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INVERSE SHADOWING IN GEOMETRIC LORENZ FLOWS

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Abstract. We introduce the inverse shadowing property of geometric Lorenz flows and prove that the geometric Lorenz flows do not have the inverse shadowing property.

1. Introduction

The geometric Lorenz model is one of the important examples in dynamical systems, which was studied in the initial stages by Guckenheimer and Williams [6], Afraimovich, Bykov and Shilnikov [1] and Yorke and Yorke [18]. Their aim was to construct topologically a simple mechanism which can give results similar to that of the parametrized ODE system in $\mathbb{R}^3$ presented experimentally by Lorenz, a meteorologist at MIT [13]. He was interested in the foundations of long-range weather forecasting and observed typical characters of chaotic motions in butterfly-shaped attractors, for some parameter values. The question whether or not the original Lorenz system for such parameter values has the same structure as the geometric Lorenz model has been unsolved for more than 30 years. Tucker [14] answered this question affirmatively, that is, for classical parameters, the original Lorenz system has a robust strange attractor which is given by the same rules as for the geometric Lorenz model. From these facts, we know that the geometric Lorenz model is crucial in the study of Lorenz dynamical systems. Viana [15] gave us introductory information about works in the fields of Lorenz strange attractors.

Komuro [10] showed that the geometric Lorenz flows do not satisfy the (parameterfixed) shadowing property except in very restricted cases, and

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recently Kiriki and Soma [8] proved that for a definite class of geometric Lorenz flows, the flows has the parameter-shifted shadowing property.

The inverse shadowing property which is a "dual" notion of shadowing was introduced by Corless and Pilyugin in [4], and the qualitative theory of dynamical systems with the property was developed by various authors [2, 3, 4, 7, 9, 11, 12].

In this paper, we introduce the inverse shadowing property of geometric Lorenz flows and we prove that the geometric Lorenz flows do not have the inverse shadowing property.

2. Preliminaries

The Lorenz geometric flows is abstracted by Lorenz equation. It is a three parameter of differential equations, that is

$$\dot{x} = -\sigma x + \sigma y, \quad \dot{y} = rx - y - xz, \quad \dot{z} = -bz + xy$$

where σ, r and b are three real positive parameters.

In fact, Geometric Lorenz models were constructed by J. Guckenheimer and R. F. Williams [6]. These models are flows in 3-dimensions for which one can rigorously prove the existence of an attractor that contains an equilibrium point of the flow together with regular solutions. Moreover, for almost every pair of nearby initial condition the corresponding solutions move away from each other exponentially fast as they converge to the attractor.

Let $\Sigma_{\pm}$ denote the components of $\Sigma \setminus \Gamma$ with $(\pm 1, 0) \in \Sigma_{\pm}$. A map $L : \Sigma \setminus \Gamma \to [-1, 1]$ said to be a *Lorenz map* if it is a piecewise C^1-diffeomorphism with has the form

$$L(x, y) = (\alpha(x), \beta(x, y))$$

where $\alpha : [-1, 1] \setminus \{0\} \to [-1, 1]$is a piecewise c^1 map with symmetric property $\alpha(-x) = -\alpha(x)$ and satisfying

$$\begin{cases} \lim_{x \to 0+} \alpha(x) = -1, \alpha(1) < 1 \\ \lim_{x \to 0+} \alpha'(x) = \infty, \alpha'(x) > \sqrt{2}, \forall x \in (0, 1] \end{cases}$$

(see Figure 1(a)), and $\beta : \Sigma \setminus \Gamma \to [-1, 1]$ is a contraction in the y-direction.

Moreover, it is required the images $L(\Sigma_+), L(\Sigma_-)$ are mutually disjoint cusps in Σ, where the vertices v_+, v_- of $L(\sum_{\pm})$ are contained in $\{\mp 1\} \times [-1, 1]$ respectively; see Figure 1(b).

