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ON A LOTKA-VOLTERRA TYPE SIMPLE FOOD-CHAIN MODEL

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Abstract. In this paper, we study a Lotka-Volterra type simple food chain model. We investigate the positive coexistence of the steady states to the model and give some results for the extinction of species under certain assumptions which can be interpreted as *Domino effect* and *Biological control*. The methods of a decoupling operator and the fixed point index theory on a positive cone are used as well as the comparison argument. Numerical evidences for our results also are provided.

1. Introduction

Of concern is to study the following Lotka-Volterra type simple food chain model:

$$
(1.1)\quad \begin{cases} u_t - d_1\Delta u = u(a_1 - b_{11}u - b_{12}v) \\ v_t - d_2\Delta v = v(a_2 + b_{21}u - b_{22}v - b_{23}w) \\ w_t - d_3\Delta w = w(a_3 + b_{32}v - b_{33}w) & \text{in } \Omega \times [0,T), \\ (u,v,w) = (0,0,0) & \text{on } \partial\Omega \times [0,T), \\ (u(x,0), v(x,0), w(x,0)) = (\widetilde{u}(x), \widetilde{v}(x), \widetilde{w}(x)) & \text{in } \overline{\Omega} \end{cases}
$$

in a bounded domain Ω of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $T \in (0,\infty)$. Here the constants d_i, b_{ij} with $(i,j) \neq (1,3),(3,1)$ are positive and a_i may change the signs for $i,j = 1,2,3$. u_i for $i = 1,2,3$ represent the densities of three interacting species. The model (1.1) describes predator-prey interactions among three species, more precisely, species v is a predator only on u and w preys only on v. This is the so-called simple food-chain model. The domain with homogeneous Dirichlet boundary conditions we consider indicates the region with a hostile environment on the boundary.

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Food-chain models have been studied on both spatially homogeneous situation ([5]) and spatially inhomogeneous case ([3, 12]) for last two decades. It is known in literatures that the dynamics of three species model is much more complicated than that of the two species model relatively. (See [3, 4, 5, 9, 12] and the references therein.) Even for ODE system, the dynamics for the behavior of positive solutions is very complicated ([5]). Other works for three-species model with predator-prey interacting type with diffusions can be found in [7, 10].

Our studies mainly focus on the existence of positive solutions to the steady states of system (1.1) and the extinction of species under certain conditions. First, we investigate the positive coexistence of the steady-states to system (1.1):

$$\begin{cases} -d_1\Delta u = u(a_1 - b_{11}u - b_{12}v) & \\ -d_2\Delta v = v(a_2 + b_{21}u - b_{22}v - b_{23}w) & \\ -d_3\Delta w = w(a_3 + b_{32}v - b_{33}w) & \text{in } \Omega, \\ (u, v, w) = (0, 0, 0) & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We give sufficient and necessary conditions for the existence of positive solutions to system (1.2). The method employed is a decoupling operator and the fixed point index theory on a positive cone. The role of diffusions for the positive coexistence is also discussed in view of our coexistence theorem.

Biological control is man's use of a specially chosen living organism to control a particular pest ([8]). Such organism could be a predator, parasite or disease that attacks certain harmful insect. One of features of simple food-chain model is the so-called *Domino effect*, namely, if one species dies out, then all the other species at higher level also die out. In literature, such biological features can be occurred on the simple spatial food chain model with ratio-dependent Michaelis-Menten functional response, i.e., ratio-dependence plays an important role not only in producing the extinction of prey species and so the collapse of the system, but also in making certain biological processes for spatial homogeneous case ([8]).

In this paper, we show that such features of biological aspects could be observed on the classical Lotka-Volttera model which has spatial inhomogeneity of species under the boundary with a hostile environment. To achieve this goal, we will study the extinction of species under certain assumptions using the comparison method. This results could be interpreted as some biological phenomena, *Biological control* and *Domino effect* on the time-dependent system (1.1).

This paper is organized as follows. In Section 2, we present some known results which are useful in the later sections. In Section 3, we give sufficient and necessary conditions for the existence of positive solutions of (1.2) and discuss the role of diffusions in the existence and the nonexistence of positive solutions of the steady state to the model. We obtain the extinction results in Section 4 and give some biological aspects for our results. Finally the numerical examples for our results are illustrated.

2. Preparations

In this section, we state some known results which are useful through this article.

Eigenvalue Problem. Denote the principal eigenvalue of the problem:

$$\begin{cases} d\Delta\phi + a(x)\phi = \lambda\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

by $\lambda_1(d\Delta + a(x))$, where $a(x) \in L^\infty(\Omega)$ and $d > 0$. Note that $\lambda_1(d\Delta + a(x))$ is decreasing with respect to d and increasing with respect to $a(x)$.

