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LIFTING T-STRUCTURES AND THEIR DUALS

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Abstract. We define and study a concept of T^f-space for a map, which is a generalized one of a T-space, in terms of the Gottlieb set for a map. We show that X is a T^f-space if and only if $G(\Sigma B; A, f, X) = [\Sigma B, X]$ for any space B. For a principal fibration $E_k \to X$ induced by $k : X \to X'$ from $\epsilon : PX' \to X'$, we obtain a sufficient condition to having a lifting $T^{\bar{f}}$-structure on E_k of a T^f-structure on X . Also, we define and study a concept of co-T^g-space for a map, which is a dual one of T^f-space for a map. We obtain a dual result for a principal cofibration $i_r : X \to C_r$ induced by $r : X' \to X$ from $\iota : X' \to cX'$.

1. Introduction

In [1], Aguade introduced a T-space as a space X having the property that the evaluation fibration $\Omega X \to X^{S^1} \to X$ is fibre homotopically trivial. It is easy to show that any H-space is a T-space. However, there are many T-spaces which are not H-spaces in [16]. Let ΣX denotes the reduced suspension of X, and ΩX denotes the based loop space of X. Let τ be the adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$. The symbols e and e' denote $\tau^{-1}(1_{\Omega X})$and $\tau(1_{\Sigma X})$ respectively. In [16], Woo and Yoon showed that the concept of T-space is closely related by the Gottlieb set $G(A, X)$, which is the set of homotopy classes of cyclic maps from A to X as follows; X is a T-space if and only if $G(\Sigma B, X) = [\Sigma B, X]$ for any space B. Also, we introduced and showed [16] that a concept of co-T-space as a dual one of T-space, which is closely related by the dual Gottlieb set $DG(X, A)$ which is the set of homotopy classes of cocyclic maps from X to A as follows; X is a co-T-space if and only if $DG(X; \Omega B) = [X, \Omega B]$ for any space B. In [12], Oda introduced the concept of f-cyclic map as a generalization of that

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of cyclic map. We called [21] the set of all homotopy classes of f-cyclic maps from B to X as the Gottlieb set $G(B;A,f,X)$ for a map $f:A\to X$. In general, $G(B,X)\subset G(B;A,f,X)\subset[B,X]$ for any map $f:A\to X$ and any space B. However, it is known [19] that $G(S^5,S^5\times S^5)\cong 2Z\oplus 2Z\neq G(S^5;S^5,i_1,S^5\times S^5)\cong 2Z\oplus Z\neq[S^5,S^5\times S^5]\cong Z\oplus Z$, where $i_1:S^5\to S^5\times S^5$ is the inclusion. In [18], we introduced the set of all homotopy classes of g-cocyclic maps from X to B as the dual Gottlieb set $DG(X,g,A;B)$ for a map $g:X\to A$. In general, $DG(X,B)\subset DG(X,g,A;B)\subset[X,B]$ for any map $g:X\to A$ and any space B. We also showed [20] that $DG(S^n\times S^n,K(Z,n))\neq DG(S^n\times S^n,p_1,S^n;K(Z,n))\neq[S^n\times S^n,K(Z,n)]$ for all n, where $p_1:S^n\times S^n\to S^n$ is the projection.

In this paper, we introduce a T^f-space for a map $f:A\to X$ as a space X having the property that $e:\Sigma\Omega X\to X$ is f-cyclic, that is, there is a T^f-structure $F:\Sigma\Omega X\times A\to X$ on X. We show that X is a T^f-space if and only if $G(\Sigma B;A,f,X)=[\Sigma B,X]$ for any space B. There is an example which is a T^f-space for a map $f:A\to X$, but not T-space. We can also obtain, from some properties of T^f-spaces, that for any $x\in\pi_n(S^2),\ \alpha\in\pi_k(S^2),\ [x,\alpha]=0$ for all $n\geq 3,\ k\geq 1$. It is known [16] that if X dominates A and X is a T-space, then A is a T-space. This fact can be generalized as follows. If X is a T^i-space for a map $i:A\to X$ and $i:A\to X$ has a left homotopy inverse $r:X\to A$, then A is a T-space. Moreover, let $p_k:E_k\to X$ be a principal fibration induced by $k:X\to X'$ from $\epsilon:PX'\to X'$. Let $F:\Sigma\Omega X\times A\to X$ be a T^f-structure on X. When can we have a $T^{\bar f}$-structure $\bar F:\Sigma\Omega E_k\times E_l\to E_k$ on E_k such that $p_k\bar F\sim F(\Sigma\Omega p_k\times p_l):E_k\times E_l\to X$? We can obtain an answer of the above question as follows. If X is a T^f-space with T^f-structure $F:\Sigma\Omega X\times A\to X$ and X' is a $T^{f'}$-space with $T^{f'}$-structure $F':\Sigma\Omega X'\times A'\to X'$ such that $kF\sim F'(\Sigma\Omega k\times l):\Sigma\Omega X\times A\to X'$, then there exists a $T^{\bar f}$-structure $\bar F:\Sigma\Omega E_k\times E_l\to E_k$ on E_k such that $p_k\bar F\sim F(\Sigma\Omega p_k\times p_l):\Sigma\Omega E_k\times E_l\to X$. As a corollary, we can obtain a sufficient condition to be E_k a T-space when X and X' are T-spaces.

