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THE LOWER BOUNDS FOR THE HYPERBOLIC METRIC ON BLOCH REGIONS

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Abstract. Let X be a hyperbolic region in the complex plane C such that the hyperbolic metrix $\lambda_X(w)|dw|$ exists. Let $R(X) = \sup\{\delta_X(w) : w \in X\}$ where $\delta_X(w)$ is the euclidean distance from w to ∂X. Here ∂X is the boundary of X. A hyperbolic region X is called a Bloch region if $R(X) < \infty$. In this paper, we obtain lower bounds for the hyperbolic metric on Bloch regions in terms of the distance to the boundary.

1. Introduction

Now we give a brief introduction to the hyperbolic (or Poincaré) metric. We refer the reader to ([2], [10] and [11]) for further details. The hyperbolic metric on the unit disk $D = \{z : |z| < 1\}$ is defined by

$$\lambda_D(z)|dz| = \frac{|dz|}{1 - |z|^2}.$$

A region X in the complex plane C is called hyperbolic if $C - X$ contains at least two points. By X, we always mean a hyperbolic region in C. If a region X is hyperbolic then by the uniformization theorem [2, p. 142] there exists an analytic universal covering projection p of D onto X. If X is simply connected then p is just a conformal function of D onto X. The collection of all analytic universal covering projections of D onto X consists of the function $p \circ T$ where $T \in \mathrm{Aut}(D)$, the group of conformal automorphisms of D. Recall that $T \in \mathrm{Aut}(D)$ if and only if there exists $a \in D$ and $\theta \in R$ such that

$$T(z) = e^{i\theta}\frac{z - a}{1 - \overline{a}z}.$$

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The density $\lambda_X(z)$ of the hyperbolic metric $\lambda_X(z)|dz|$ is defined by

$$\lambda_X(p(z))|p'(z)| = \lambda_D(z)$$

where $p : D \to X$ is any analytic universal covering projection. The density of the hyperbolic metric is independent of the choice of the analytic universal covering projection p since $|T'(z)|(1-|T(z)|^2)^{-1} = (1-|z|^2)^{-1}$ for $z \in D$. The hyperbolic density is positive and real-analytic on X. Because of this we may select any w in X and assume that $p(0) = w$ then $\lambda_X(w)|p'(0)| = 1$. It has constant Gaussian curvature -4; this means that

$$k(w, \lambda_X) = -\frac{\triangle \log \lambda_X(w)}{\lambda_X^2(w)} = -4,$$

where $\triangle$ is the Laplacian. An important property of the hyperbolic metric is the principle of hyperbolic metric due to R. Nevanlinna [8, p. 50]. We shall need this principle in the following form

$$\lambda_X(f(z))|f'(z)| \leq \lambda_D(z),$$

where $f : D \to X$ is analytic and z is any point in D, and equality holds if and only if f is an analytic universal covering projection of D onto X. When we try to obtain a general lower bound of λ_X, the most important thing is probably the lower bound of that for the special domain $X_0 = C - \{0, 1\}$. The following inequality was proved by Hempel [5] and Jenkins [6] independently :

$$\lambda_{X_0}(z) \geq \frac{1}{2|z|(|\log|z|| + c_1)},$$

where $c_1 = 1/2\lambda_{X_0}(-1) = \Gamma(1/4)^4/4\pi^2$, $\Gamma(x)$ is the usual Gamma function. And also Sugawa [11] proved that, for the $\lambda_{X_0}(z)|dz|$, the lower estimates

$$\lambda_0(z) \geq \frac{c_2}{|z|^{3/4}|z-1|^{1/2}}$$

can be obtained. Here $c_2 = \sqrt{2}\lambda_{X_0}(-1) = 2\sqrt{2}\pi^2/\Gamma(1/4)^4$.

Let $\delta_X(w)$ denote the euclidean distance from w to ∂X. Hence ∂X is the boundary of X. Then $\delta_X(w)^{-1}$ is called the density of the quasi hyperbolic metric. We shall make use of estimates of the hyperbolic metric in terms of the quasi hyperbolic metric. Let $R(X) = \sup\{\delta_X(w) : w \in X\}$. Roughly speaking, $R(X)$ is the radius of the largest disk in X. A hyperbolic region X is called a Bloch region if $R(X) < \infty$. From the definition of the Bloch regions it is easy to see that X is a Bloch region if and only if X does not contain arbitrarily large disk.