The following two lemmas are found in [11].

Lemma 2.1. *Let $a(x) \in C^\alpha(\overline{\Omega})$ and $u \geq 0$, $u \not\equiv 0$ in Ω with $u = 0$ on $\partial\Omega$.*

(i) If $0 \not\equiv (\Delta + a(x))u \geq 0$, then $\lambda_1(\Delta + a(x)) > 0$.

(ii) If $0 \not\equiv (\Delta + a(x))u \leq 0$, then $\lambda_1(\Delta + a(x)) < 0$.

(iii) If $(\Delta + a(x))u \equiv 0$, then $\lambda_1(\Delta + a(x)) = 0$.

Lemma 2.2. *Let $a(x) \in L^\infty(\Omega)$ and M be a positive constant such that $a(x) + M > 0$ for all $x \in \overline{\Omega}$.*

(i) If $\lambda_1(d\Delta + a(x)) > 0$, then $r[\frac{1}{d}(-\Delta + M)^{-1}(a(x) + M)] > 1$.

(ii) If $\lambda_1(d\Delta + a(x)) < 0$, then $r[\frac{1}{d}(-\Delta + M)^{-1}(a(x) + M)] < 1$.

Here $r(T)$ represents the spectral radius of a linear map $T : E \to E$ on a Banach space.

Scalar Equation. For the following scalar equation:

$$\begin{cases} -d\Delta u = u(a(x) - b(x)u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $a(x), b(x) \in C(\Omega)$, the next theorem is well-known. One can refer [6, 13].

THEOREM 2.3. *(i) If $\lambda_1(d\Delta + a(x)) \leq 0$, then (2.1) has no positive solutions. Moreover, the trivial solution is globally asymptotically stable.*

(ii) If $\lambda_1(d\Delta + a(x)) > 0$, then (2.1) has a unique positive solution which is globally asymptotically stable. In this case, the trivial solution is unstable.

Fixed Point Index Let E be a Banach space and let A be a Fréchet differentiable compact operator in E which maps a closed convex set W into itself. In [1], a fixed point index $index_W(A, U)$ can be defined on each open subset U of W where boundary contains no fixed points of A. The index is defined through the Leray-Schauder degree

$$index_W(A, U) = deg_X(I - A \circ \gamma, \gamma^{-1}(u), 0),$$

where I is the identity map in E and $\gamma : E \to W$ is a retraction of W. Due to the fact that the positive cone P is a retract of the Banach space E, it is possible to define a fixed point index for compact maps A which are defined in the positive cone P. This fixed point index is equivalent to the Leray-Schauder degree.

The following lemma can be found in [1].

LEMMA 2.4. *Let $f : \overline{P}_\rho \to P$ be a compact map such that $f(0) = 0$ where $\overline{P}_\rho :=$ closure of $\{u \in \overline{P} : ||u|| < \rho\}$. Suppose that f has a right derivative $f'_+(0)$ at zero such that 1 is not an eigenvalue of $f'_+(0)$ corresponding to a positive eigenfunction. Then there exists a constant $\sigma_0 \in (0, \rho]$ such that for every $\sigma \in (0, \sigma_0]$,*
(i) $index_P(f, P_\sigma) = 1$ if $f'_+(0)$ has no positive eigenfunction corresponding to an eigenvalue greater than one;
(ii) $index_P(f, P_\sigma) = 0$ if $f'_+(0)$ has a positive eigenfunction corresponding to an eigenvalue greater than one.

3. Coexistence of steady-states

In this section, we give sufficient and necessary conditions for the positive coexistence of (1.2).

In view of Theorem 2.3, three semi-trivial solutions u_0, v_0, w_0 of (1.2) could exist. More precisely, when other two species v and w are absent, u_0 denote the unique positive solution of the equation, $-d_1\Delta u = u(a_1 - b_{11}u)$ in Ω and $u = 0$ on $\partial\Omega$ if $a_1/d_1 > \lambda_1(-\Delta)$. Similarly, in case that other two species are absent, v_0 and w_0 are the unique positive solutions in the second and third equations if $a_2/d_2 > \lambda_1(-\Delta)$ and $a_3/d_3 > \lambda_1(-\Delta)$ hold, respectively.

