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MODULES THAT SUBMODULES LIE OVER A SUMMAND

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Abstract. Let M be a nonzero module. M has the property that every submodule of M lies over a direct summand of M. We study some properties of such a module. The endomorphism ring of such a module is also studied. The relationships of such a module to the semi-regular modules, and to the semi-perfect modules are described.

Through out this paper, rings are associative rings with identity, all modules are unitary left R-modules, and module homomorphisms are on the right of their arguments. We freely use terminologies and nations of F. Kasch [2].

Let M be any module, a submodule K of M is said to be small in M if $K + N \neq M$ for every submodule $N \neq M$. The Jacobson radical of a ring R will be denoted by $J(R)$ and it is easily verified that $J(R)x$ is small in M for each $x \in M$.

The following submodules of M are equal :
(1) the intersection of all the maximal submodules of M,
(2) the sum of all the small submodules of M, and
(3) $\{x \in M \mid Rx \text{ is small in } M\}$.

This submodule is called the *radical* of M and will be denoted by $rad\, M$. If $\alpha : M \longrightarrow N$ is an R-homomorphism, it is well known that $(rad\, M)\alpha \subseteq rad\, N$. A submodule N of a module M is said to *lie over* a summand of M if there exists a direct decomposition $M = P \oplus Q$ with $P \subseteq N$ and $Q \cap N$ small in M. A projective cover of a module K is a R-epimorphism $P \longrightarrow K \longrightarrow O$ with small kernel where P is projective.

(S) Let M be a nonzero module. Every submodule of M lies over a summand of M.

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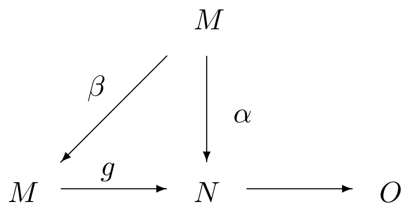
If $E = End_R(M)$, we say that M is *self-projective*, if $M\gamma \subseteq M\alpha$, $\gamma, \alpha \in E$, implies that $\gamma \in E\alpha$. If M is a module, then M is *quasi-projective* in case for each epimorphism $g : M \longrightarrow N$ and each homomorphism $\alpha : M \longrightarrow N$, there is an R-homomorphism $\beta : M \longrightarrow M$ such that the diagram

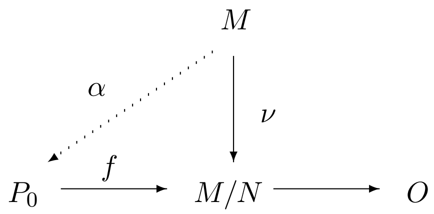


commutes.

Let R be a ring and $_R$M be a right R-module.
(a) M is called semi-perfect if M is projective and every epimorphic image of M has a projective cover [1,2].
(b) M is called complemented if every submodule of M has an addition complement in M [2].

THEOREM 0.1. *A projective module M is semiperfect if and only if M satisfies (S).*

Proof. Assume that M is a semi-perfect module. Let N be a submodule of M and $\nu : M \longrightarrow M/N$ a natural homomorphism. Let $P_0 \longrightarrow M/N$ be a projective cover of N. Then there exists a commutative diagram



Since ν is an epimorphism, we have $P_0 = Im\alpha + ker\, f$. Since $ker\, f$ is small in P_0, $P_0 = Im\alpha$, i.e., α is an epimorphism. Since P_0 is projective, α splits

$$M = P_1 \oplus ker\, \alpha.$$

Then $\alpha_1 = \alpha \mid P_1 : P_1 \longrightarrow P_0$ is an isomorphism. $ker\, \alpha_1 f = ker\, \nu \cap P_1$ is small in P$_1$. $N \cap P_1$ is small in M. $M = P_1 \oplus P_2$ where $P_2 = ker\, \alpha$. $P_2 = ker\, \alpha \subset ker\, \sigma = N$. N lies over a summand of M.

Conversely assume that M satisfies (S). If N is a submodule of M, then N lies over a summand of M. There exists a direct decomposition $M = P \oplus Q$ with $P \subseteq N$ and $Q \cap N$ is small in M. Let $g : Q \longrightarrow M/N$ be the natural epimorphism. Then Q is a projective cover of M/N. Since every epimorphic image of M has a projective cover, M is semi-perfect. □

Proposition 0.1.1. *Let M be a module which satisfies (S). Then M is complemented.*

Proof. Let N be a submodule of M. There is a direct decomposition $M = A \oplus B$ with $A \subseteq N$ and $B \cap N$ is small in M. Thus $M = N + B$ and $B \cap N$ is small in M. Therefore B is a complement of N in M. If M has the largest submodule , i.e. a proper submodule which contains all other proper submodules, then M is called a local module. Let M be a non-projective local module. M satisfies (S) but M is not semi-perfect. In this case $rad\, M$ is small in M. Let N be a submodule of M. Then $N \subset rad\, M$. or $N = M$. If $N \subset rad\, M$, then $M = 0 \oplus H$ and $M \cap N = N \subset rad\, M$. Since $rad\, M$ is small in M, N is small in M. □

A module M is called a *semi-regular* module if Rx lies over a projective summand of M, for each $x \in M$. Let $x \notin rad\, M$ and $x \in M$ where M is the module above. Then $Rx = M$. Since M is not projective, M is not semi-regular.

