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JORDAN–VON NEUMANN TYPE FUNCTIONAL INEQUALITIES

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ABSTRACT. It is shown that $f : \mathbb{R} \to \mathbb{R}$ satisfies the following functional inequalities

$$|f(x) + f(y)| \leq |f(x + y)|, \tag{0.1}$$

$$|f(x) + f(y)| \leq |2f(\frac{x + y}{2})|, \tag{0.2}$$

$$|f(x) + f(y) - 2f(\frac{x - y}{2})| \leq |2f(\frac{x + y}{2})|, \tag{0.3}$$

respectively, then the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the Cauchy functional equation, the Jensen functional equation and the Jensen quadratic functional equation, respectively.

1. Introduction and preliminaries

Gilányi [3] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\| \tag{1.1}$$

then f satisfies the quadratic functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y).$$

See also [6]. Gilányi [4] and Fechner [2] proved the generalized Hyers–Ulam stability of the functional inequality (1.1). Park, Cho and Han [5] investigated the Jordan–von Neumann type Cauchy–Jensen additive mappings and prove their stability, and Cho and Kim [1] proved the generalized Hyers–Ulam stability of the Jordan–von Neumann type Cauchy–Jensen additive mappings.

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In Section 2, we prove that if $f : \mathbb{R} \to \mathbb{R}$ satisfies the additive functional inequalities (0.1) and (0.2), respectively, then the function f is Cauchy additive and Jensen additive, respectively, and that if $f : \mathbb{R} \to \mathbb{R}$ satisfies the quadratic functional inequality (0.3), then the function f is Jensen quadratic.

Throughout this paper, let $\mathbb{R}$ denote the field of real numbers.

2. Jordan–von Neumann type additive functional inequalities

In this section, we investigate the Cauchy additive inequality and the Jensen additive inequality.

THEOREM 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that*

$$|f(x) + f(y)| \le |f(x + y)| \tag{2.1}$$

for all $x, y \in \mathbb{R}$. Then f satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Proof. Letting $x = y = 0$ in (2.1), we get

$$|2f(0)| \le |f(0)|.$$

So $f(0) = 0$. Letting $y = -x$ in (2.1), we get

$$|f(x) + f(-x)| \le |f(0)| = 0$$

for all $x \in \mathbb{R}$. Hence $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

First of all, we show that $|f(x) + f(y)| = |f(x + y)|$ for all $x, y \in \mathbb{R}$.

We divide into four cases.

(i) Case 1. $f(x) \ge 0$ and $f(y) \ge 0$.

It follows from (2.1) that

$$f(x) + f(y) \le |f(x + y)|.$$

Replacing x by $x + y$ and y by $-y$ in (2.1), we get

$$|f(x + y) + f(-y)| \le |f(x)| = f(x).$$

So

$$|f(y)| + |f(x + y) + f(-y)| \le f(x) + |f(y)| = f(x) + f(y).$$

By the triangle inequality,

$$\begin{aligned} |f(y)| + |f(x + y) + f(-y)| &\ge |f(y) + (f(x + y) + f(-y))| \\ &= |f(x + y) + f(y) + f(-y)| = |f(x + y)|. \end{aligned}$$

Thus

$$|f(x+y)| \leq f(x)+f(y).$$

So

$$|f(x+y)| = f(x)+f(y).$$

(ii) Case 2. $f(x) < 0$ and $f(y) < 0$.

Replacing x by $-x$ and y by $-y$, we get

$$|f(-x)+f(-y)| = f(-x-y).$$

By (i),

$$|-f(x)-f(y)| = |-f(x+y)|.$$

So

$$|f(x)+f(y)| = |f(x+y)|.$$

(iii) Case 3. $f(x) \geq 0, f(y) \leq 0$ and $f(x+y) \geq 0$.

It follows from (2.1) that

$$f(x)+f(y) \leq |f(x+y)|.$$

Replacing x by $x+y$ and y by $-y$ in (2.1), we get

$$|f(x+y)+f(-y)| \leq |f(x)|.$$

So

$$|f(x+y)-f(y)| \leq f(x).$$

For the case $f(x+y) \geq 0$,

$$f(x+y)-f(y) \leq f(x).$$

Thus

$$f(x+y) \leq f(x)+f(y).$$

Since

$$f(x)+f(y) \leq f(x+y),$$
$$f(x)+f(y) = f(x+y).$$

So

$$|f(x)+f(y)| = |f(x+y)|.$$

For the case $f(x+y) < 0$, replacing x by $x+y$ and y by $-x$ in (2.1),

$$|f(x+y)-f(x)| \leq |f(y)|.$$

Since

$$f(x)-f(x+y) \leq -f(y),$$
$$0 \leq f(x)+f(y) \leq f(x+y) < 0,$$

which is a contradiction.

Similarly, for the case $f(x) \leq 0$, $f(y) \geq 0$, $f(x) + f(y) \geq 0$, one can show that

$$|f(x) + f(y)| = |f(x + y)|.$$

(iv) Case 4. $f(x) \geq 0, f(y) \leq 0$ and $f(x + y) < 0$.