On the other hand, we introduce a dual one of the above concept, co-T^g-space for a map $g:X\to A$ as a space X having the property that $e':X\to\Omega\Sigma X$ is a g-cocyclic, that is, there is a co-T^g-structure $\theta:X\to\Omega\Sigma X\vee A$. We show that X is a co-T^g-space if and only if $DG(X,g,A;\Omega B)=[X,\Omega B]$ for any space B. It is known [16] that if X dominates A and X is a co-T-space, then A is a co-T-space. This fact can be generalized as follows. If X is a co-T^r-space for a map $r:X\to A$ and $r:X\to A$ has a right homotopy inverse $i:A\to X$, then

A is a co-T-space. Moreover, let $i_r : X \to C_r$ be a principal cofibration induced by $r : X' \to X$ from $\iota : X' \to cX'$. Let $\theta : X \to \Omega\Sigma X \vee A$ be a co-T^g-structure on X. When can we have a co-$T^{\bar{g}}$-structure on C_r such that $(\Omega\Sigma i_r \vee i_s)\theta \sim \bar{\theta} i_r : X \to \Omega\Sigma C_r \vee C_s$? Then we show that if X is a co-T^g-space with co-T^g-structure $\theta : X \to \Omega\Sigma X \vee A$ and X' is a co-$T^{g'}$-space with co-$T^{g'}$-structure $\theta' : X' \to \Omega\Sigma X' \vee A'$ such that $(\Omega\Sigma r \vee s)\theta' \sim \theta r : X' \to \Omega\Sigma X \vee A$, then there exists a co-$T^{\bar{g}}$-structure $\bar{\theta} : C_r \to \Omega\Sigma C_r \vee C_s$ on C_r such that $(\Omega\Sigma i_r \vee i_s)\theta \sim \bar{\theta} i_r : X \to \Omega\Sigma C_r \vee C_s$. As a corollary, we can obtain a sufficient condition to be C_r a co-T-space when X and X' are co-T-spaces.

2. Lifting T^f-structures

Let $f : A \to X$ be a map. A based map $g : B \to X$ is called *f-cyclic* [12] if there is a map $\phi : B \times A \to X$ such that the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\phi} & X \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle \nabla} \\ A \vee B & \xrightarrow{(f\vee g)} & X \vee X \end{array}$$

is homotopy commute, where $j : A \vee B \to A \times B$ is the inclusion and $\nabla : X \vee X \to X$ is the folding map. We call such a map ϕ an *associated map* of a f-cyclic map g. Clearly, g is f-cyclic iff f is g-cyclic. In the case $f = 1_X : X \to X$, a map $g : B \to X$ is called cyclic [15]. We denote the set of all homotopy classes of f-cyclic maps from B to X by $G(B; A, f, X)$ which is called the *Gottlieb set for a map* $f : A \to X$. In the case $f = 1_X : X \to X$, we called such a set $G(B; X, 1, X)$ as the *Gottlieb set*, denoted by $G(B; X)$. In particular, $G(S^n; A, f, X)$ will be denoted by $G_n(A, f, X)$. Gottlieb [3,4] introduced and studied the *evaluation subgroups* $G_n(X) = G_n(X, 1, X)$ of $\pi_n(X)$.

In general, $G(B; X) \subset G(B; A, f, X) \subset [B, X]$ for any map $f : A \to X$ and any space B. However, there is an example [19] such that $G(B, X) \neq G(B; A, f, X) \neq [B, X]$. Thus we know that for any map $f : A \to X$, any cyclic map $g : B \to X$ is f-cyclic, but the converse does not hold.

The next proposition is an immediate consequence from the definition.

Proposition 2.1.
(1) *For any maps* $f : A \to X$, $\theta : C \to A$ *and any space* B, $G(B; A, f, X) \subset G(B; C, f\theta, X)$.
(2) $G(B, X) = G(B; X, 1_X, X) \subset G(B; A, f, X) \subset G(B; A, *, X) = [B, X]$ *for any spaces* X, A *and* B.
(3) $G(B, X) = \cap\{G(B; A, f, X) | f : A \to X$ *is a map and* A *is a space*$\}$.
(4) *If* $h : C \to A$ *is a homotopy equivalence, then* $G(B; A, f, X) = G(B; C, fh, X)$.
(5) *For any map* $k : X \to Y$, $k_\#(G(B; A, f, X)) \subset G(B; A, kf, Y)$.
(6) *For any map* $k : X \to Y$, $k_\#(G(B, X)) \subset G(B; X, k, Y)$.
(7) *For any map* $s : C \to B$, $s^\#(G(B; A, f, X)) \subset G(C; A, f, X)$.

The following proposition says that T-spaces are completely characterized by the Gottlieb sets.

Proposition 2.2. [16] *X is a T-space if and only if* $G(\Sigma B, X) = [\Sigma B, X]$ *for any space* B.

Aguade showed [1] that X is a T-space if and only if $e : \Sigma\Omega X \to X$ is cyclic. Now, for a map $f : A \to X$, we would like to introduce new spaces which can be characterized by the Gottlieb sets for a map $f : A \to X$.

Definition 2.3. *A space X is called a T^f-space for a map $f : A \to X$ if there is a map, T^f-structure on X,* $F : \Sigma\Omega X \times A \to X$ *such that* $Fj \sim \nabla(e \vee f)$, *where* $j : \Sigma\Omega X \vee A \to \Sigma\Omega X \times A$ *is the inclusion.*

Clearly, any T-space means a T^1-space. A space X is called an *H^f-space for a map* $f : A \to X$ [20] if there is a map, H^f-structure on X, $F : X \times A \to X$ such that $Fj \sim \nabla(1 \vee f)$, where $j : X \vee A \to X \times A$ is the inclusion. We can easily show that any H^f-space for a map $f : A \to X$ is a T^f-space for a map $f : A \to X$ for we can take a T^f-structure $F' = F(e \times 1) : \Sigma\Omega X \times A \to X$, where $F : X \times A \to X$ is an H^f-structure on X.

The following theorem says that a T^f-space can be characterized by the Gottlieb sets for a map $f : A \to X$.

Theorem 2.4. *X is a T^f-space for a map $f : A \to X$ if and only if* $G(\Sigma B; A, f, X) = [\Sigma B, X]$ *for any space* B.

