JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **20**, No. 3, September 2007

THE STUDY OF THE SYSTEM OF NONLINEAR WAVE EQUATIONS

Tacksun Jung* and Q-Heung Choi **

Abstract. We show the existence of the positive solution for the system of the following nonlinear wave equations with Dirichlet boundary conditions

$$u_{tt} - u_{xx} + av^{+} = s\phi_{00} + f,$$
$$v_{tt} - v_{xx} + bu^{+} = t\phi_{00} + g,$$
$$u(\pm\frac{\pi}{2}, t) = v(\pm\frac{\pi}{2}, t) = 0,$$

where $u_{+} = \max\{u, 0\}$, $s,\ t \in R$, ϕ_{00} is the eigenfunction corresponding to the positive eigenvalue $\lambda_{00} = 1$ of the eigenvalue problem $u_{tt} - u_{xx} = \lambda_{mn}u$ with $u(\pm\frac{\pi}{2}, t) = 0$, $u(x, t + \pi) = u(x, t) = u(-x, t) = u(x, -t)$ and f, g are π-periodic, even in x and t and bounded functions in $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\phi_{00} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g\phi_{00} = 0$.

1. Introduction and statement of main result

In this paper we show the existence of the positive solution of the system of the following nonlinear wave equations with Dirichlet boundary conditions

$$(1.1)\qquad \begin{cases} u_{tt} - u_{xx} + av^{+} &= s\phi_{00} + f, \\ v_{tt} - v_{xx} + bu^{+} &= t\phi_{00} + g, \\ u(\pm\frac{\pi}{2}, t) &= v(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t + \pi) &= u(x, t) = u(-x, t) = u(x, -t), \\ v(x, t + \pi) &= v(x, t) = v(-x, t) = v(x, -t), \end{cases}$$

Received July 16, 2007.
2000 Mathematics Subject Classification: Primary 35B10, 35L05, 35L20.
Key words and phrases: system of nonlinear wave equations, positive solution, Dirichlet boundary condition.
**Research supported in part by Inha University Research Grant.

eigenfunction corresponding to the positive eigenvalue $\lambda_{00} = 1$ of the eigenvalue problem $u_{tt} - u_{xx} = \lambda_{mn} u$ with $u(\pm\frac{\pi}{2}, t) = 0$, $u(x, t+\pi) = u(x,t) = u(-x,t) = u(x,-t)$. We assume that f, g are π-periodic, even in x and t and bounded functions in $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ with $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\phi_{00} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g\phi_{00} = 0$. The system can be rewritten by

$$
(1.2) \qquad \begin{cases} U_{tt} - U_{xx} + AU^+ & = \binom{s\phi_{00}}{t\phi_{00}}, \\ U(\pm\frac{\pi}{2}, t) & = 0, \\ U(x, t+\pi) & = U(x,t) = U(-x,t) = U(x,-t), \end{cases}
$$

where $U = \binom{u}{v}$, $U^+ = \binom{u^+}{v^+}$, $U_{tt} - U_{xx} = \binom{u_{tt}-u_{xx}}{v_{tt}-v_{xx}}$, $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \in M_{2\times 2}(R)$. Let us define the Hilbert space spanned by eigenfunctions as follows:
The eigenvalue problem for $u(x,t)$,

$$u_{tt} - u_{xx} = \lambda u \qquad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}) \times R,$$

$$u(\pm\frac{\pi}{2}, t) = 0, \qquad u(x, t+\pi) = u(x,t) = u(-x,t) = u(x,-t)$$

has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^2 - 4m^2 \qquad (m, n = 0, 1, 2, \ldots)$$

and corresponding normalized eigenfunctions ϕ_{mn} $(m, n \geq 0)$ given by

$$\phi_{0n} = \frac{\sqrt{2}}{\pi} \cos(2n+1)x \qquad \text{for } n \geq 0,$$

$$\phi_{mn} = \frac{2}{\pi} \cos 2mt \cdot \cos(2n+1)x \qquad \text{for } m > 0, n \geq 0.$$