Replacing x by $-x$ and y by $-y$ in (2.1), we get

$$|f(-x) + f(-y)| = |f(-x - y)|.$$

By (iii),

$$| - f(x) - f(y)| = | - f(x + y)|.$$

So

$$|f(x) + f(y)| = |f(x + y)|.$$

Similarly, for the case $f(x) \leq 0$, $f(y) \geq 0$, $f(x) + f(y) < 0$, one can show that

$$|f(x) - f(y)| = |f(x + y)|.$$

Thus

$$|f(x) + f(y)| = |f(x + y)|$$

for all $x, y \in \mathbb{R}$.

Next, we show that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Since $|f(x+y)| = |f(x)+f(y)|$, $f(x+y) = f(x)+f(y)$ or $f(x+y) = -f(x) - f(y)$.

Assume that there are $x, y \in \mathbb{R}$ such that $f(x + y) = -f(x) - f(y)$ with $f(x + y) \neq 0$.

We divide into four cases.

(i) Case 1. $f(x) \geq 0$ and $f(y) \geq 0$.

Since $f(x + y) = -f(x) - f(y) \leq 0$, $f(x + y) < 0$. Replacing x and y by $x + y$ and $-y$ in (2.1), respectively, we get

$$|f(x + y) - f(y)| = |f(x + y) + f(-y)| = |f(x)| = f(x).$$

Thus $f(y)-f(x+y) = f(x)$. Since $f(y)-f(x) = f(x+y) = -f(x)-f(y)$, $f(y) = 0$.

Similarly, we can show that $f(x) = 0$. So $f(x+y) = -f(x)-f(y) = 0$, which is a contradiction.

(ii) Case 2. $f(x) \leq 0$ and $f(y) \leq 0$.

Note that $f((-x) + (-y)) = -f(-x) - f(-y)$, $f(-x) = -f(x) \geq 0$ and $f(-y) = -f(y) \geq 0$. By (i), we can get $f(x + y) = 0$, which is a contradiction.

(iii) Case 3. $f(x) \geq 0, f(y) \leq 0$ and $f(x + y) \geq 0$.

Since $f(x + y) = -f(x) - f(y) \leq 0$, $f(x + y) < 0$. Replacing x and y by $x+y$ and $-x$ in (2.1), respectively, we get $|f(x+y)+f(-x)| = |f(y)|$.

Thus $f(x) - f(x+y) = -f(y)$. So $f(x+y) = f(x) + f(y) = -f(x+y)$, which is a contradiction.

(iv) Case 4. $f(x) \geq 0, f(y) \leq 0$ and $f(x+y) \leq 0$.

Since $f(x+y) = -f(x) - f(y) \geq 0$, $f(x+y) < 0$. Replacing x and y by $x+y$ and $-y$ in (2.1), respectively, we get $|f(x+y)+f(-y)| = |f(x)|$. Thus $f(x+y) - f(y) = f(x)$. So $f(x+y) = f(x) + f(y) = -f(x+y)$, which is a contradiction.

Therefore,

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$, as desired. □

COROLLARY 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(2x) = 2f(x)$ such that*

$$|f(x) + f(y)| \leq |2f(\frac{x+y}{2})|$$

for all $x, y \in \mathbb{R}$. Then f satisfies $2f(\frac{x+y}{2}) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Proof. Since $f(2x) = 2f(x)$, the inequality

$$|f(x) + f(y)| \leq |2f(\frac{x+y}{2})|$$

is equivalent to the inequality

$$|f(x) + f(y)| \leq |2f(\frac{x+y}{2})| = |f(x+y)|$$

for all $x, y \in \mathbb{R}$. By Theorem 2.1,

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$, and

$$2f(\frac{x+y}{2}) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$, as desired. □

3. Jordan–von Neumann type quadratic functional inequalities

In this section, we investigate the Jensen quadratic inequality.

THEOREM 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(2x) = 4f(x)$ such that*

$$|f(x) + f(y) - 2f(\frac{x-y}{2})| \leq |2f(\frac{x+y}{2})| \tag{3.1}$$

for all $x, y \in \mathbb{R}$. *Then* f *satisfies* $2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) = f(x) + f(y)$ *for all* $x, y \in \mathbb{R}$.

Proof. Putting $x = 0$ in $f(2x) = 4f(x)$, we get $f(0) = 4f(0)$. So $f(0) = 0$.

Letting $y = -x$ in (3.1), we get

$$|f(x) + f(-x) - 2f(x)| \leq |f(0)| = 0$$

for all $x \in \mathbb{R}$. So

$$f(-x) = f(x) \tag{3.2}$$

for all $x \in \mathbb{R}$.

It follows from (3.1) that

$$(f(x) + f(y) - 2f(\frac{x-y}{2}))^2 \leq 4f(\frac{x+y}{2})^2$$

for all $x, y \in \mathbb{R}$. Hence

$$\begin{aligned}(f(x) + f(y))^2 + 4f(\frac{x-y}{2})^2 - 4(f(x)\\ +f(y))f(\frac{x-y}{2}) \leq 4f(\frac{x+y}{2})^2\end{aligned} \tag{3.3}$$

for all $x, y \in \mathbb{R}$.