Proof. Suppose that X is a T^f-space for a map $f : A \to X$. Then there is a map $F : \Sigma\Omega X \times A \to X$ such that $Fj \sim \nabla(e \vee f)$, where $j : \Sigma\Omega X \vee A \to \Sigma\Omega X \times A$ is the inclusion. Let $g \in [\Sigma B, X]$. Consider the map $G = F(\Sigma\tau(g) \times 1) : \Sigma B \times A \to X$. Then $Gj \sim \nabla(g \vee f)$ and $g \in G(\Sigma B; A, f, X)$. On the other hand, suppose that $G(\Sigma B; A, f, X) =$

$[\Sigma B, X]$ for any space B. Take $B = \Omega X$ and consider the map $e : \Sigma\Omega X \to X$. Since $e \in G(\Sigma\Omega X; A, f, X)$, we know that the map e is f-cyclic and X is a T^f-space for a map $f : A \to X$. □

It is known [16] that if X dominates A and X is a T-space, then A is a T-space. This fact can be generalized as the following corollary.

Corollary 2.5. *Let X be a T^i-space for a map $i : A \to X$.*
(1) *If $i : A \to X$ has a left homotopy inverse $r : X \to A$, then A is a T-space.*
(2) *If $i : A \to X$ has a right homotopy inverse $r : X \to A$, then X is a T-space.*

Proof. (1) Let B be any space. It is sufficient to show that $[\Sigma B, A] \subset G(\Sigma B, A)$ for any space B. Since X is a T^i-space for $i : A \to X$, we know, from Theorem 2.4, that $G(\Sigma B; A, i, X) = [\Sigma B, X]$. Thus we have, from Proposition 2.1(5), that $[\Sigma B, A] = r_*[\Sigma B, X] = r_*(G(\Sigma B; A, i, X)) \subset G(\Sigma B; A, ri, A) = G(\Sigma B, A, 1, A) = G(\Sigma B, A)$. Thus A is a T-space. (2) We show that $[\Sigma B, X] \subset G(\Sigma B, X)$ for any space B. By Theorem 2.4 and Proposition 2.1(1), we can obtain that $[\Sigma B, X] = G(\Sigma B; A, i, X) \subset G(\Sigma B; X, ir, X) = G(\Sigma B; X, 1, X) = G(\Sigma B, X)$. Thus we know, from Proposition 2.2, that X is a T-space. □

From Proposition 2.1(2),(3), Proposition 2.2 and Theorem 2.4, we have the following corollary.

Corollary 2.6. *X is a T-space if and only if for any space A and any map $f : A \to X$, X is a T^f-space for a map $f : A \to X$.*

Definition 2.7. *For a map $f : A \to X$, $P(\Sigma B; A, f, X) = \{\alpha \in [\Sigma B, X] | \; [f_\#(\beta), \alpha] = 0$ for any space C and any map $\beta \in [\Sigma C, A]\}$. A space X is called a GW^f-space for a map $f : A \to X$ if for any space B, $P(\Sigma B; A, f, X) = [\Sigma B, X]$.*

Proposition 2.8. *$G(\Sigma B; A, f, X) \subset P(\Sigma B; A, f, X)$ for any space B.*

Proof. Let $[h] \in G(\Sigma B; A, f, X)$. Then there is a map $H : A \times \Sigma B \to X$ such that $Hj \sim \nabla(f \vee h)$, where $j : A \vee \Sigma B \to A \times \Sigma B$ is the inclusion. Let C be a space and $\beta = [g] \in [\Sigma C, A]$. Then consider the map $F = H(g \times 1) : \Sigma C \times \Sigma B \overset{(g\times 1)}{\to} A \times \Sigma B \overset{H}{\to} X$. Then $Fj' \sim \nabla(fg \vee h)$, where $j' : \Sigma C \vee \Sigma B \to \Sigma C \times \Sigma B$ is the inclusion. Thus we have $[f_\#(\beta), [h]] = 0$ and $[h] \in P(\Sigma B; A, f, X)$. □

Corollary 2.9. *If X is a T^f-space for a map $f : A \to X$, then X is a GW^f-space for $f : A \to X$.*

Consider the natural pairing $\mu : S^3/S^1 = S^2 \times S^3 \to S^3/S^1 = S^2$. Thus we know that the Hopf map $\eta : S^3 \to S^2$ is cyclic. Thus η is e-cyclic and e is η-cyclic, that is, S^2 is a T^η-space. Thus we know that S^2 is a GW^η-space for $\eta : S^3 \to S^2$. On the other hand, it is known [16] that H-spaces and T-spaces are equivalent in the category of spheres. Thus we know that S^2 is not a T-space. Moreover, it is known [14] that $\eta_\# : \pi_n(S^3) \to \pi_n(S^2)$, $\eta_\#(\beta) = \eta \circ \beta$, is an isomorphism for $n \geq 3$. Thus we have the following example.

Example 2.10.
(1) *S^2 is a T^η-space, but not T-space.*
(2) *For any $x \in \pi_n(S^2)$, $\alpha \in \pi_k(S^2)$ $(n \geq 3,\ k \geq 1)$, $[x, \alpha] = 0$.*

Let $f : A \to X$, $f' : A' \to X'$, $l : A \to A'$, $k : X \to X'$ be maps. Then a pair of maps $(k, l) : (X, A) \to (X', A')$ is called *a map from f to f'* if the following diagram is commutative;

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle l}\downarrow & & {\scriptstyle k}\downarrow \\ A' & \xrightarrow{f'} & X'. \end{array}$$

It will be denoted by $(k, l) : f \to f'$.