As another consequence of Theorem 2.3, the following operator $S : C^{2,\alpha}(\Omega) \to C^{2,\alpha}(\Omega)$ is well-defined for some $0 < \alpha < 1$:

$$Sv := \begin{cases} u_v & \text{if } a_1/d_1 > \lambda_1(-\Delta + (b_{12}/d_1)v), \\ 0 & \text{otherwise,} \end{cases}$$

where u_v is the unique positive solution of the equation, $-d_1\Delta u = u(a_1 - b_{11}u - b_{12}v)$ in Ω and $u = 0$ on $\partial\Omega$ when $a_1/d_1 > \lambda_1(-\Delta + (b_{12}/d_1)v)$. Similarly, we can define another operator $T : C^{2,\alpha}(\Omega) \to C^{2,\alpha}(\Omega)$ by

$$Tv := \begin{cases} w_v & \text{if } a_3/d_3 > \lambda_1(-\Delta - (b_{32}/d_3)v) > 0, \\ 0 & \text{otherwise} \end{cases}$$

from the equation, $-d_3\Delta w = w(a_3 + b_{32}v - b_{33}w)$ in Ω and $w = 0$ on $\partial\Omega$. For these operators S and T, we have the following lemma from the result in [6].

LEMMA 3.1. *(i) The operators S and T are continuous in sense of $C^{2,\alpha}(\Omega) \to C^{2,\alpha}(\Omega)$ for some $0 < \alpha < 1$.*

(ii) If $v_1 \geq v_2 \not\equiv v_1$, then either $u_{v_1} < u_{v_2}$ or $u_{v_1} \equiv u_{v_2} \equiv 0$. With respect to T, either $w_{v_1} > w_{v_2}$ or $w_{v_1} \equiv w_{v_2} \equiv 0$.

When we use the fixed point index, one of the important things is the estimation of an *a priori* bound of solutions to the system. For the equation (1.2), one can easily obtain the following lemma by using the strong maximum principle.

LEMMA 3.2. *The nonnegative solution (u, v, w) of (1.2) has an a priori bound;*

$$u(x) \leq Q_1,\ v(x) \leq Q_2,\ w(x) \leq Q_3,$$

where

$$Q_1 := \max\{\frac{a_1}{b_{11}}, 0\}, \quad Q_2 := \max\{\frac{a_1b_{21} + a_2b_{11}}{b_{11}b_{22}}, 0\}$$

and

$$Q_3 := \max\{\frac{a_1b_{21}b_{32} + a_2b_{11}b_{32} + a_3b_{11}b_{22}}{b_{11}b_{22}b_{33}}, 0\}.$$

Now by the definition of operators S and T, replacing u, w by Sv and Tv, respectively, in the second equation of (1.2), we have

$$\text{(3.1)} \qquad \begin{cases} -d_2\Delta v = v(a_2 + b_{21}Sv - b_{22}v - b_{23}Tv) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

If we define the positive compact operator A by

$$Av := \frac{1}{d_2}(-\Delta + M)^{-1}((a_2 + b_{21}Sv - b_{22}v - b_{23}Tv + Md_2)v),$$

where $M > \frac{1}{d_2}(-a_2 + b_{22}(Q_2 + 1) + b_{23}Q_3)$ for all $v \in [0, Q_2 + 1]$, then the fixed point of A is a solution of (3.1).

We use the following notations to calculate the fixed point index of A.

(i) $E := C^2(\overline{\Omega})$,

(ii) $P := \{v \in E : 0 \leq v(x), x \in \overline{\Omega}\}$,

(iii) $P_\rho := \{v \in P : v(x) \leq \rho\}$, $\rho := Q_2 + 1$.

LEMMA 3.3. *$index_P(A, P_\rho) = 1$.*

Proof. Note that ∂P_ρ does not contain the fixed point of A and so $index_P(A, P_\rho)$ is well-defined. Define an operator $A_\mu : \overline{P_\rho} \to P$ by

$$A_\mu v := \frac{1}{d_2}(-\Delta + M)^{-1}[(\mu(a_2 + b_{21}Sv - b_{22}v - b_{23}Tv) + Md_2)v],$$

where $\mu \in [0, 1]$. Then $A = A_1$ and a fixed point v of A_μ if and only if v is a solution of the equation:

$$\begin{cases} -d_2\Delta v = \mu v(a_2 + b_{21}Sv - b_{22}v - b_{23}Tv) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

for each μ. One can easily verify that every fixed point v of A_μ has an *a priori* bound(i.e., $v(x) \leq Q_2$ in $\overline{\Omega}$), and so A_μ can have a fixed point in $P_\rho \setminus \partial P_\rho$. Using the homotopy invariance property of index, $index_P(A, P_\rho) = index_P(A_1, P_\rho) = index_P(A_0, P_\rho)$. Since (3.2) has only the trivial solution 0 when $\mu = 0$ and $r(A_0'(0)) < 1$(this can be checked simply by the simple calculation and Lemma 2.2 (ii)), one may conclude $index_P(A_0, P_\rho) = 1$ by Lemma 2.4 (i). $\square$

