu_A) \subseteq \Delta_T$ and $\mathrm{Im}(\gamma_A) \subseteq \Delta_S$. Hence $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of X.

Proposition 3.5. *Let T be a t-norm and S be a s-norm on $[0, 1]$. If IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of X, then we have*

$$\mu_A(0) \geq \mu_A(x) \text{and } \gamma_A(0) \leq \gamma_A(x)$$

for all $x \in X$.

Proof. For every $x \in X$, we have

$$\mu_A(0) = \mu_A(x * x) \geq T(\mu_A(x), \mu_A(x)) = \mu_A(x),$$

and

$$\gamma_A(0) = \gamma_A(x * x) \leq S(\gamma_A(x), \gamma_A(x)) = \gamma_A(x).$$

This completes the proof. □

Proposition 3.6. *Let T be a t-norm and S be a s-norm. If IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of X, then the set*

$$X_A = \{x \in X \mid \mu_A(x) = \mu_A(0), \gamma_A(x) = \gamma_A(0)\}$$

is a subalgebra of BCK-algebra X.

Proof. Let T be a t-norm and S be a s-norm. Let $x, y \in X_A$. Then $\mu_A(x) = \mu_A(y) = \mu_A(0)$ and $\gamma_A(x) = \gamma_A(y) = \gamma_A(0)$. Since $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of X, it follows that

$$\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)) = T(\mu_A(0), \mu_A(0)) = \mu_A(0),$$

$$\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y)) = S(\gamma_A(0), \gamma_A(0)) = \gamma_A(0)$$

so that $\mu_A(x * y) = \mu_A(0)$ and $\gamma_A(x * y) = \gamma_A(0)$. Thus $x * y \in X_A$, and consequently X_A is a subalgebra of BCK-algebra X. □

Let $A = (\mu_A, \gamma_A)$ be an IFS in X and let $\alpha \in [0, 1]$. Then the sets

$$U(\mu_A; \alpha) := \{x \in X : \mu_A(x) \geq \alpha\}$$

and

$$L(\gamma_A; \alpha) := \{x \in X : \gamma_A(x) \leq \alpha\}$$

are called a *μ-level α-cut* and a *γ-level α-cut* of A, respectively.

THEOREM 3.7. *Let T be a t-norm and S be a s-norm and let IFS $A = (\mu_A, \gamma_A)$ be an intuitionistic (T, S)-normed fuzzy subalgebra of X and $\alpha \in [0, 1]$. then we have*

(i) if $\alpha = 1$, then the upper level set $U(\mu_A; \alpha)$ is either empty or a subalgebra of X.

(ii) if $\alpha = 0$, then the lower level set $L(\gamma_A; \alpha)$ is either empty or a subalgebra of X.

(iii) if $T = \min$, then the upper level set $U(\mu_A; \alpha)$ is either empty or subalgebra of X.

(iv) if $S = \max$, then the lower level set $L(\gamma_A; \alpha)$ is either empty or subalgebra of X.

Proof. (i) Suppose that $\alpha = 1$ and let $x, y \in U(\mu_A; \alpha)$. Then $\mu_A(x) \geq \alpha = 1$ and $\mu_A(y) \geq \alpha = 1$. It follows that $\mu_A(x*y) \geq T(\mu_A(x), \mu_A(y)) \geq T(1, 1) = 1$ so that $x * y \in U(\mu_A; \alpha)$. Hence $U(\mu_A; \alpha)$ is a subalgebra of X when $\alpha = 1$.

(ii) Suppose that $\alpha = 0$ and let $x, y \in L(\gamma_A; \alpha)$. Then $\gamma_A(x) \leq \alpha = 0$ and $\gamma_A(y) \leq \alpha = 0$. It follows that $\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y)) \leq S(0, 0) = 0$ so that $x * y \in L(\gamma_A; \alpha)$. Hence $L(\gamma_A; \alpha)$ is a subalgebra of X when $\alpha = 0$.

(iii) Assume that $T = \min$ and let $x, y \in U(\mu_A; \alpha)$. Then

$$\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)) = \min(\mu_A(x), \mu_A(y)) \geq \min(\alpha, \alpha) = \alpha$$

for all $\alpha \in [0, 1]$. Hence $x*y \in U(\mu_A; \alpha)$ and so $U(\mu_A; \alpha)$ is a subalgebra of X.