Given maps $f : A \to X$, $f' : A' \to X'$, let $(k, l) : f \to f'$ be a map from f to f'. Let PX' and PA' be the spaces of paths in X' and A' which begin at $*$ respectively. Let $\epsilon_{X'} : PX' \to X'$ and $\epsilon_{A'} : PA' \to A'$ be the fibrations given by evaluating a path at its end point. Let $p_k : E_k \to X$ be the fibration induced by $k : X \to X'$ from $\epsilon_{X'}$. Let $p_l : E_l \to A$ induced by $l : A \to A'$ from $\epsilon_{A'}$. Then there is a map $\bar{f} : E_l \to E_k$ such that the following diagram is commutative

$$\begin{array}{ccc} E_l & \xrightarrow{\bar{f}} & E_k \\ {\scriptstyle p_l}\downarrow & & {\scriptstyle p_k}\downarrow \\ A & \xrightarrow{f} & X, \end{array}$$

where $E_l = \{(a, \xi) \in A \times PA' | l(a) = \epsilon(\xi)\}$, $E_k = \{(x, \eta) \in X \times PX' | k(x) = \epsilon(\eta)\}$, $\bar{f}(a, \xi) = (f(a), f' \circ \xi)$, $p_k(x, \eta) = x$, $p_l(a, \xi) = a$.

Definition 2.11. *Let X be a T^f-space with T^f-structure $F : \Sigma\Omega X \times A \to X$. A map $(k, l) : f \to f'$ is called a T^f-primitive with respect to*

F if there is an associate map $F' : \Sigma\Omega X' \times A' \to X'$ of $e_{X'}$-cyclic map f' such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} \Sigma\Omega X \times A & \xrightarrow{F} & X \\ {\scriptstyle \Sigma\Omega k \times l}\big\downarrow & & {\scriptstyle k}\big\downarrow \\ \Sigma\Omega X' \times A' & \xrightarrow{F'} & X'. \end{array}$$

The following lemmas are standard.

LEMMA 2.12. *A map $l : C \to X$ can be lifted to a map $C \to E_k$ if and only if $kl \sim *$.*

LEMMA 2.13. [5] *Given maps $g_i : A_i \to E_k,\ i = 1,\ 2$ and $g : A_1 \times A_2 \to E_k$ satisfying $p_k g|_{A_i} \sim p_k g_i,\ i = 1,\ 2,$ then there is a map $h : A_1 \times A_2 \to E_k$ such that $p_k h = p_k g$ and $h|_{A_i} \sim g_i, i = 1,\ 2.$*

THEOREM 2.14. *If X is a T^f-space with T^f-structure $F : \Sigma\Omega X \times A \to X$ and $(k, l) : f \to f'$ is a T^f-primitive with respective to F, then there exists a $T^{\bar{f}}$-structure $\bar{F} : \Sigma\Omega E_k \times E_l \to E_k$ on E_k such that the following diagram is homotopy commutative;*

$$\begin{array}{ccc} \Sigma\Omega E_k \times E_l & \xrightarrow{\bar{F}} & E_k \\ {\scriptstyle \Sigma\Omega p_k \times p_l}\big\downarrow & & {\scriptstyle p_k}\big\downarrow \\ \Sigma\Omega X \times A & \xrightarrow{F} & X. \end{array}$$

Proof. Since $(k, l) : f \to f'$ is a T^f-primitive with respect to F, there is a map $F' : \Sigma\Omega X' \times A' \to X'$ such that $kF \sim F'(\Sigma\Omega k \times l) : \Sigma\Omega X \times A \to X'$. Then $kF(\Sigma\Omega p_k \times p_l) \sim F'(\Sigma\Omega k \times l)(\Sigma\Omega p_k \times p_l) = F'(\Sigma\Omega(k \circ p_k) \times l \circ p_l) \sim F'(* \times *) \sim * : \Sigma\Omega E_k \times E_l \to X'$. From Lemma 2.12, there is a lifting $\tilde{F} : \Sigma\Omega E_k \times E_l \to E_k$ of $F(\Sigma\Omega p_k \times p_l) : \Sigma\Omega E_k \times E_l \to X$, that is, $p_k\tilde{F} = F(\Sigma\Omega p_k \times p_l)$. Then $p_k \circ \tilde{F}|_{\Sigma\Omega E_k} \sim F|_{\Sigma\Omega X} \circ \Sigma\Omega p_k \sim p_k \circ e_{E_k}$ and $p_k \circ \tilde{F}|_{E_l} \sim F|_A \circ p_l \sim f \circ p_l = p_k \circ \bar{f}$. Thus we have, from Lemma 2.13, that there is a map $\bar{F} : \Sigma\Omega E_k \times E_l \to E_k$ such that $p_k\bar{F} = p_k\tilde{F} = F(\Sigma\Omega p_k \times p_l)$ and $\bar{F}|_{\Sigma\Omega E_k} \sim e_{E_k},\ \bar{F}|_{E_l} \sim \bar{f}$. This proves the theorem. □

Taking $f = 1_X,\ f' = 1_{X'}$ and $l = k$, we can obtain the following corollary.

COROLLARY 2.15. *Let X and X' be T-spaces with T^1-structures $E : \Sigma\Omega X \times X \to X$ and $E' : \Sigma\Omega X' \times X' \to X'$ respectively. If $k : X \to X'$ is a map satisfying $kE \sim E'(\Sigma\Omega k \times 1) : \Sigma\Omega X \times X \to X'$, then there is an T^1-structure $\bar{E} : \Sigma\Omega E_k \times E_k \to E_k$ on E_k such that $p_k\bar{E} \sim E(\Sigma\Omega p_k \times p_k) : \Sigma\Omega E_k \times E_k \to X$.*

In 1951, Postnikov [13] introduced the notion of the Postnikov system as follows; *A Postnikov system for X (or homotopy decomposition of X)* $\{X_n, i_n, p_n\}$ consists of a sequence of spaces and maps satisfying (1) $i_n : X \to X_n$ induces an isomorphism $(i_n)_\# : \pi_i(X) \to \pi_i(X_n)$ for $i \leq n$. (2) $p_n : X_n \to X_{n-1}$ is a fibration with fiber $K(\pi_n(X), n)$. (3) $p_n i_n \sim i_{n+1}$. It is well known fact [11] that if X is a 1-connected space having a homotopy type of CW-complex, then there is a Postnikov system $\{X_n, i_n, p_n\}$ for X such that $p_{n+1} : X_{n+1} \to X_n$ is the fibration induced from the path space fibration over $K(\pi_{n+1}(X), n+2)$ by a map $k^{n+2} : X_n \to K(\pi_{n+1}(X), n+2)$.