Proposition 0.1.2. *The following statements are equivalent:*

(1) If $N \leq M$ is a submodule, there exists $\gamma : M \longrightarrow N$ such that $\gamma^2 = \gamma$ and $N(1-\gamma)$ is small in M.

(2) N lies over a summand of M.

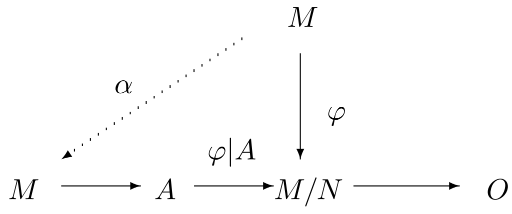
Proof. (1)⇒(2) $M = M\gamma + M(1-\gamma)$. Let $x \in M\gamma \cap M(1-\gamma)$. Then $x = m_1\gamma = m_2(1-\gamma)$ $m_1\gamma = m_1\gamma^2 = m_2(1-\gamma)\gamma = 0$. Thus $x = 0$. Therefore $M = M\gamma \oplus M(1-\gamma)$ and $M\gamma \leq N$. This means that $N \cap M(1-\gamma) = N(1-\gamma)$ is small in M.

(2)⇒(1). Let $M = A \oplus B$, $A \leq N$ and $N \cap B$ be small in M. Then $\gamma : M \longrightarrow N$ is the natural projection onto A. Therefore $\gamma^2 = \gamma$ $N(1-\gamma) = N \cap B$ is small in M. □

Proposition 0.1.3. *Let M be a quaci-projective module. Suppose $M = A + B$ where A and B are submodules and A is a direct semmand of M. There exists a submodule $Q \subseteq B$ such that $M = A \oplus Q$.*

Proof. Let $\gamma^2 = \gamma : M \longrightarrow M$ be any projection with $M\gamma = A$. If $\varphi : M \longrightarrow M/B$ is the natural map let $\alpha : M \longrightarrow M$ satisfy $\alpha\gamma\varphi = \varphi$. Define $\delta = \gamma + (1-\gamma)\alpha\gamma$. Then $\delta^2 = \delta$, $M\delta = M\gamma = A$ and $ker\,\delta = M(1-\delta) = M(1-\gamma)(1-\alpha\gamma) \leq ker\,\varphi = B$. Let $Q = ker\,\delta$. Then $M = A \oplus Q$ and $Q \leq B$.

The definition of $\sigma[M]$ projective module is in [4].



□

PROPOSITION 0.1.4. *Let M be $\sigma[M]$ projective module. M satisfies (S) if and only if M is complemented.*

Proof. The necessity of the condition is by proposition 2.

Conversely, let $A \subseteq M$ be a submodule of M and K be a complement of A in M. Then $A + K = M$. Since M is complemented, K has a complement in M. So the argument in [2, p. 277] goes through to show that K is a direct summand of M. $K + A = M$. There exits a summand $B \leq A$ such that $M = K \oplus B$, $A \cap K$ is small in M since K is a complement of A. Thus A lies over a direct summand of M. □

The following is clear from definition.

PROPOSITION 0.1.5. *If a module M satisfies (S), then every submodule of M satisfies (S).*

THEOREM 0.2. *Let M be a module and let $\varphi : M \longrightarrow M/radM$ be the natural homomorphism.*

if M satisfies (S), then

(1) $M/radM$ is semi-simple.

(2) If M is quaci-projective, and $M\varphi = A \oplus B$, then there exists a decomposition $M = P \oplus Q$ such that $P\varphi = A$ and $Q\varphi = B$.

If M is quasi-projective and $radM$ is small in M, then the converse holds.

Proof. Let A be a submodule of $M/radM$. There exists a submodule P of M such that $P\varphi = A$. Since M satisfies (S), there are submodules C and D of M such that $M = C \oplus D$ where $C \leq P$ and $P \cap D$ is small in M. Then $C\varphi = P\varphi = A$ and $M/radM = C\varphi \oplus D\varphi = A \oplus D\varphi$. A is a direct summand of $M/radM$. Thus $M/radM$ is semi-simple. Now assume that $M\varphi = A \oplus B$. Choose N such that $N\varphi = A$. Then $M = N + B\varphi^{-1}$. Since M satisfies (S), there exist submodules C and D of M such that $M = C \oplus D$ where $C \leq N$ and $D \cap N$ is small in M. Since $D \cap N \leq radM$, $M = C + B\varphi^{-1}$. By proposition 4, there exists a submodule $Q \subseteq B\varphi^{-1}$ such that $M = C \oplus Q$. Clearly $C\varphi = N\varphi = A$ and $Q\varphi = B$.