LEMMA 3.4. *Assume $\min\{a_1/d_1, a_3/d_3\} > \lambda_1(-\Delta)$ and $a_2/d_2 > \lambda_1(-\Delta - (b_{21}/d_2)u_0 + (b_{23}/d_2)w_0)$. Then $index_P(A, 0) = 0$.*

Proof. By the calculation, $A'(0) = \frac{1}{d_2}(-\Delta + M)^{-1}(a_2 + b_{21}u_0 - b_{23}w_0 + Md_2)$. Suppose that 1 is an eigenvalue of $A'(0)$ corresponding to a positive eigenfunction ϕ. Then we have

$$\begin{cases} -d_2\Delta\phi = \phi(a_2 + b_{21}u_0 - b_{23}w_0) & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 2.1 (iii), $\lambda_1(d_2\Delta + a_2 + b_{21}u_0 - b_{23}w_0) = 0$, which contradicts the assumption, and so 1 is not an eigenvalue of $A'(0)$ corresponding to a positive eigenfunction. Using Lemma 2.2 (i), we get $r(A'(0)) > 1$, and thus one may conclude that $A'(0)$ has a positive eigenfunction corresponding to an eigenvalue greater than one by Krein-Rutman theorem. By Lemma 2.4 (ii), there is a $\sigma_0 \in (0, \rho]$ such that $index_P(A, P_\rho) = 0$ for every $\sigma \in (0, \sigma_0]$. Since 0 is isolated, there is a $\delta > 0$ such that 0 is the only fixed point of A in P_δ. Finally, by taking $\sigma < \min\{\sigma_0, \delta\}$, we have $index_P(A, 0) = index_P(A, P_\sigma) = 0$. □

In the proof of above lemma, if $w_0 \equiv 0$, then the following holds.

COROLLARY 3.5. *Assume* $a_3/d_3 \le \lambda_1(-\Delta) < a_1/d_1$. *If* $a_2/d_2 > \lambda_1(-\Delta - (b_{21}/d_2)u_0)$, *then* $index_P(A, 0) = 0$.

By using Lemma 3.3-3.4 and corollary 3.5, we have the following.

LEMMA 3.6. *Assume* $a_1/d_1 > \lambda_1(-\Delta)$.
Case 1 : $a_3/d_3 \le \lambda_1(-\Delta)$.
If $a_2/d_2 > \lambda_1(-\Delta - (b_{21}/d_2)u_0)$, *then A has a positive fixed point.*
Case 2 : $a_3/d_3 > \lambda_1(-\Delta)$.
If $a_2/d_2 > \lambda_1(-\Delta - (b_{21}/d_2)u_0 + (b_{23}/d_2)w_0)$, *then A has a positive fixed point.*

REMARK 3.7. (i) In Case 1 of the above lemma, there always exists a positive fixed point of A when $a_2/d_2 > \lambda_1(-\Delta)$ since $\lambda_1(d_2\Delta + a_2 + b_{21}u_0) > \lambda_1(d_2\Delta + a_2) > 0$.

(ii) Whether the second species v can survive alone or not, A has a positive fixed point if $a_2/d_2 > \lambda_1(-\Delta - (b_{21}/d_2)u_0)$ for Case 1 and $a_2/d_2 > \lambda_1(-\Delta - (b_{21}/d_2)u_0 + (b_{23}/d_2)w_0)$ for Case 2 are satisfied.

Now we have the following coexistence theorem.

THEOREM 3.8. *Assume* $a_1/d_1 > \lambda_1(-\Delta)$.
Case 1 : $\min\{a_2/d_2, (a_1 - b_{12}Q_2)/d_1\} > \lambda_1(-\Delta)$, *where Q_2 is a priori bound for solution v as defined in Lemma 3.2.*

If $a_3/d_3 > \lambda_1(-\Delta)$, *then (1.2) always has at least one positive coexistence. In the case of* $a_3/d_3 \le \lambda_1(-\Delta)$, *there exists a positive coexistence if* $a_3/d_3 > \lambda_1(-\Delta - (b_{32}/d_3)v_0)$.

Case 2 : $a_2/d_2 \le \lambda_1(-\Delta) < a_3/d_3$.
(1.2) has a positive coexistence if and only if $a_2/d_2 > \lambda_1(-\Delta - (b_{21}/d_2)u_0)$.

Case 3 : $\max\{a_2/d_2, a_3/d_3\} \le \lambda_1(-\Delta)$.
If $a_2/d_2 > \lambda_1(-\Delta - (b_{21}/d_2)u_0)$, *then (1.2) has a positive coexistence for sufficiently large* b_{32}. *In this case, the condition* $a_2/d_2 > \lambda_1(-\Delta - (b_{21}/d_2)u_0)$ *is necessary for the positive coexistence.*

Proof. In all cases, there is a positive fixed point v of A by Lemma 3.6, and so we need to show $Sv := u_v > 0$ and $Tv := w_v > 0$ to finish the proof.