THEOREM 2.16. *Let A and X be spaces having the homotopy type of 1-connected countable CW-complexes, and $\{A_n, i'_n, p'_n\}$ and $\{X_n, i_n, p_n\}$ be Postnikov systems for A and X respectively. If X is a T^f-space with T^f-structure $F : \Sigma\Omega X \times A \to X$, then there exists a T^{f_n}-structure $F_n : \Sigma\Omega X_n \times A_n \to X_n$ for each stage X_n such that*

$$\begin{array}{ccc} \Sigma\Omega X_n \times A_n & \xrightarrow{F_n} & X_n \\ {\scriptstyle \Sigma\Omega p_n \times p'_n}\downarrow & & \downarrow{\scriptstyle p_n} \\ \Sigma\Omega X_{n-1} \times A_{n-1} & \xrightarrow{F_{n-1}} & X_{n-1}, \end{array}$$

where f_n is an induced map from f, and all the pair of k-invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$ are T^{f_n}-primitive with respect to F_n, where $\tilde{f}_\# : K(\pi_{n+1}(A), n+2) \to K(\pi_{n+1}(X), n+2)$ is the induced map by $f : A \to X$.

Proof. Clearly $\{\Sigma\Omega X_n \times A_n, \Sigma\Omega i_n \times i'_n, \Sigma\Omega p_n \times p'_n\}$ is a Postnikov system for $\Sigma\Omega X \times A$. Then we have, by Kahn's result [7,Theorem 2.2], that there are families of maps $f_n : A_n \to X_n$ and $F_n : \Sigma\Omega X_n \times A_n \to X_n$ such that $p_n f_n = f_{n-1} p'_n$ and $i_n f \sim f_n i'_n$, and $p_n F_n = F_{n-1}(\Sigma\Omega p_n \times p'_n)$ and $i_n F \sim F_n(\Sigma\Omega i_n \times i'_n)$ for $n = 2, 3, \cdots$ respectively, and $k_X^{n+2} f_n \sim \tilde{f}_\# k_A^{n+2} : A_n \to K(\pi_{n+1}(X), n+2)$ and $k_X^{n+2} F_n \sim \tilde{F}_\#(k_{\Sigma\Omega X}^{n+2} \times k_A^{n+2}) : X_n \times A_n \to K(\pi_{n+1}(X), n+2)$, where $k_A^{n+2} : A_n \to K(\pi_{n+1}(A), n+2)$, $k_X^{n+2} : X_n \to K(\pi_{n+1}(X), n+2)$ and $k_{\Sigma\Omega X}^{n+2} : \Sigma\Omega X_n \to K(\pi_{n+1}(\Sigma\Omega X), n+2)$ are k-invariants of A, X and $\Sigma\Omega X$ respectively, $\tilde{f}_\# : K(\pi_{n+1}(A), n+2) \to K(\pi_{n+1}(X), n+2)$ and $\tilde{F}_\# : K(\pi_{n+1}(\Sigma\Omega X), n+2) \times K(\pi_{n+1}(A), n+2) \approx K(\pi_{n+1}(\Sigma\Omega X \times A), n+2) \to K(\pi_{n+1}(X), n+2)$ are the induced maps by $f : A \to X$ and $F : X \times A \to X$ respectively. Since $F|_{\Sigma\Omega X} \sim e$ and $F_n|_{A_n} \sim f_n$, we know, from Kahn's another result [8, Theorem 1.2], that $F_{n|\Sigma\Omega X_n} =$

$(F|_{\Sigma\Omega X})_n \sim e$ and $F_{n|A_n} = (F|_A)_n \sim f_n$. Thus there exists an T^{f_n}-structure $F_n : \Sigma\Omega X_n \times A_n \to X_n$ for each stage X_n such that

$$\begin{array}{ccc} \Sigma\Omega X_n \times A_n & \xrightarrow{F_n} & X_n \\ {\scriptstyle \Sigma\Omega p_n \times p'_n}\big\downarrow & & {\scriptstyle p_n}\big\downarrow \\ \Sigma\Omega X_{n-1} \times A_{n-1} & \xrightarrow{F_{n-1}} & X_{n-1}, \end{array}$$

where f_n is an induced map from f, and all the pair of k-invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$ are T^{f_n}-primitive with respect to F_n, where $\tilde{f}_\# : K(\pi_{n+1}(A), n+2) \to K(\pi_{n+1}(X), n+2)$ is the induced map by $f : A \to X$. □

In fact, the above theorem follows from Theorem 2.14 if we can show that all the pair of k-invariants $(k_X^{n+2}, k_A^{n+2}) : f_n \to \tilde{f}_\#$ are T^{f_n}-primitive with respect to F_n.

We can obtain an equivalent condition for E_k is a $T^{\bar{f}}$-space for $\bar{f}$.

Theorem 2.17. *Let $(k, l) : f \to f'$ be a map. Then E_k is a $T^{\bar{f}}$-space for $\bar{f} : E_l \to E_k$ if and only if there is a map $G : \Sigma\Omega E_k \times E_l \to X$ such that $Gj \sim \nabla(p_k \circ e \vee p_k \circ \bar{f})$ and $kG \sim *$, where $j : \Sigma\Omega E_k \vee E_l \to \Sigma\Omega E_k \times E_l$ is the inclusion.*