Typically, there is no explicit formula for the density $\lambda_X(w)$ of the hyperbolic metric, so estimates are useful. However, there are few results that deal explicity with the size of the hyperbolic metric. Let us survey some of these. Ahlfors ([1], [2]) gave analytic bounds in case X is the trice punctured sphere. Often, one is interested in bounds for $\lambda_X(w)$ in terms of the geometric quantity $\delta_X(w)$. The upper bound $\lambda_X(w)\delta_X(w) \leq 1$ is a direct consequence of Schwarz's lemma [7, p. 45]. On the other hand, for any hyperbolic simply connected region X, we have $1/4 \leq \lambda_X(w)\delta_X(w)$ [7, p. 45]. Blevins [3] obtained a sharp lower bound for simply connected regions that are bounded by a quasiconformal circle. The upper bound of $\lambda_X(w)\delta_X(w)$ is sharp but the lower bound 1/4 is not.

We are interested in obtaining a lower bound for $\lambda_X(z)$ in terms of $\delta_X(z)$ that is valid even if the boundary of X has isolated points. If $X = \{z : 0 < |z-a| < R\}$, then

$$\lambda_X(z) = \frac{1}{2|z-a|[\log R - \log|z-a|]}.$$

Then $\delta_X(z) = |z-a|$ for $0 < |z-a| \leq R/2$ and so

$$\lambda_X(z) = \frac{1}{2\delta_X(z)[\log R - \log\delta_X(z)]}$$

for $z \to a$. This example also shows explicity that $\lambda_X(z)\delta_X(z)$ has no positive lower bound as $z \to a$. For a general hyperbolic region we consider the possibility of finding a lower bound of the form

$$\lambda_X(z) \geq \frac{1}{2\delta_X(z)[\log b - \log\delta_X(z)]}$$

where b is a positive constant. Such a bound is implicit in Ahlfors' method for determining a lower bound for the Landau constant [1]. Since bounds for the hyperbolic metric are relatively scare, it seems worthwhile to make explicit these bounds.

In this paper, we will establish lower bounds for the hyperbolic density $\lambda_X(w)$ by various powers of the distance to the boundary ∂X, in this case when X is a Bloch region.

2. Main theorem

In this Section, we assume that $f : D \to C$ is an analytic universal covering projection.

LEMMA 2.1. *Let X be a Bloch region. Then for any $x \in (0,1)$,*

$$|f'(0)| \leq \frac{\delta_X(w)^{1-x}(G^2 - \delta_X(w)^{2x})}{xG},$$

where $w = f(0)$ and G is a constant.

Proof. For $x \in (0,1)$, we consider the conformal metric $\rho(w)|dw|$ in X given by

$$\rho(w)|dw| = \frac{xG|dw|}{R(w)^{1-x}(G^2 - R(w)^{2x})}. \tag{1}$$

Here G is a constant and is given by satisfying (1). And also $R(w)$ is the radius of the largest unramified disk about w in the image of f. Since f is a universal covering projection, $\delta_X(w)$ is the radius of the largest unramified disk. It follows that $R(w) = \delta_X(w)$.

We will show that $\rho(f(z))|f'(z)||dz|$ is an ultrahyperbolic metric on D for G sufficiently larger. The inequality $\rho(f(z)) \leq \lambda_X(f(z))$ will then follow from Ahlfor's generalization of Schwarz's Lemma ([1], [2, p. 13]). Since $\delta_X(f(z))$ is a continuous function, it is clear that $\rho(f(z))|f'(z)||dz|$ is a positive continuous metric on D.