Case 1 : Since $v \le Q_2$, we have $\lambda_1(d_1\Delta + a_1 - b_{12}v) > \lambda_1(d_1\Delta + a_1 - b_{12}Q_2) > 0$, and so $Sv := u_v > 0$ by the definition of the operator S. If $a_3/d_3 > \lambda_1(-\Delta)$, then $\lambda_1(d_3\Delta + a_3 + b_{32}v) > \lambda_1(d_3\Delta + a_3) > 0$, and thus $Tv := w_v > 0$. In the case of $a_3/d_3 \le \lambda_1(-\Delta)$, suppose that $Tv := w_v \equiv 0$. Then since $-d_2\Delta v_0 = v_0(a_2 - b_{22}v_0) \le v_0(a_2 + b_{21}u - b_{22}v_0)$ in Ω, $v_0 \le v$ holds by the comparison principle and so we have $\lambda_1(d_3\Delta + a_3 + b_{32}v) \ge \lambda_1(d_3\Delta + a_3 + b_{32}v_0) > 0$, which derives a contradiction to the definition of T.

Case 2 : <Sufficiency> Since $a_3/d_3 > \lambda_1(-\Delta)$, we have $a_3/d_3 > \lambda_1(-\Delta - (b_{32}/d_3)v)$, and thus $Tv := w_v > 0$. To show that $Sv > 0$, contrariwise, suppose $Sv := u_v \equiv 0$. Then $a_2/d_2 = \lambda_1(-\Delta + (b_{22}/d_2)v + (b_{23}/d_2)w_v)$ by Lemma 2.1 (iii), and so $\lambda_1(d_2\Delta + a_2) > \lambda_1(d_2\Delta + a_2 - b_{22}v - b_{23}w_v) = 0$ which is a contradiction.

<Necessity> If (1.2) has a positive solution (u, v, w), then $\lambda_1(d_2\Delta + a_2 + b_{21}u - b_{22}v - b_{23}w) = 0$ by Lemma 2.1 (iii). Since $-d_1\Delta u_0 = u_0(a_1 - b_{11}u_0) \ge u_0(a_1 - b_{11}u_0 - b_{12}v)$ in Ω, we have $u \le u_0$ by the comparison principle. Therefore using the monotonicity property of principal eigenvalue, one has the desired result.

Case 3 : The proof of $Sv > 0$ is the same as Case 2. Observe that the fixed point v of A does not depend on the constant b_{32}, and so we can easily find a sufficiently large constant b_{32} such that $a_3/d_3 > \lambda_1(-\Delta - (b_{32}/d_3)v)$ which implies $Tv > 0$. The proof of the final assertion is also virtually the same as in Case 2. □

The next corollary shows how the diffusions plays a role in the existence and the nonexistence of positive solutions to system (1.2).

In the following corollary, we assume $a_1/d_1 > \lambda_1(-\Delta)$ as in Theorem 3.8.

COROLLARY 3.9. *(i) There are positive constants K_1 and δ_1 such that (1.2) has a positive solution if $d_1, d_2, d_3 < K_1$ and $b_{12} < \delta_1$.*

(ii) There are positive constants K_2, δ_2, $\underline{d_3}$ and $\overline{d_3}$ with $\underline{d_3} < \overline{d_3}$ such that (1.2) has a positive solution for every $d_3 \in [\underline{d_3}, \overline{d_3})$ if $d_1, d_2 < K_2$ and $b_{12} < \delta_2$.

(iii) There are positive constants K_3, $\underline{d_2}$ and $\overline{d_2}$ with $\underline{d_2} < \overline{d_2}$ such that (1.2) has a positive solution for every $d_2 \in [\underline{d_2}, \overline{d_2})$ and has no positive solution for every $d_2 \ge \overline{d_2}$ if $d_1, d_3 < K_3$.

(iv) For sufficiently large b_{32}, there are positive constants K_4, K_5, $\underline{d_2}$ and $\overline{d_2}$ with $\underline{d_2} < \overline{d_2}$ such that (1.2) has a positive solution for every $d_2 \in [\underline{d_2}, \overline{d_2})$ if $d_1 < K_4$ and $d_3 \geq K_5$.

(v) There are positive constants K_6 and K_7 such that (1.2) has a positive solution if $d_1 < K_7$ and $d_2 \geq K_6$.