Proof. Suppose that E_k is a $T^{\bar{f}}$-space for $\bar{f} : E_l \to E_k$. Then there is a map $\bar{F} : \Sigma\Omega E_k \times E_l \to E_k$ such that $\bar{F}j' \sim \nabla(e \vee \bar{f})$. Let $G = p_k\bar{F} : \Sigma\Omega E_k \times E_l \to X$. Then $Gj \sim \nabla(p_k \circ e \vee p_k \circ \bar{f})$, where $j : \Sigma\Omega E_k \vee E_l \to \Sigma\Omega E_k \times E_l$ is the inclusion. Since G has a lifting $\bar{F}$, by Lemma 2.12, we know that $kG \sim *$. On the other hand, suppose there is a map $G : \Sigma\Omega E_k \times E_l \to X$ such that $Gj \sim \nabla(p_k \circ e \vee p_k \circ \bar{f})$ and $kG \sim *$, where $j : \Sigma\Omega E_k \vee E_l \to \Sigma\Omega E_k \times E_l$ is the inclusion. Since $kG \sim *$, there is a map $H : \Sigma\Omega E_k \times E_l \to E_k$ such that $p_kH \sim G$. For maps $e : \Sigma\Omega E_k \to E_k$ and $\bar{f} : E_l \to E_k$, we can easily know that $p_kH_{|\Sigma\Omega E_k} \sim p_k \circ e_{E_k}$ and $p_kH_{|E_l} \sim p_k \circ \bar{f}$. Thus we have, from Lemma 2.13, that there is a map $\bar{F} : \Sigma\Omega E_k \times E_l \to E_k$ such that $p_k\bar{F} = p_kH$ and $\bar{F}_{|\Sigma\Omega E_k} \sim e$ and $\bar{F}_{E_l} \sim \bar{f}$. Thus we know that E_k is a $T^{\bar{f}}$-space for $\bar{f} : E_l \to E_k$. □

Now we can obtain the converse of Theorem 2.14 under some conditions as follows;

Theorem 2.18. *Suppose that there are maps $s_k : X \to E_k$ and $s_l : A \to E_l$ such that $p_ks_k \sim 1_X$ and $p_ls_l \sim 1_A$. If there exists a $T^{\bar{f}}$-structure $\bar{F} : \Sigma\Omega E_k \times E_l \to E_k$ on E_k such that the following diagram* *is homotopy commutative;*

$$\begin{array}{ccc} \Sigma\Omega E_k \times E_l & \xrightarrow{\bar{F}} & E_k \\ {\scriptstyle \Sigma\Omega p_k \times p_l}\downarrow & & {\scriptstyle p_k}\downarrow \\ \Sigma\Omega X \times A & \xrightarrow{F} & X, \end{array}$$

then X is a T^f-space with T^f-structure $F : \Sigma\Omega X \times A \to X$.

Proof. Since E_k is a $T^{\bar{f}}$-space for $\bar{f} : E_l \to E_k$, there is a map $G : \Sigma\Omega E_k \times E_l \to X$ such that $Gj \sim \nabla(p_k \circ e \vee p_k \circ \bar{f})$ and $kG \sim *$, where $j : \Sigma\Omega E_k \vee E_l \to \Sigma\Omega E_k \times E_l$ is the inclusion. Consider the map $F = G(\Sigma\Omega s_k \times s_l) : \Sigma\Omega X \times A \to X$. Then $Fj' \sim \nabla(e \vee f)$ and $kF(\Sigma\Omega p_k \times p_l) \sim *$, where $j' : \Sigma\Omega X \vee A \to \Sigma\Omega X \times A$ is the inclusion. Thus we know that X is a T^f-space with T^f-structure $F : \Sigma\Omega X \times A \to X$. □

3. Extending co-T^g-structures

Let $g : X \to A$ be a map. A based map $f : X \to B$ is called *g-coclic* [12] if there is a map $\theta : X \to A \vee B$ such that the following diagram is homotopy commutative;

$$\begin{array}{ccc} X & \xrightarrow{\theta} & A \vee B \\ {\scriptstyle \Delta}\downarrow & & {\scriptstyle j}\downarrow \\ X \times X & \xrightarrow{(g\times f)} & A \times B, \end{array}$$

where $j : A \vee B \to A \times B$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. We call such a map θ a *coassociated map* of a g-cocyclic map f.

In the case $g = 1_X : X \to X$, $f : X \to B$ is called *cocyclic* [15]. Clearly any cocyclic map is a g-cocyclic map and also $f : X \to B$ is g-cocyclic iff $g : X \to A$ is f-cocyclic. The *dual Gottlieb set* $DG(X, g, A; B)$ *for a map* $g : X \to A$ is the set of all homotopy classes of g-cocyclic maps from X to B. In the case $g = 1_X : X \to X$, we called such a set $DG(X, 1, X; B)$ as the *dual Gottlieb set*, denoted by $DG(X; B)$, that is, the dual Gottlieb set is exactly same with the dual Gottlieb set for the identity map. In particular, $DG(X, g, A; K(\pi, n))$ will be denoted by $G^n(X, g, A; \pi)$. Haslam [5] introduced and studied the *coevaluation subgroups* $G^n(X; \pi)$ of $H^n(X; \pi)$. $G^n(X; \pi)$ is defined to be the set of all homotopy classes of cocyclic maps from X to $K(\pi, n)$.

In general, $DG(X;B) \subset DG(X,g,A;B) \subset [X,B]$ for any map $g : X \to B$ and any space B. However, there is an example in [18] such that $DG(X,B) \neq DG(X,g,A;B) \neq [X,B]$.

The next proposition is an immediate consequence from the definition.

Proposition 3.1.
(1) *For any maps* $g : X \to A$, $h : A \to B$ *and any space* C, $DG(X,g,A;C) \subset DG(X,hg,B;C)$.
(2) $DG(X,B) = DG(X,1_X,X;B) \subset DG(X,g,A;B) \subset DG(X,*,A;B) = [X,B]$ *for any spaces* X, A *and* B.
(3) $DG(X,B) = \cap\{DG(X,g,A;B) | g : X \to A$ *is a map and* A *is a space*$\}$.
(4) *If* $h : A \to B$ *is a homotopy equivalence, then* $DG(X,g,A;C) = DG(X,hg,B;c)$.
(5) *For any map* $k : Y \to X$, $k^{\#}(DG(X,g,A;B)) \subset DG(Y,gk,A;B)$.
(6) *For any map* $k : Y \to X$, $k^{\#}(DG(X;B)) \subset DG(Y,k,X;B)$.
(7) *For any map* $s : B \to C$, $s_{\#}(DG(X,g,A;B)) \subset DG(X,g,A;C)$.