Let n be a fixed integer and define

$$\lambda_n^+ = \inf_m \{\lambda_{mn} : \lambda_{mn} > 0\} = 4n+1,$$

$$\lambda_n^- = \sup_m \{\lambda_{mn} : \lambda_{mn} < 0\} = -4n-3.$$

Letting $n \to \infty$, we obtain that $\lambda_n^+ \to +\infty$ and $\lambda_n^- \to -\infty$. We can check easily that the eigenvalues in the interval (-15,9) are given by

$$\lambda_{32} = -11 < \lambda_{21} = -7 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{11} = 5.$$

Let Q be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and H_0 the Hilbert space defined by

$$H_0 = \{u \in L^2(Q) |\ u \text{ is even in } x \text{ and } t\}.$$

The set of functions $\{\phi_{mn}\}$ is an orthonormal basis in H_0. Let us denote an element u, in H_0, by

$$u = \sum h_{mn}\phi_{mn}.$$

We define a Hilbert space H as follows

$$H = \{u \in H_0 : \ \sum |\lambda_{mn}| h_{mn}^2 < \infty\}.$$

Then this space is a Banach space with norm

$$\|u\|^2 = [\sum |\lambda_{mn} h_{mn}^2|]^{\frac{1}{2}}.$$

Let us set $E = H \times H$. We endow the Hilbert E with the norm

$$\|(u, v)\|_E^2 = \|u\|^2 + \|v\|^2.$$

We are looking for the weak solutions of (1.1) in H, that is, (u, v) satisfying the equation

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (u_{tt} - u_{xx})z + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (v_{tt} - v_{xx})w + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (A(u^+, v^+), (z, w)) - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [s\phi_{00} + f]z$$

$$- \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [t\phi_{00} + g]w = 0 \qquad \forall (z, w) \in E,$$

where $u = \sum c_{mn}\phi_{mn}$, $v = \sum d_{mn}\phi_{mn}$ with $u_{tt} - u_{xx} = \sum \lambda_{mn} c_{mn}\phi_{mn} \in H$, $v_{tt} - v_{xx} = \sum \lambda_{mn} d_{mn}\phi_{mn} \in H$ i.e., with $\sum c_{mn}^2\lambda_{mn}^2 < \infty$, $\sum d_{mn}^2\lambda_{mn}^2 < \infty$, which implies u, $v \in H$. Now we state the main result:

Theorem 1.1. *(Existence of a positive solution)*
Assume that

$$\lambda_{mn}^2 - ab \neq 0, \qquad \text{for all } m, n \text{ with } (m, n) \neq (0, 0), \tag{1.3}$$

$$a < 0, \qquad b < 0. \tag{1.4}$$

Then for each f, $g \in H$ such that f and g are π-periodic, even in x and t and bounded functions with $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\phi_{00} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g\phi_{00} = 0$ if $\lambda_{00}^2 - ab < 0$, then there exists (s_0, t_0) with $s_0 < 0$ and $t_0 < 0$ such that the system (1.1) has a positive solution for each $s < s_0$ and $t < t_0$, and if $\lambda_{00}^2 - ab > 0$, then there exists (s_1, t_1) with $s_1 > 0$ and $t_1 > 0$ such that the system (1.1) has a positive solution for each $s > s_1$ and $t > t_1$.

2. Proof of Theorem 1.1

We have some properties. Since $|\lambda_{mn}| \geq 1$ for all m, n, we have that

LEMMA 2.1. *(i)* $\|u\| \geq \|u\|_{L^2(Q)}$, *where* $\|u\|_{L^2(Q)}$ *denotes the* L^2 *norm of* u.
(ii) $\|u\| = 0$ *if and only if* $\|u\|_{L^2(Q)} = 0$.
(iii) $u_{tt} - u_{xx} \in H$ *implies* $u \in H$.

LEMMA 2.2. *Suppose that* c *is not an eigenvalue of* L, $Lu = u_{tt} - u_{xx}$, *and let* $f \in H_0$. *Then we have* $(L-c)^{-1}f \in H$.