Proof. Take the constants K_i, δ_1, δ_2, $\underline{d_2}$, $\overline{d_2}$, $\underline{d_3}$ and $\overline{d_3}$ for $i = 1, \cdots, 7$ as the following:

- $\delta_1, \delta_2 := a_1/Q_2$;
- $\underline{d_2} := a_2/\lambda_1(-\Delta)$, $\underline{d_3} := a_3/\lambda_1(-\Delta)$;
- $\overline{d_2}$ and $\overline{d_3}$ such that $a_2/\overline{d_2} = \lambda_1(-\Delta - (b_{21}/\overline{d_2})u_0)$ and $a_3/\overline{d_3} = \lambda_1(-\Delta - (b_{32}/\overline{d_3})v_0)$;
- $K_1 := \min\{(a_1 - \delta_1 Q_2)/\lambda_1(-\Delta), a_2/\lambda_1(-\Delta), a_3/\lambda_1(-\Delta)\}$;
- $K_2 := \min\{a_1/\lambda_1(-\Delta), a_2/\lambda_1(-\Delta)\}$;
- $K_3 := \min\{a_1/\lambda_1(-\Delta), a_3/\lambda_1(-\Delta)\}$;
- $K_4, K_7 := a_1/\lambda_1(-\Delta)$, $K_5 := \underline{d_3}$, $K_6 := \overline{d_2}$.

Then the results (i), (ii) follow from **Case 1**; (iii) from **Case 2**; and (iv), (v) from **Case 3** of Theorem 3.8. □

4. Extinction result

In this section, we obtain the extinction result for simple food-chain models which can be interpreted as some biological phenomena, the so-called *Biological control* and *Domino effect* on the time-dependent system (1.1). We also illustrate numerical evidences of our results.

THEOREM 4.1. *Consider (1.1) with $(\widetilde{u}(x), \widetilde{v}(x), \widetilde{w}(x)) > (0,0,0)$. Let u_0, w_0 denote the semi-trivial solutions of system (1.2). (For details, see Section 3.)*

(i) If $a_i/d_i < \lambda_1(-\Delta)$ for $i = 1,2,3$, then $u(x,t), v(x,t), w(x,t) \to 0$ uniformly as $t \to \infty$.

(ii) Assume that $a_1/d_1 > \lambda_1(-\Delta)$.

Case 1 : *$a_3/d_3 < \lambda_1(-\Delta)$.*

If $a_2/d_2 < \lambda_1(-\Delta - (b_{21}/d_2)u_0)$, then $u(x,t) \to u_0$ and $v(x,t)$, $w(x,t) \to 0$ uniformly as $t \to \infty$.

Case 2 : *$a_3/d_3 > \lambda_1(-\Delta)$.*

If $a_2/d_2 \leq \lambda_1(-\Delta - (b_{21}/d_2)u_0 + (b_{23}/d_2)w_0)$, then $u(x,t) \to u_0$, $v(x,t) \to 0$ and $w(x,t) \to w_0$ uniformly as $t \to \infty$.

(iii) Assume that $\min\{a_1/d_1, a_2/d_2\} > \lambda_1(-\Delta)$. If $a_3/d_3 < \lambda_1(-\Delta)$, then there exists a sufficiently small b_{32} such that $w(x,t) \to 0$ uniformly as $t \to \infty$.

Proof. (i) Let (u, v, w) be the associated global solution of the system (1.1) with an initial condition $(\widetilde{u}(x), \widetilde{v}(x), \widetilde{w}(x)) \in int(P) \times int(P) \times int(P)$ and U the unique solution of the initial value problem:

$$\text{(4.1)} \qquad \begin{cases} U_t - d_1 \Delta U = U(a_1 - b_{11} U) & \text{in } \Omega \times (0, \infty), \\ U = 0 & \text{on } \partial\Omega \times (0, \infty), \\ U(x, 0) = \widetilde{u}(x) & \text{in } \Omega. \end{cases}$$

Then $0 \le u(x,t) \le U(x,t)$ since $U_t - d_1 \Delta U = U(a_1 - b_{11} U) \ge U(a_1 - b_{11} U - b_{12} v)$ in $\Omega \times (0, \infty)$. Furthermore, since $a_1/d_1 < \lambda_1(-\Delta)$, we have $U(x,t) \to 0$ uniformly as $t \to \infty$ by Theorem 2.3 (i), and thus $u(x,t) \to 0$ uniformly as $t \to \infty$. Let $\epsilon > 0$ be given. Then there exists $T_\epsilon \ge 0$ such that $u(x,t) \le \epsilon$ for all $t \ge T_\epsilon$. Thus the following inequality holds:

$$v_t - d_2 \Delta v = v(a_2 + b_{21} u - b_{22} v - b_{23} w) \le v(a_2 + \epsilon b_{21} - b_{22} v) \text{ for all } t \ge T_\epsilon.$$