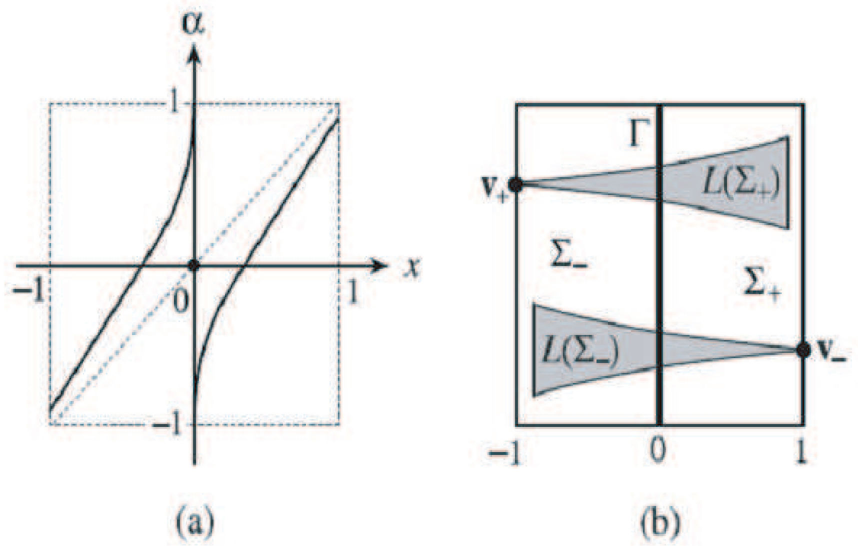

FIGURE 1

Let us identify Σ with $\{(x, y, 1) \in R^3 : |x|, |y| \leq 1\}$ and Γ with$\{(0, y, 1) \in R^3 : |y| \leq 1\}$

DEFINITION 2.1. A C^1-vector field X_L on R^3 is said to be a *geometric Lorenz vector field controlled by* a Lorenz map $L : \Sigma \backslash \Gamma \to \Sigma$ if it satisfies the following conditions (i) and (ii) :

(i) For any point (x, y, z) in a neighborhood of the origin O of R^3, X_L is given by

$$(\dot{x}, \dot{y}, \dot{z}) = (\lambda_1 x, -\lambda_2 y, -\lambda_3 z),$$

where λ_i are positive numbers satisfying $\lambda_3 < \lambda_1 < \lambda_2$

(ii) All forward orbit of X_L starting from $\Sigma \setminus \Gamma$ will return to Σ and the first return mal is L.

Note then that O is singular point of saddle type, the local unstable manifold of O is tangent to the x-axis, and the local stable manifold of O is tangent to the yz-plane.

DEFINITION 2.2. A C^1-map $\varphi_L : R^3 \times R \to R$ is the *geometric Lorenz flow controlled by* L if it generated by X_L, that is,

$$\varphi_L(X, O) = X, \quad \text{and} \quad (\partial/\partial t)(X, t) = X_L(\varphi_L(X, t)).$$

Note that the closure $\bigcup_{z\in\Sigma\setminus\Gamma}\varphi_L(Z,[0,\infty)) \subset R^3$ is homeomorphic to the genus two solid handlebody as in illustrated in Figure 2, which is called a *trapping region* of φ_L and denote by T_{φ_L} or T_L.

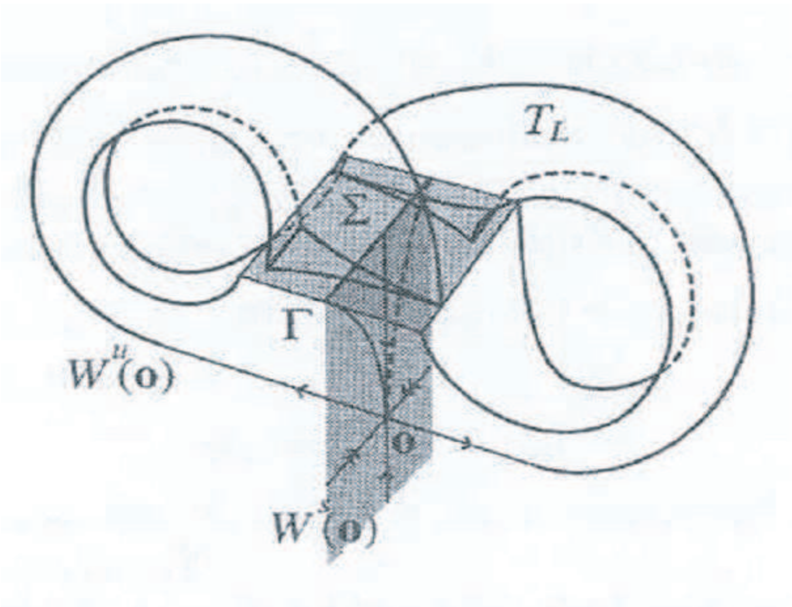

FIGURE 2

DEFINITION 2.3. For $\delta > 0$ a sequence $\{x_n\}_{n\geq 0} \subset I = [-1,1]$ is called a *δ-pseudo-orbit* of α if

$$|\alpha(x_n) - x_{n+1}| \leq \delta, \qquad \text{for any } n \geq 0.$$

Here, if $x_n = 0$, then $x_{n+1} \in [-1,-1+\delta) \cup (1-\delta, 1]$.