Proof. When n is fixed, λ_n^+ and λ_n^- were defined in section 1:

$$\lambda_n^+ = 4n+1,$$
$$\lambda_n^- = -4n-3.$$

We see that $\lambda_n^+ \to +\infty$ and $\lambda_n^- \to -\infty$ as $n \to \infty$. Hence the number of elements in the set $\{\lambda_{mn} : \ |\lambda_{mn}| < |c|\}$ is finite, where λ_{mn} is an eigenvalue of L. Let

$$f = \sum h_{mn}\phi_{mn}.$$

Then

$$(L-c)^{-1}f = \sum \frac{1}{\lambda_{mn}+c} h_{mn}\phi_{mn}.$$

Hence we have the inequality

$$\|(L-c)^{-1}f\| = \sum |\lambda_{mn}| \frac{1}{(\lambda_{mn}+c)^2} h_{mn}^2 \leq C \sum h_{mn}^2$$

for some C, which means that

$$\|(L-c)^{-1}f\| \leq C_1 \|f\|_{L^2(Q)}, \qquad C_1 = \sqrt{C}.$$

□

LEMMA 2.3. *Assume that the conditions (1.3) and (1.4) hold. Then the system*

$$(2.1) \qquad \begin{cases} u_{tt} - u_{xx} + av & = s\phi_{00}, \\ v_{tt} - v_{xx} + bu & = t\phi_{00}, \\ u(\pm\frac{\pi}{2}, t) & = v(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) & = v(x,t) = v(-x,t) = v(x,-t) \end{cases}$$

has a unique solution $(u_*, v_*) \in E$, *which is of the form*

$$u_* = [\frac{-a}{\lambda_{00}} \frac{-bs+t}{\lambda_{00}^2 - ab} + \frac{s}{\lambda_{00}}]\phi_{00},$$

$$v_* = [\frac{-bs + t\lambda_{00}}{\lambda_{00}^2 - ab}]\phi_{00}.$$

Proof. We note that (u_*, v_*) is a solution of the system (2.1). □

LEMMA 2.4. *Assume that the conditions (1.3) and (1.4) hold. Then the system*

$$U_{tt} - U_{xx} + AU = 0, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix} \in E,$$

$$U(\pm\frac{\pi}{2}, t) = 0,$$

$$U(x, t+\pi) = U(x,t) = U(-x,t) = U(x,-t)$$

has only a trivial solution $U(x,t) = \binom{0}{0}$.

Proof. We assume that there exists a nontrivial solution $U = (u, v) \in E$ of (2.2) of the form $u = \phi_{mn}$ and $v = \phi_{m'n'}$. The equation

$$L\begin{pmatrix} \phi_{mn} \\ \phi_{m'n'} \end{pmatrix} + A\begin{pmatrix} \phi_{mn} \\ \phi_{m'n'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is equivalent to the equation

$$\begin{pmatrix} \lambda_{mn}\phi_{mn} \\ \lambda_{m'n'}\phi_{m'n'} \end{pmatrix} + \begin{pmatrix} a\phi_{m'n'} \\ b\phi_{mn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus when $mn \neq m'n'$, we have a contradiction since ϕ_{mn} and $\phi_{m'n'}$ are linearly independent. When $mn = m'n'$, we have $\lambda_{mn} + a = 0$ and $\lambda_{mn} + b = 0$, which means that $\lambda_{mn}^2 - ab = 0$. These contradicts the assumption (1.3). □

LEMMA 2.5. *Assume that the conditions (1.3)and (1.4) hold and f, $g \in H$ are π-periodic, even in x and t and bounded functions with* $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f\phi_{00} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g\phi_{00} = 0$. *Then the system*

$$(2.2) \qquad \begin{cases} u_{tt} - u_{xx} + av & = f, \\ v_{tt} - v_{xx} + bu & = g, \\ u(\pm\frac{\pi}{2}, t) & = v(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) & = v(x,t) = v(-x,t) = v(x,-t) \end{cases}$$

has a unique solution $(\breve{u}, \breve{v}) \in E$.