Replacing y by $-y$ in (3.3), we get

$$(f(x) + f(-y))^2 + 4f(\frac{x+y}{2})^2 - 4(f(x) + f(-y))f(\frac{x+y}{2}) \leq 4f(\frac{x-y}{2})^2$$

for all $x, y \in \mathbb{R}$. By (3.2),

$$\begin{aligned}(f(x) + f(y))^2 + 4f(\frac{x+y}{2})^2 - 4(f(x)\\ +f(y))f(\frac{x+y}{2}) \leq 4f(\frac{x-y}{2})^2\end{aligned} \tag{3.4}$$

for all $x, y \in \mathbb{R}$.

It follows from (3.3) and (3.4) that

$$2(f(x) + f(y))^2 - 4(f(x) + f(y))(f(\frac{x-y}{2}) + f(\frac{x+y}{2})) \leq 0$$

for all $x, y \in \mathbb{R}$. So

$$(-2f(x) - 2f(y))(2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) - f(x) - f(y)) \leq 0$$

for all $x, y \in \mathbb{R}$. Thus

$$(-f(x) - f(y))(2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) - f(x) - f(y)) \leq 0 \tag{3.5}$$

for all $x, y \in \mathbb{R}$.

Replacing x by $\frac{-x-y}{2}$ and y by $\frac{-x+y}{2}$ in (3.3), we get

$$\begin{aligned}(f(\frac{-x-y}{2}) &+ f(\frac{-x+y}{2}))^2 + 4f(\frac{-y}{2})^2 - 4(f(\frac{-x-y}{2}) \\ &+ f(\frac{-x+y}{2}))f(\frac{-y}{2}) \leq 4f(\frac{-x}{2})^2\end{aligned}$$

for all $x, y \in \mathbb{R}$. Since $f(-x) = f(x)$,

$$\begin{aligned}(f(\frac{x+y}{2}) + f(\frac{x-y}{2}))^2 + 4f(\frac{y}{2})^2 - 4(f(\frac{x+y}{2}) \\ +f(\frac{x-y}{2}))f(\frac{y}{2}) \leq 4f(\frac{x}{2})^2\end{aligned} \tag{3.6}$$

for all $x, y \in \mathbb{R}$.

Replacing x by $\frac{-x-y}{2}$ and y by $\frac{x-y}{2}$ in (3.3), we get

$$\begin{aligned}(f(\frac{-x-y}{2}) &+ f(\frac{x-y}{2}))^2 + 4f(\frac{-x}{2})^2 - 4(f(\frac{-x-y}{2}) \\ &+ f(\frac{x-y}{2}))f(\frac{-x}{2}) \leq 4f(\frac{-y}{2})^2\end{aligned}$$

for all $x, y \in \mathbb{R}$. Since $f(-x) = f(x)$,

$$\begin{aligned}(f(\frac{x+y}{2}) + f(\frac{x-y}{2}))^2 + 4f(\frac{x}{2})^2 \\ -4(f(\frac{x+y}{2}) + f(\frac{x-y}{2}))f(\frac{x}{2}) \leq 4f(\frac{y}{2})^2\end{aligned} \tag{3.7}$$

for all $x, y \in \mathbb{R}$.

It follows from (3.6) and (3.7) that

$$2(f(\frac{x+y}{2}) + f(\frac{x-y}{2}))^2 - 4(f(\frac{x+y}{2}) + f(\frac{x-y}{2}))(f(\frac{x}{2}) + f(\frac{y}{2})) \leq 0$$

for all $x, y \in \mathbb{R}$. So

$$(2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}))(f(\frac{x+y}{2}) + f(\frac{x-y}{2}) - 2f(\frac{x}{2}) - 2f(\frac{y}{2})) \leq 0$$

for all $x, y \in \mathbb{R}$. Since $f(\frac{y}{2}) = \frac{1}{4}f(y)$,

$$\begin{aligned}(2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}))(2f(\frac{x+y}{2}) \\ +2f(\frac{x-y}{2}) - f(x) - f(y)) \leq 0\end{aligned} \tag{3.8}$$

for all $x, y \in \mathbb{R}$.

It follows from (3.5) and (3.8) that

$$(2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) - f(x) - f(y))^2 \leq 0$$

for all $x, y \in \mathbb{R}$.

Therefore,

$$2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) - f(x) - f(y) = 0$$

for all $x, y \in \mathbb{R}$. So

$$2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$. □

REMARK 3.1. Under the assumption $f(2x) = 4f(x)$ for all $x \in \mathbb{R}$, the inequality (3.1) is equivalent to the inequality

$$|2f(x) + 2f(y) - f(x-y)| \leq |f(x+y)| \tag{3.9}$$

for all $x, y \in \mathbb{R}$. Gilányi [3] proved that if $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality (3.9) then

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

holds for all $x, y \in \mathbb{R}$. Since $f(\frac{x}{2}) = \frac{1}{4}f(x)$ for all $x \in \mathbb{R}$,

$$2f(\frac{x+y}{2}) + 2f(\frac{x-y}{2}) = f(x) + f(y)$$

holds for all $x, y \in \mathbb{R}$.

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