It is well known [5] that $G^n(X;\pi)$ is a subgroup of $H^n(X;\pi)$. Moreover, it is also shown [10] that if B is an H-group, then $DG(X,B)$ is a subgroup of $[X,B]$.

But we do not know whether $DG(X,g,A;B)$ is a group.

A space X is called a *co-T-space* [16] if $e' : X \to \Omega\Sigma X$ is cocyclic. The following proposition says that co-T-spaces are completely characterized by the dual Gottlieb sets.

Proposition 3.2. [16] *X is a co-T-space if and only if* $DG(X,\Omega B) = [X,\Omega B]$ *for any space* B.

Now, for a map $g : X \to A$, we would like to introduce new spaces which can be characterized by the dual Gottlieb sets for a map $g : X \to A$.

Definition 3.3. *A space* X *is called a co-T^g-space for a map* $g : X \to A$ *if there is a map, a co-T^g-structure,* $\theta : X \to \Omega\Sigma X \vee A$ *such that* $j\theta \sim (e' \times g)\Delta$, *where* $j : \Omega\Sigma X \vee A \to \Omega\Sigma X \times A$ *is the inclusion and* $\Delta : X \to X \times X$ *is the diagonal map.*

The following proposition says that co-T^g-spaces are completely characterized by the dual Gottlieb sets for a map $g : X \to A$.

Proposition 3.4. [18] *X is a co-T^g-space for a map* $g : X \to A$ *if and only if* $DG(X,g,A;\Omega B) = [X,\Omega B]$ *for any space* B.

It is clear, from Proposition 3.1(2) and the above propositions, that any co-T-space is a co-T^g-space for any map $g : X \to A$. It is known [18] that if X dominates A and X is a co-T-space, then A is a co-T-space. This fact can be generalized as follows;

COROLLARY 3.5. *Let X be a co-T^r-space for a map $r : X \to A$.*
(1) *If $r : X \to A$ has a right homotopy inverse $i : A \to X$, then A is a co-T-space.*
(2) *If $r : X \to A$ has a left homotopy inverse $i : A \to X$, then X is a co-T-space.*

Proof. (1) Let B be any space. It is sufficient to show that $[A, \Omega B] \subset DG(A, \Omega B)$. Since X is a co-T^r-space for a map $r : X \to A$, we have that $DG(X, r, A; \Omega B) = [X; \Omega B]$. Thus we know, from Proposition 3.1(5), that $[A, \Omega B] = i^{\#}[X, \Omega B] = i^{\#} DG(X, r, A; \Omega B) \subset DG(A, ri, A; \Omega B) = DG(A, 1, A; \Omega B) = DG(A, \Omega B)$. (2) For any space B, we can obtain, from Proposition 3.4 and Proposition 3.1(1), that $[X, \Omega B] = DG(X, r, A; \Omega B) \subset DG(X, ir, X; \Omega B) = DG(X, 1, X; \Omega B) = DG(X, \Omega B)$. □

Given maps $g : X \to A$, $g' : X' \to A'$, let $(s, r) : g' \to g$ be a map from g' to g, that is, the following diagram is commutative;

$$\begin{array}{ccc} X' & \xrightarrow{g'} & A' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{g} & A. \end{array}$$

It is a well known fact that $Y \xrightarrow{\iota} cY \to \Sigma Y$ is a cofibration, where $\iota(y) = [y, 1]$. Let $i_r : X \to C_r$ be the cofibration induced by $r : X' \to X$ from $\iota_{X'} : X' \to cX'$. Let $i_s : A \to C_s$ be the cofibration induced by $s : A' \to A$ from $\iota_{A'} : A' \to cA'$. Then there is a map $\bar{g} : C_t \to C_s$ such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{g} & A \\ {\scriptstyle i_r}\downarrow & & \downarrow{\scriptstyle i_s} \\ C_r & \xrightarrow{\bar{g}} & C_s, \end{array}$$

where $C_t = cX' \amalg X/[x', 1] \sim t(x')$, and $C_s = cA' \amalg A/[a', 1] \sim s(a')$, $\bar{g} : C_t \to C_s$ is given by $\bar{g}([x', t]) = [g'(x'), t]$ if $[x', t] \in cX'$ and $\bar{g}(x) = g(x)$ if $x \in X$, $i_r(x) = x$, $i_s(a) = a$.

DEFINITION 3.6. *Let X be a co-T^g-space with co-T^g-structure $\theta : X \to \Omega\Sigma X \vee A$. Then a map $(s, r) : g' \to g$ is called a co-T^g-primitive* *with respect to* $\theta : X \to \Omega\Sigma X \vee A$ *if there is a coassociate map* $\theta' : X' \to \Omega\Sigma X' \vee A'$ *of* e'*-cocyclic map* g' *such that the following diagram is homotopy commutative;*

$$\begin{array}{ccc} X' & \xrightarrow{\theta'} & \Omega\Sigma X' \vee A' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \Omega\Sigma r \vee s} \\ X & \xrightarrow{\theta} & \Omega\Sigma X \vee A. \end{array}$$

The following lemmas are standard.

LEMMA 3.7. *Let* $f : X \to B$ *be a map. Then there is a map* $h : C_r \to B$ *such that* $hi_r = f$ *if and only if* $fr \sim *$.