Now let V_ϵ be the solution of the equation:

$$\text{(4.2)} \qquad \begin{cases} (V_\epsilon)_t - d_2 \Delta V_\epsilon = V_\epsilon(a_2 + \epsilon b_{21} - b_{22} V_\epsilon) & \text{in } \Omega \times (T_\epsilon, \infty), \\ V_\epsilon = 0 & \text{on } \partial\Omega \times (T_\epsilon, \infty), \\ V_\epsilon(x, T_\epsilon) = v(x, T_\epsilon) & \text{in } \Omega. \end{cases}$$

Using the comparison theorem, we get $0 \le v(x,t) \le V_\epsilon(x,t)$ for $t \ge T_\epsilon$. If $a_2/d_2 < \lambda_1(-\Delta)$, then $(a_2 + \epsilon b_{21})/d_2 \le \lambda_1(-\Delta)$ for sufficiently small ϵ, and so $V_\epsilon(x,t) \to 0$ uniformly as $t \to \infty$ which implies $v(x,t) \to 0$ uniformly as $t \to \infty$. Similarly, one can show $w(x,t) \to 0$ uniformly as $t \to \infty$.

(ii) **Case 1 :** In the proof of (i), we have $0 < u(x,t) \le U(x,t)$. We also have $U(x,t) \to u_0(x)$ uniformly as $t \to \infty$ by Theorem 2.3 (ii). Let $\epsilon > 0$ be so small that $(a_1 - \epsilon)/d_1 > \lambda_1(-\Delta)$, then there exists $T_\epsilon \ge 0$ such that

$$\text{(4.3)} \qquad u(x,t) \le u_0(x) + \epsilon \text{ for all } t \ge T_\epsilon.$$

Since $v_t - d_2 \Delta v = v(a_2 + b_{21} u - b_{22} v - b_{23} w) \le v(a_2 + b_{21}\epsilon + b_{21} u_0 - b_{22} v)$, similarly as in the proof of (i), one can show that $v(x,t) \to 0$ uniformly as $t \to \infty$. Hence there is T'_ϵ such that $b_{12} v \le \epsilon$ for all $t \ge T'_\epsilon$, and thus $u_t - d_1 \Delta u = u(a_1 - b_{11} u - b_{12} v) \ge u(a_1 - \epsilon - b_{11} u)$ for $t \ge T'_\epsilon$. Let U_ϵ be the solution of the equation:

$$\text{(4.4)} \qquad \begin{cases} (U_\epsilon)_t - d_1 \Delta U_\epsilon = U_\epsilon(a_1 - \epsilon - b_{11} U_\epsilon) & \text{in } \Omega \times (T'_\epsilon, \infty), \\ U_\epsilon = 0 & \text{on } \partial\Omega \times (T'_\epsilon, \infty), \\ U_\epsilon(x, T'_\epsilon) = u(x, T'_\epsilon) & \text{in } \Omega. \end{cases}$$

Then by the comparison argument, we have

$$u(x,t) \geq U_\epsilon(x,t) \text{ for all } t \geq T'_\epsilon. \tag{4.5}$$

Now, by using the continuity, $U_\epsilon(x,t) \to U(x,t)$ as $\epsilon \to 0$, where $U(x,t)$ is the unique solution of the initial value problem (4.1). Note that $U(x,t) \to u_0(x)$ uniformly as $t \to \infty$. From the equations (4.3) and (4.5), we conclude that $u(x,t) \to u_0(x)$ as $t \to \infty$. One also has $w(x,t) \to 0$ uniformly as $t \to \infty$ since $a_3/d_3 < \lambda_1(-\Delta)$, which finally gives $\|(u(x,t), v(x,t), w(x,t)) - (u_0(x), 0, 0)\|_{C(\overline{\Omega})\times C(\overline{\Omega})\times C(\overline{\Omega})} \to 0$ as $t \to \infty$.

The proofs of (ii) Case 2 and (iii) can be shown similarly. □

REMARK 4.2. Theorem 4.1 (i) shows that one can observe *Domino effect* on the diffusive Lotka-Volterra type simple food-chain model, i.e., if one species dies out, then all the species at higher trophic level die out as well. The Case 1 of Theorem 4.1 (ii) can be interpreted as *Biological control* by assuming b_{32} is sufficiently large. In system (1.1), if we consider u, v, w as the densities of plant, pest and top predator (control agent), respectively, only plant can survive in this situation(sometimes, called as a mutual extinction). Note that $w(x,t)$ can survive until $v(x,t)$ dies out because the inter-specific coefficient b_{32} is sufficiently large. We will call this situation *Biological control: Type 1.* In system (1.1), if we regard u, v, w as the densities of plant, pest, natural enemy of pest which can survive alone, then the Case 2 of Theorem 4.1 (ii) can also imply the natural successful biological control, which we call this *Biological control: Type 2.*

In view of our results in this article, we conclude that the extinction of certain species in the spatial simple food-chain interactions heavily depends on the sign of the principal eigenvalue of the linearized operator from the equation in the system. So the shape and the size of the given domain play an important role in the biological control of the density of certain species and the domino effect as well as the coexistence of simple food-chain interacting species.