Conversely assume that (1) and (2) hold and $radM$ is small. Let N be a submodule of M. Then there exists a direct summand Q of M such that $M\varphi = N\varphi \oplus Q\varphi$. Since $radM$ is small, this means $N \cap Q$ is small and $M = N + Q$. By proposition 4, $M = P \oplus Q$ where $P \subseteq N$. □

Proposition 0.2.1. *Let M be quasi-projective and $M = M_1 \oplus M_2$ a direct sum of modules M_1, M_2. If $M_1$and M_2 satisfies (S), then so is M.*

Proof. Let $N \leq M$. We show that there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is small in M.

Case(1). If $M_1 \cap (N + M_2) = 0$, then $N \leq M_2$. Since M_2 satisfies (S), there exists $B_1 \leq N$ such that $M_2 = B_1 \oplus B_2$ and $N \cap B_2$ is small in M_2. Hence $M = M_1 \oplus B_1 \oplus B_2$ and $N \cap B_2$ is small in M_2. So $N \cap B_2$ is small in M.

Case(2). If $M_1 \cap (N + M_2) \neq 0$, then M_1has a decomposition $M_1 = A_1 \oplus A_2$ such that $A_1 \leq M_1 \cap (N + M_2)$ and $M_1 \cap (N + M_2) \cap A_2 = A_2 \cap (N + M_2)$ is small in M. Then $M = A_1 \oplus A_2 \oplus M_2 = N + (M_2 \oplus A_2)$. Assume $M_2 \cap (N + A_2) = 0$. Since $N \cap A_2 \leq A_2$ and A_2satisfies (S) by proposition 6, A_2 has a decomposition $A_2 = C_1 \oplus C_2$ such that $C_1 \leq N \cap A_2$ and $N \cap A_2 \cap C_2 = N \cap C_2$ is small in M_1. Then $M = (A_1 \oplus C_1) \oplus (C_2 \oplus M_2) = N + (C_2 + M_2)$. Since M is quasi-projective, there exists $N' \leq N$ such that $M = N' \oplus C_2 \oplus M_2$. Since $M_2 \cap (N + A_2) = 0$, we have $N \cap (C_2 \oplus M_2) = N \cap C_2$ is small in M_1.

Assume $M_2 \cap (N + A_2) \neq 0$. Then M_2 has a decomposition $M_2 = B_1 \oplus B_2$, such that $B_1 \leq M_2 \cap (N + A_2)$ and $B_2 \cap (N + A_2)$ is small in M_2. Then $M = N + A_2 + B_2 = (A_1 \oplus B_1) \oplus (A_2 \oplus B_2)$. Since M is quasi-projective, there exists $N' \leq N$ such that $M = N' \oplus A_2 \oplus B_2$. Since $B_2 \cap (N + A_2)$ is small in M, $N \cap (A_2 \oplus B_2)$ is small in M. □

Proposition 0.2.2. *Let M be a module such that $M \neq radM$.*

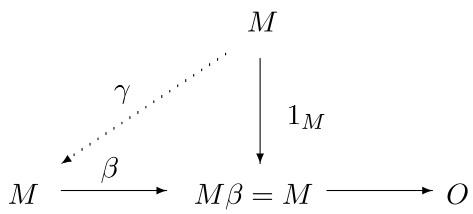
(a) M satisfies (S) and M is indecomposable if and only if M is local.

(b) moreover if M is self-projective and satisfies (a), then $End(M)$ is local.

Proof. (a) assume that M is a local module. M has the unique maximal submodule $radM$. Since $radM$ is small in M, M satisfies (S) and M is indecomposable. Conversely M satisfies (S) and M is indecomposable. Let N be a proper submodule of M. Then $M = A \oplus B$, $A \leq N$ and $B \cap N$ is small in M. Since $A \neq M$, $B = M$. $N = B \cap N \subset radM$. $radM$ is the unique maximal submodule of M. M is local.

(b) By (a), if $x \notin radM$, then $Rx = M$.

Let $A = \{\alpha \in EndM \mid M\alpha \subseteq radM\}$. A is an ideal of $EndM$. Let $\beta \in EndM$ and $\beta \notin A$, $M\beta = M$.



Since M is self-projective, $\gamma\beta = 1_M$. β has a left inverse. Thus $EndM$ is a local ring. □

THEOREM 0.3. [3] *Let M be a module. Write $E = EndM$ and put $\{\alpha \in E \mid M\alpha$ is small in $M\}$.*

If M is direct-projective, then $A \subseteq J(E)$, moreover E is semi-regular and $A = J(E)$ if and only if $M\alpha$ lies over a summand of M for all $\alpha \in E$.

COROLLARY 0.4. *Let M be a direct projective module. If M satisfies (S), then $E = EndM$ is semi-regular and $A = J(E)$.*

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