LEMMA 3.8. [17] *Let* $g_t : C_r \to B_t (t = 1, 2)$ *and* $g : C_r \to B_1 \vee B_2$ *a map such that* $p_t j g i_k \sim g_t i_r (t = 1, 2)$*, where* $j : B_1 \vee B_2 \to B_1 \times B_2$ *is the inclusion and* $p_t : B_1 \times B_2 \to B_t, t = 1, 2$ *are projections. Then there is a map* $h : C_r \to B_1 \vee B_2$ *such that* $gi_r = hi_r$ *and* $p_t j' h \sim g_t (t = 1, 2)$*, where* $j' : B_1 \vee B_2 \to B_1 \times B_2$ *is the inclusion.*

THEOREM 3.9. *If* X *is a co-*T^g*-space with co-*T^g*-structure* $\theta : X \to \Omega\Sigma X \vee A$ *and* $(s, r) : g' \to g$ *is a co-*T^g*-primitive with respect to* θ*, then there exists a co-*$T^{\bar{g}}$*-structure* $\bar{\theta} : C_r \to \Omega\Sigma C_r \vee C_s$ *on* C_r *satisfying commutative diagram*

$$\begin{array}{ccc} C_r & \xrightarrow{\bar{\theta}} & \Omega\Sigma C_r \vee C_s \\ {\scriptstyle i_r}\uparrow & & \uparrow{\scriptstyle \Omega\Sigma i_r \vee i_s} \\ X & \xrightarrow{\theta} & \Omega\Sigma X \vee A. \end{array}$$

Proof. Since $(s, r) : g' \to g$ is a co-T^g-primitive with respect to θ, then there is a map $\theta' : X' \to \Omega\Sigma X' \vee A'$ satisfying commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\theta'} & \Omega\Sigma X' \vee A' \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \Omega\Sigma r \vee s} \\ X & \xrightarrow{\theta} & \Omega\Sigma X \vee A. \end{array}$$

Then we have that $(\Omega\Sigma i_r \vee i_s)\theta r \sim (\Omega\Sigma i_r \vee i_s)(\Omega\Sigma r \vee s)\theta' \sim (\Omega\Sigma(i_r \circ r) \vee i_s \circ s)\theta \sim *$. Thus we know, from Lemma 3.7, that there is a map $\tilde{\theta} : C_r \to \Omega\Sigma C_r \vee C_s$ such that $\tilde{\theta} i_r = (\Omega\Sigma i_r \vee i_s)\theta$. Then $p_1 j \tilde{\theta} i_r = p_1 j (\Omega\Sigma i_r \vee i_s)\theta \sim p_1(\Omega\Sigma i_r \times i_s)(e' \times g)\Delta \sim e'_{C_r} \circ i_r$ and $p_2 j \tilde{\theta} i_r \sim p_2(\Omega\Sigma i_r \times i_s)(e' \times g)\Delta \sim i_s \circ g = \bar{g} \circ i_r$. Thus we have, from Lemma 3.8, that there is a map $\bar{\theta} : C_r \to \Omega\Sigma C_r \vee C_s$ such that $\bar{\theta} i_r = \tilde{\theta} i_r = (\Omega\Sigma i_r \vee i_s)\theta$

and $p_1 j\bar{\theta} \sim e'$, $p_2 j\bar{\theta} \sim \bar{g}$, where $j : \Omega\Sigma C_r \vee C_s \to \Omega\Sigma C_r \times C_s$ is the inclusion. □

Taking $g = 1_X$, $g' = 1_{X'}$ and $s = r$, we can get the following corollary.

COROLLARY 3.10. *Let X and X' be co-T-spaces with co-T^1-structures $\theta : X \to \Omega\Sigma X \vee X$ and $\theta' : X' \to \Omega\Sigma X' \vee X'$ respectively. If $r : X' \to X$ is a map satisfying $(\Omega\Sigma r \vee r)\theta' \sim \theta r : X' \to \Omega\Sigma X \vee X$, then there is a co-T^1-structure $\bar{\theta} : C_r \to \Omega\Sigma C_r \vee C_r$ on C_r such that $(\Omega\Sigma i_r \vee i_r)\theta \sim \bar{\theta} i_r : X \to \Omega\Sigma C_r \vee C_r$.*

In 1959, Eckmann and Hilton [2] introduced a dual concept of Postnikov system as follows; *A homology decomposition of X* consists of a sequence of spaces and maps $\{X_n, q_n, i_n\}$ satisfying (1) $q_n : X_n \to X$ induces an isomorphism $(q_n)_* : H_i(X_n) \to H_i(X)$ for $i \le n$. (2) $i_n : X_n \to X_{n+1}$ is a cofibration with cofiber $M(H_{n+1}(X), n)$(a Moore space of type $(H_{n+1}(X), n)$). (3) $q_n \sim q_{n+1} \circ i_n$. It is known by [6] that if X be a 1-connected space having the homotopy type of CW complex, then there is a homology decomposition $\{X_n, q_n, i_n\}$ of X such that $i_n : X_n \to X_{n+1}$ is the principal cofibration induced from $\iota : M(H_{n+1}(X), n) \to cM(H_{n+1}(X), n)$ by a map $r : M(H_{n+1}(X), n) \to X_n$ which is called the dual Postnikov invariants.

From Theorem 3.9, we have the following corollary.

COROLLARY 3.11. *Let X and A be spaces having the homotopy type of 1-connected countable CW-complexes, and $\{X_n, q_n, i_n\}$ and $\{A_n, q'_n, i'_n\}$ be homology decompositions for X and A respectively. If X is a co-T^g-space with co-T^g-structure $\theta : X \to \Omega\Sigma X \vee A$ and for each $n \ge 2$, the pair of r daul invariants $(r^n_A, r^n_X) : \tilde{g}_* \to g_n$ are co-T^{g_n}-primitive with respect to $\theta_n : X_n \to \Omega\Sigma X_n \vee A_n$, where $\tilde{g}_* : M(H_{n+1}(X), n) \to M(H_{n+1}(A), n)$ and g_n are induced maps from $g : X \to A$, then there exists a co-$T^{g_{n+1}}$-structure on X_{n+1} such that $(i'_{n+1}, i_{n+1}) : g_n \to g_{n+1}$ is a co-$T^{g_{n+1}}$-primitive with respect to $\theta_n : X_n \to X_n \vee A_n$.*

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