(iv) Assume that $S = \max$ and let $x, y \in L(\gamma_A; \alpha)$. Then

$$\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y)) = \max(\gamma_A(x), \gamma_A(y)) \leq \max(\alpha, \alpha) = \alpha$$

for all $\alpha \in [0, 1]$. Hence $x * y \in L(\gamma_A; \alpha)$ and so $L(\gamma_A; \alpha)$ is a subalgebra of X. □

If μ is a fuzzy set in X and θ is a mapping from X into itself, we define a mapping $\mu[\theta] : X \to [0, 1]$ by $\mu[\theta](x) = \mu(\theta(x))$ for all $x \in X$.

THEOREM 3.8. *Let T be a t-norm and S be a s-norm. Let θ be an endomorphism of X. If $A=(\mu_A,\gamma_A)$ is an intuitionistic (T,S)-normed fuzzy subalgebra of X, then $B=(\mu_A[\theta],\gamma_A[\theta])$ is an intuitionistic (T,S)-normed fuzzy subalgebra of X.*

Proof. For any $x,y\in X$, we have

$$\mu_A[\theta](x*y)=\mu_A(\theta(x*y))=\mu_A(\theta(x)*\theta(y))$$

$$\geq T(\mu_A(\theta(x)),\mu_A(\theta(y)))=T(\mu_A[\theta](x),\mu_A[\theta](y)).$$

Similarly, we have for any $x,y\in X$, we have

$$\gamma_A[\theta](x*y)=\gamma_A(\theta(x*y))=\gamma_A(\theta(x)*\theta(y))$$

$$\leq S(\gamma_A(\theta(x)),\gamma_A(\theta(y)))=S(\gamma_A[\theta](x),\gamma_A[\theta](y)).$$

This completes the proof. □

THEOREM 3.9. *Let T be a t-norm and S be a s-norm let $A=(\mu_A,\gamma_A)$ be an* IFS *in X such that the non-empty sets $U(\mu_A;\alpha)$ and $L(\gamma_A;\alpha)$ are subalgebras of X for all $\alpha\in[0,1]$. Then $A=(\mu_A,\gamma_A)$ is an intuitioinistic (T,S)-normed fuzzy subalgebra of X.*

Proof. Suppose that there exists $x_0,y_0\in X$ such that

$$\mu_A(x_0*y_0)<T(\mu_A(x_0),\mu_A(y_0)).$$

Taking $\alpha_0:=\frac{1}{2}(\mu_A(x_0*y_0)+T(\mu_A(x_0),\mu_A(y_0)))$, then

$$\begin{aligned}\min(\mu_A(x_0),\mu_A(y_0))&\geq T(\mu_A(x_0),\mu_A(y_0))\\&\geq\alpha_0>\mu_A(x_0*y_0).\end{aligned}$$

It follows that $x_0,y_0\in U(\mu_A;\alpha_0)$ and $x_0*y_0\notin U(\mu_A;\alpha_0)$. Hence μ_A satisfies the inequality $\mu_A(x*y)\geq T(\mu_A(x),\mu_A(y))$ for all $x,y\in X$. Similarly, suppose that there exists $x_0,y_0\in X$ such that

$$\gamma_A(x_0*y_0)>S(\gamma_A(x_0),\gamma_A(y_0)).$$

Taking $\beta_0:=\frac{1}{2}(\gamma_A(x_0*y_0)+S(\gamma_A(x_0),\gamma_A(y_0)))$, then

$$\max(\gamma_A(x_0),\gamma_A(y_0))\leq S(\gamma_A(x_0),\gamma_A(y_0))\leq\beta_0<\gamma_A(x_0*y_0).$$

It follows that $x_0,y_0\in L(\gamma_A;\beta_0)$ and $x_0*y_0\notin L(\gamma_A;\beta_0)$. This is a contradiction and hence γ_A satisfies the inequality $\gamma_A(x*y)\leq S(\gamma_A(x),\gamma_A(y))$ for all $x,y\in X$. □

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