DEFINITION 2.4. For $\delta > 0$ a mapping $\varphi : I \to I^{\overline{Z^+}}$ is said to be a *δ-method* for α if $\varphi(x)$ is a δ-pseudo-orbit of α and $\varphi(x)_0 = x$, where, $\overline{Z^+} = Z^+ \cup \{0\}$ and $\varphi(x)_0 = x$ denote the 0th components of $\varphi(x)$.

DEFINITION 2.5. We say that a mapping $\alpha : [-1,1] \setminus \{0\} \to [-1,1]$ has the *inverse shadowing property* if for each $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo-orbit of α and φ for α, for any $x \in I - C$, $\exists\, y \in I$ such that

$$|\alpha^n(x) - \varphi(y)_n| \leq \epsilon,\ \forall\, n \geq 0,$$

where, $C = \bigcup_{j=0}^{\infty} \alpha^- j(0)$.

Remark 2.1. [10] Note that C is countable and dense subset in I.

Theorem 2.1. *A mapping α does not have the inverse shadowing property.*

Proof. To prove this, it is sufficient to show that $\exists\ \epsilon_0 > 0$ such that for any $n > 0$, there is a $\frac{1}{n}$-method φ_n for α and $x_n \in I - C$ such that $\forall\ y \in I, \exists\ k \geq 0$ such that $|\alpha^k(x_n) - \varphi_n(y)_k| \geq \epsilon_0$.

Let $0 < \epsilon_0 < \frac{1}{10}$ and n be arbitrary. Take $x_n \in [-1, -1+\epsilon_0) \cap C^c$. Define a $\frac{1}{n}$-method $\varphi : I \to I^{\overline{Z+}}$ is as follows : $\forall w \in [-1, -1+2\epsilon_0]$

$$\varphi_n(w) = \{\varphi_n(w)_0, \varphi_n(w)_1, \varphi_n(w)_2, \ldots\}$$

such that

(i) $\varphi_n(w)_0 = w$,

(ii) $|\varphi_n(w)_1 - \alpha(w)| \leq \frac{1}{2n}$ satisfying $\varphi_n(w)_k \in C$, i.e., $\exists\ k_0(w) > 0$ such that $\alpha^{k_0(w)}(\varphi_n(w)_1) = 0$

(iii) if $\alpha^{k_0}(x_n) \notin [-\epsilon_0, \epsilon_0]$,
$\varphi_n(w)_j = \alpha^{j-1}(\varphi_n(w)_1), \ \ 1 \leq j \leq k_0 + 1$

(iv) if $\alpha^{k_0}(x_n) \in [-\epsilon_0, \epsilon_0] \setminus \{0\}$:

(a) if $-\epsilon_0 \leq \alpha^{k_0}(x_n) < 0$ then

$$\begin{cases} \varphi_n(w)_i = \alpha^{i-1}(\varphi_n(w)_1), \ 1 \leq i \leq k_0, \\ \varphi_n(w)_{k_0+1} = -1 \end{cases}$$

(b) if $0 \leq \alpha^{k_0}(x_n) < \varepsilon_0$ then

$$\begin{cases} \varphi_n(w)_i = \alpha^{i-1}(\varphi_n(w)_1), \ 1 \leq i \leq k_0, \\ \varphi_n(w)_{k_0+1} = 1. \end{cases}$$

Then for all $y \in I$, there exists $k \geq 0$ such that

$$|\alpha^k(x_n) - \varphi_n(y)_k| \geq \epsilon_0.$$

In fact, if $y \in I \setminus [-1, -1+2\varepsilon_0]$, then $|x_n - y| \geq \varepsilon_0$, for some $k = 0$. Assume that $y \in [-1, -1+2\varepsilon_0]$. Then there exists $k_0(y) > 0$ such that

$$\alpha^{k_0(y)}(\varphi_n(y)_1) = 0.$$

If $\alpha^{k_0}(x_n) \notin [-\varepsilon_0, \varepsilon_0]$ then

$$|\varphi_n(y)_{k_0+1} - \alpha_0^k(x_n)| \geq \varepsilon_0,$$

for some $k_0 + 1 \geq 0$. If $\alpha^{k_0}(x_n) \notin [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$ then

$$|\varphi_n(y)_{k_0+1} - \alpha_0^k(x_n)| > \varepsilon_0,$$

for some $k_0 + 1 \geq 0$. Therefore, α does not have the inverse shadowing property. □