To show that $\rho(f(z))|f'(z)||dz|$ is an ultrahyperbolic metric on D, we must exhibit a supporting metric at each point $f(z_1)$ of X. This is a metric $\rho_1(f(z))|f'(z)||dz|$ defined in a neighborhood of $f(z_1)$ with constant curvature -4 such that $\rho_1(f(z)) \leq \rho(f(z))$ for $f(z) \to f(z_1)$ with equality at $f(z_1)$. Given $f(z_1) \in X$, select $a \in \partial X$ with $|f(z_1) - a| = \delta_X(f(z_1))$. Then

$$R(f(z)) = \delta_X(f(z)) \leq |f(z) - a| < R(X)$$

for $f(z) \to f(z_1)$ with equality at $f(z_1)$. At $z \in D$, a supporting metric is given by $\rho_1(f(z))|f'(z)||dz|$ with

$$\rho_1(f(z))|f'(z)| = \frac{xG|f'(z)|}{|f(z)-a|^{1-x}(G^2 - |f(z)-a|^{2x})}.$$

If the function $h(t) = t^{(1-x)}(G^2 - t^{2x})$ is increasing then the inequality

$$R(f(z)) = \delta_X(f(z)) \leq |f(z) - a| < R(X)$$

for $f(z) \to f(z_1)$ yields $\rho_1(f(z)) \leq \rho(f(z))$ for $f(z) \to f(z_1)$ with equality at $f(z_1)$.

Since the hyperbolic metric $\rho(f(z))|f'(z)||dz|$ has constant curvature -4, it is a supporting metric for $\rho(f(z))|f'(z)||dz|$ at $f(z_1)$. In order to have $\rho_1(f(z)) \leq \rho(f(z))$, we need the function $t^{(1-x)}(G^2 - t^{2x})$ to be increasing. This will be the case provided

$$G^2 > \frac{(1+x)R(X)^{2x}}{1-x}.$$

Hence $\rho(f(z))|f'(z)||dz|$ is ultrahyperbolic metric for the chosen of G. Therefore

$$\rho(f(z))|f'(z)| \leq (1-|z|^2)^{-1}.$$

At $z = 0$ we have the result. □ □

Theorem 2.2. *Let X be a Bloch region. Then for any $x \in (0,1)$,*

$$\lambda_X(w) \geq \frac{x\sqrt{(1-x)/(1+x)}}{\delta_X(w)^{1-x}R(X)}$$

where $f(0) = w \in X$.

Proof. We will derive our results from Lemma 2.1. Since $xG|f'(0)| \leq G^2\delta_X(w)^{1-x}$, we have

$$\lambda_X(w) = \frac{1}{|f'(0)|} \geq \frac{x}{G\delta_X(w)^{1-x}},$$

and the result follows by letting $G \to \sqrt{(1+x)/(1-x)}R(X)^x$. □ □

Remark. Ahlfors first used his method to establish a lower bound for the size of unramified disk in the image of analytic functions $f : D \to C$ normalized by $|f'(0)| = 1$. He showed the estimate $\sqrt{3}/4$ and with basically the same argument one can prove the bound $1/4$ when f so normalized is locally univalent. These bounds are not sharp. In general it is not true that there is a positive constant $c(X)$ such that $\lambda_X(z)\delta_X(z) \geq c(X)$. For example, if $D_1 = D - \{0\}$ then $\delta_{D_1}(w) = |w|$ for $0 < |w| \leq 1/2$ and

$$\lambda_{D_1}(w)\delta_{D_1}(w) = \frac{1}{2|w|\log(1/|w|)}|w| = -\frac{1}{2\log|w|}$$

for $0 < |w| \leq 1/2$ so that $\lambda_{D_1}(w)\delta_{D_1}(w) \to 0$ as $w \to 0$. Beardon and Pommerenke [4] and Pommerenke [6] have obtained a necessary and sufficient condition for the existence of such a positive contant $c(X)$. They introduce the function

$$\beta_X(w) = \inf\left\{\left|\log\left|\frac{w-a}{b-a}\right|\right| : a,\ b \in \partial X,\ |w-a| = \delta_X(w)\right\},$$

and show that

$$\lambda_X(w)\delta_X(w) \geq \frac{1}{2\sqrt{2}(k+\beta_X(w))}$$

with $k = 4 + \log(3 + 2\sqrt{2})$.