Proof. Let $\delta > 0$ and $\delta > \max\{a, b\}$. Let us consider the modified system

$$(2.3)\qquad \begin{cases} u_{tt} - u_{xx} & +av + \lambda_{00}u + \delta u = f, \\ v_{tt} - v_{xx} & +bu + \lambda_{00}v + \delta v = g, \\ u(\pm\frac{\pi}{2}, t) & = v(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) & = v(x,t) = v(-x,t) = v(x,-t). \end{cases}$$

Let us set

$$L_\delta U = U_{tt} + AU + \lambda_{00}U + \delta U, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix}.$$

The system (2.3) is invertible. Thus there exists an inverse operator $L_\delta^{-1} : L^2(Q) \times L^2(Q) \to E$ which is a linear and compact operator such that $(u, v) = L_\delta^{-1}(f, g)$. Thus we have that if (u, v) is a solution of (2.2) if and only if

$$(2.4)\qquad (u, v) = L_\delta^{-1}((f, g) + \lambda_{00}(u, v) + \delta(u, v)).$$

Thus we have

$$(I - (\lambda_{00} + \delta)L_\delta^{-1})((f, g) + \lambda_{00}(u, v) + \delta(u, v)) = (f, g).$$

By the conditions (1.3) and (1.4), $\frac{1}{\lambda_{00}+\delta} \notin \sigma(L_\delta^{-1})$. Since L_δ^{-1} is a compact operator, the system (2.4) has a unique solution, thus the system (2.2) has a unique solution. □

PROOF OF THEOREM 1.1 By Lemma 2.3 and Lemma 2.5, $(u_* + \check{u}, v_* + \check{v})$ is a solution of the system

$$(2.5)\qquad \begin{cases} u_{tt} - u_{xx} + av & = s\phi_{00} + f, \\ v_{tt} - v_{xx} + bu & = t\phi_{00} + g, \\ u(\pm\frac{\pi}{2}, t) & = v(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) & = u(x,t) = u(-x,t) = u(x,-t), \\ v(x, t+\pi) & = v(x,t) = v(-x,t) = v(x,-t), \end{cases}$$

where $u_* = [\frac{-a}{\lambda_{00}} \frac{-bs+t}{\lambda_{00}^2 - ab} + \frac{s}{\lambda_{00}}]\phi_{00}$ and $v_* = [\frac{-bs+t\lambda_{00}}{\lambda_{00}^2 - ab}]\phi_{00}$. Therefore if $\lambda_{00}^2 - ab < 0$, then there exists (s_0, t_0) with $s_0 < 0$ and $t_0 < 0$ such that $u_* + \check{u} > 0$ and $v_* + \check{v} > 0$ for each $s < s_0$ and $t < t_0$, and if $\lambda_{00}^2 - ab > 0$, then there exists (s_1, t_1) with $s_1 > 0$ and $t_1 > 0$ such that $u_* + \check{u} > 0$ and $v_* + \check{v} > 0$ for each $s > s_1$ and $t > t_1$.

References

[1] T. Bartsch and M. Klapp, *Critical point theory for indefinite functionals with symmetries,* J. Funct. Anal., 107-136 (1996).

[2] A. Castro and A. C. Lazer, *Critical point theory and the number of solutions of a nonlinear Dirichlet problem,* Ann. Mat. Pura Appl. **120**, no. 4, 113-137 (1979).

[3] K. C. Chang, *Solutions of asymptotically linear operator equations via Morse theory,* Comm. Pure Appl. Math. **34**, 693-712 (1981).

[4] K. C. Chang, *Infinite dimensional Morse theory and multiple solution problems,* Birkhäuser (1993).

[5] Q. H. Choi and T. Jung, *An application of a variational reduction method to a nonlinear wave equation,* J. Differential Equations, **117**, 390-410 (1995).

[6] Q. H. Choi and T. Jung, *Multiple periodic solutions of a semilinear wave equation at double external resonances,* Communications in Applied Analysis **3**, no. 1, 73-84 (1999).

[7] Q. H. Choi and T. Jung, *Multiplicity results for nonlinear wave equations with nonlinearities crossing eigenvalues,* Hokkaido Mathematical Journal **24**, no. 1, 53-62 (1995).

[8] P. J. McKenna and W. Walter, *Nonlinear oscillations in a suspension bridge,* Archive for Rational Mechanics and Analysis **98**, no. 2, 167-177 (1987).

*

Department of Mathematics
Kunsan National University
Kunsan 573-701, Republic of Korea
E-mail: tsjung@kunsan.ac.kr

**

Department of Mathematics Education
Inha University
Incheon 402-751, Republic of Korea
E-mail: qheung@inha.ac.kr