Definition 2.6. For $\delta > 0$ a sequence $\{X_n\}_{n\geq 0} \subset \Sigma$ is called a *δ-pseudo-orbit* of a Lorenz map L satisfying the following condition :

(a) $|L(X_n) - X_{n+1}| \leq \delta, \ \forall\, n \geq 0$,
(b) if $X_n \in \Gamma$ then $X_{n+1} \in (N_\delta(v_+) \cap \Sigma) \cup (N_\delta(v_-) \cap \Sigma)$.

Definition 2.7. For $\delta > 0$ a mapping $\Phi : \Sigma \to \Sigma^{\overline{Z+}}$ is said to be a *δ-method* for the Lorenz map L. If $\Phi(X)$ is a δ-pseudo-orbit of L and $\Phi(X)_0 = X$.

Definition 2.8. We say that the map L has the *inverse shadowing property* if for any $\varepsilon > 0$, there is $\delta > 0$ such that for all δ-method Φ for L, for any $X = (x, y) \in \Sigma$ with $x \in C^c$, there exists $Y \in \Sigma$ such that

$$|L(X_n) - \Phi(Y)| \leq \varepsilon,$$

for all $n \geq 0$.

Theorem 2.2. *L does not have the inverse shadowing property.*

Proof. To prove this, it suffices to show that there exists $\varepsilon_0 > 0$ such that for all $n > 0$, there exists $X_n \in \Sigma$, with $[X_n]_x \in C^c$ and $\frac{1}{n}$-method Φ_n for L such that for any $Y \in \Sigma$, we can choose $k \geq 0$ such that

$$|L^k(X_n) - \Phi_n(Y)_k| \geq \varepsilon_0.$$

Take $0 < \varepsilon_0 < \min\{1/10, 1/\eta_0\}$ and $n > 0$ be arbitrary. Note that the α does not have the inverse shadowing property. Put $X_n = (x_n, 0) \in \Sigma$. Then We can define a map $\Phi_n : \Sigma \to \Sigma^{\overline{Z+}}$ by

$$\Phi_n(X) = \{\Phi_n(X)_k\}_{k\geq 0},$$

where $\Phi_n(X)_0 = x$. Then $\Phi_n(X)_k = (\Phi_n([X]_x)_k, \beta(\Phi_n(X)_{k-1}), \forall k \geq 1$. Then Φ_n is a δ-method for L. Let $Y \in \Sigma$ with $Y \in N_{\varepsilon_0}(X_n)$,that is, $|X_n - Y| \leq \varepsilon_0$ Set $[Y]_x = Y \in \Sigma$. Then, there exists $k > 0$ such that

$$|\alpha^k(X_n) - \Phi_n(Y)_k| \geq \varepsilon_0.$$

Therefore,

$$\begin{aligned} |L^k(X_n) - \Phi_n(Y)_k| &> |[L^k(X_n)]_x - [\Phi_n(Y)_k]_x| \\ &= |\alpha^k X_n - \Phi_n(Y)_k| \geq \varepsilon_0. \end{aligned}$$

So, L does not have the inverse shadowing property. □

Definition 2.9. Let φ be a geometric Lorenz flow, and $\delta,\ \tau > 0$. A mapping $\psi : \overline{R^+} \to T_\varphi$ is a (δ, τ)-*pseudo-orbit* for the flow φ if for any $t \in \overline{R^+}$,

$$|\psi(\varphi((t), s) - \varphi(s + t)| \leq \delta,$$

for $0 \leq s \leq \tau$. Here $\overline{R}^+ = R^+ \cup \{0\}$ and T_φ is a trapping region of φ_L.

Definition 2.10. We say that a mapping $\Phi : T_\varphi \times \overline{R^+} \to T_\varphi$ is a (δ, τ)-*method* for the flow φ if for any $x \in T_\varphi$, $\Phi_x(\cdot) : \overline{R^+} \to T_\varphi$ defined by $\Phi_x(t) = \Phi(x, t)$, for all $t \in \overline{R^+}$, is a (δ, τ)-pseudo-orbit for the flow. φ

Definition 2.11. The flow φ has the *inverse shadowing property* if for all $\varepsilon > 0$, there exists $\delta > 0$ and $\tau > 0$ such that (δ, τ)-method Φ, for any $X \in T_\varphi$, there exists $Y \in T_\varphi$ and $\alpha \in Rep$ for which

$$|\varphi(X, t) - \Phi(Y, \alpha(t)| \leq \varepsilon$$

for all $t \in \overline{R^+}$. Here $Rep = \{\alpha : \overline{R^+} \to \overline{R^+} \mid \alpha$ is an increasing homeomorphism with $\alpha(0) = 0\}$.