Here we have a lower bound of $\lambda_X(w)\delta_X(w)$ on a Bloch region.

THEOREM 2.3. *Let X be a Bloch region. Then*

$$\lambda_X(w)\delta_X(w) \geq \frac{1}{2[1 + \log R(X) - \log \delta_X(w)]},$$

where $w \in X$.

Proof. By Lemma 2.1 and letting $G \to \sqrt{(1+x)/(1-x)}R(X)^x$, we have

$$\begin{aligned}\frac{1}{|f'(0)|} &\geq \frac{xG}{\delta_X(w)^{1-x}(G^2 - \delta_X(w)^{2x})}\\ &\geq \sqrt{\frac{1+x}{1-x}}R(X)^x \frac{x}{\delta_X(w)^{1-x}\left(\frac{1+x}{1-x}R(X)^{2x} - \delta_X(w)^{2x}\right)}.\end{aligned}$$

Therfore

$$\frac{\delta_X(w)}{|f'(0)|} \geq \sqrt{\frac{1+x}{1-x}} R(X)^x \frac{x}{\delta_X(w)^{-x}\left(\frac{1+x}{1-x} R(X)^{2x} - \delta_X(w)^{2x}\right)}$$

$$= \sqrt{\frac{1+x}{1-x}} \left(\frac{R(X)}{\delta_X(w)}\right)^x \frac{x}{\left(\frac{1+x}{1-x}\right)\left(\frac{R(X)}{\delta_X(w)}\right)^{2x} - 1}.$$

Now it follows from the L'Hospital rule that

$$\lim_{x\to 0} \frac{x}{\left(\frac{1+x}{1-x}\right)\left(\frac{R(X)}{\delta_X(w)}\right)^{2x} - 1}$$

$$= \lim_{x\to 0} \frac{1}{\frac{2}{(1-x)^2}\left(\frac{R(X)}{\delta_X(w)}\right)^{2x} + 2\left(\frac{1+x}{1-x}\right)\left(\frac{R(X)}{\delta_X(w)}\right)^{2x} \log\left(\frac{R(X)}{\delta_X(w)}\right)}$$

$$= \frac{1}{2[1 + \log R(X) \; - \log \delta_X(w)]}.$$

By letting $x \to 0$, we have the result. $\square$ $\square$

Remark. For a Bloch region X, $\lambda_X(w) \to \infty$ when $w \to \infty$ if and only if $\delta_X(w) \to 0$ when $w \to \infty$.

References

1. L. V. Ahlfors, *An extension of Schwarz's lemma*, Trans. Amer. Math. Soc. **43** (1938), 359-364.
2. L.V. Ahlfors, *Conformal invariants. Topics in geometric function theory*, McGraw-Hill, New York, 1973.
3. D. K. Blevins, *Conformal mappings of domains bounded by quasi conformal circles*, Duke. Math. J. **40** (1973), 877-883.
4. A. F. Beardon and Ch. Pommerenke, *The Poincaré metric on the plane domains*, J. London Math Soc. **18** (1978), 475-483.
5. J. A. Hempel, *The Poincaré metric on the twice punctured plane and the theorems of Landau and Schottky*, J. London Math. Soc. **20(2)** (1979), 435-445.
6. J. A. Jenkins, *On explicit bounds in Landau's theorem (II)*, Canada. J. Math. **33** (1981), 559-562.
7. I. Kra, *Automorphic functions and kleinian groups*, W. A. Benjamin, Reading, Mass, 1972.
8. R. Nevanlinna, *Eindeutige Analytisher Funktionen*, Springer-Verlag, 1953.
9. Ch. Pommerenke, *Uniformly perfect sets and the Poincaré metric*, Arch. Math. **32** (1979), 192-199.
10. A. Yu. Solynin and M. Vuorinen, *Estimates for the hyperbolic metric of the punctured plane and applications*, Israel J. Math. **124** (2001), 29-60.

11. Toshiyuki Sugawa, *Estimates of hyperbolic metric applications to Teichmüller spaces*, Kyungbook Math J. **42** (2002), 51-60.

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