Before closing this article, we give numerical examples to illustrate the results of Theorem 4.1 (i) (*Domino effect*) and (ii) (*Biological control*). For the model (1.1), we use the following data on one dimensional domain $\Omega = (0, 4)$:

- $d_1 = d_2 = d_3 = 1$, $a_2 = 0.4$, $b_{11} = b_{12} = b_{22} = b_{23} = b_{33} = 1$,
- initial data : $\widetilde{u}(x) = 0.7 \sin \frac{\pi x}{4}$, $\widetilde{v}(x) = 0.3 \sin \frac{\pi x}{4}$, $\widetilde{w}(x) = 0.2 \sin \frac{\pi x}{4}$,
- (a) $a_1 = 0.5$, $a_3 = 0.3$, $b_{21}, b_{32} = 1$,
- (b) $a_1 = 1$, $a_3 = 0.3$, $b_{21} = 0.1$, $b_{32} = 10$,

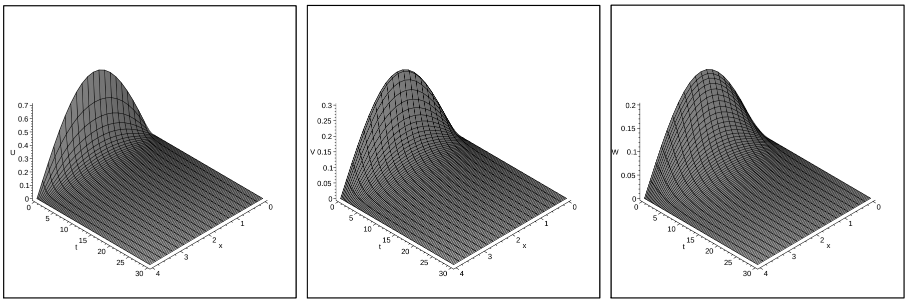

Figure 1. *Domino effect* of (1.1).

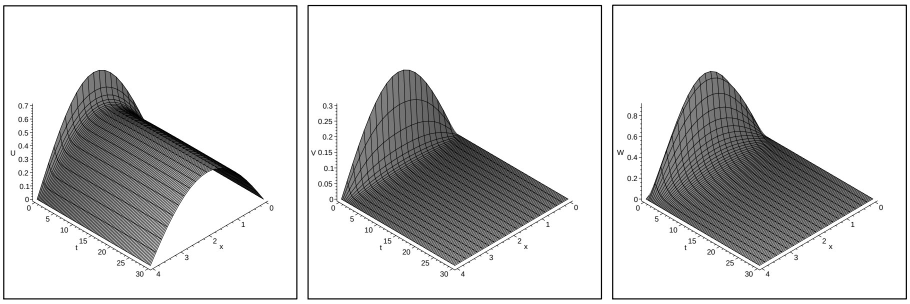

Figure 2. *Biological control: Type 1* of (1.1).

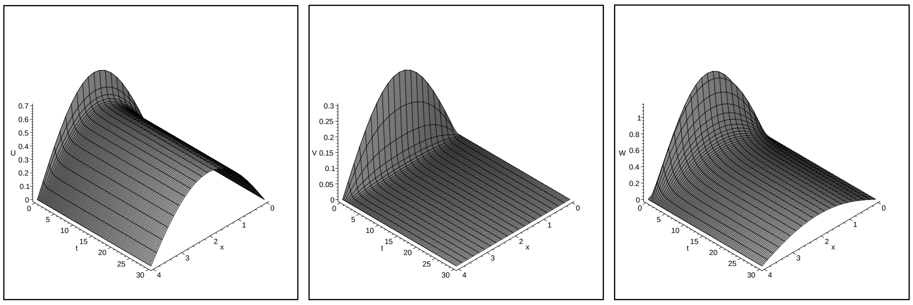

Figure 3. *Biological control: Type 2* of (1.1).

- (c) $a_1 = 1$, $a_3 = 1$, $b_{21} = 0.1$, $b_{32} = 10$.

Note that $\lambda_1(-\Delta) = (\frac{\pi}{4})^2$. The numerical values in (a) are used to illustrate *Domino effect* (see Figure 1). (b) and (c) are used to present *Biological control: Type 1* (see Figure 2) and *Biological control: Type 2* (see Figure 3), respectively.

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