Theorem 2.3. *φ does not have the inverse shadowing property.*

Proof. To show this, it is enough to show that there exists $\varepsilon_0 > 0$ such that for all $\delta > 0$ and any $\tau > 0$, there exists $(\frac{1}{n}, \tau)$-method with $0 < \frac{1}{n} < \delta$ and $x \in T_\varphi$ satisfying the following :
for all $y \in T_\varphi$ and any $\alpha \in Rep$, there exists $t \in \overline{R^+}$ such that

$$|\varphi(x, t) - \Phi(y, \alpha(t)| \geq \varepsilon_0.$$

Take $\varepsilon_0 > 0$ such that the restriction $\varphi|_{\pi(3\varepsilon_0)}$ is the linear flow

$$(e^{\lambda_1 t}x, e^{\lambda_2 t}y, e^{\lambda_3 t}z)$$

for some $-\lambda_2 < -\lambda_3 < 0 < \lambda_3 < \lambda_1$. Let $\delta > 0$ and $\tau > 0$ be arbitrary. Since L is the first return map of φ, for any $Z \in \Sigma \setminus \Gamma$, there exists $\tau_Z > 0$ such that

$$\varphi(Z, \tau_Z) = L(Z).$$

Note that there exists $\tau_1 > 0$ such that for any $X \in \Sigma$,

$$\varphi(X, (0, 5\tau_1]) \cap \Sigma = \phi.$$

Set τ^*=max$\{\tau, \tau_1\}$
Take $0 < \delta^* < 1/n < \delta$ such that $|X - Y| \leq \delta^*$, for $0 \leq t \leq \tau^*$. Choose a point $X_n \in \Sigma \setminus \Gamma$ with$[X_n]_x \in I \setminus C$. For any $Y \in N_{\varepsilon_0}(X_n) \cap T_\varphi$ there exists a time sequence $\{\tau_n(Y) \leq :\ n \geq 0\}$ such that

$$\varphi(Y, \tau_0(Y)) \in \Sigma \quad \text{with} \quad \varphi(Y, [0, \tau_0(Y)) \cap \Sigma = \phi.$$

If $[Y]_Z = 1$ then $\tau_0(Y) = 0.\varphi(Y, \tau_0(Y)) = L^n(\varphi(Y, \tau_0(Y)))$ for all $n \geq 1$. If $[\varphi(Y, \tau_0(Y))]_x \in C$, then $\Phi_n(Y, t) = \varphi(Y, t)$, $\forall t \in \overline{R^+}$. If $[\varphi(Y, \tau_0(Y))]_x \notin C$, then we choose a point $Y' \in \Sigma$ such that

$$|\varphi(Y, \tau_k(Y) - Y'| \leq \delta^*,$$

for some $\tau_k(Y) > 2\tau^*$. Since $\Phi_n(Y, t) = \varphi(Y, t)$ for $0 \leq t \leq \tau_k(Y)$ and $\varphi(Y', \ t - \tau_k(Y))$ for $\tau_k(Y) < t$, we can see that $\Phi_n(Y, \ \cdot)$ is a (δ', τ')-pseudo-orbit for φ. By above condition, we can define a (δ', τ')-method for φ,$\Phi_n : T_\varphi \to T_\varphi^{\overline{R+}}$ such that

(a) $\Phi_n | N_{\varepsilon_0(X_n)}(Y)$= Above condition.

(b) for all $Y \in N_{\varepsilon_0}(X_n)^C$ in T_φ, $\Phi_n(Y, \cdot) = \varphi(Y, \cdot)$.

Since $Y \in N_{\varepsilon_0}(X_n)$ in T_φ, for any $\alpha \in Rep$,

$$\Phi_n(\alpha(t), Y) \to 0.$$

But, $\varphi(t, X_n) \to 0$ as $t \to \infty$. This means that

$$|\varphi(X, t) - \Phi_n(Y, \alpha(t))| \geq \varepsilon_0,$$

for some $t \in \overline{R^+}$. Therefore φ does not inverse shadowing property □

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