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ON INEQUALITIES OF GRONWALL TYPE

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Abstract. In this paper, we improve the results of [9] and give an application to boundedness of the solutions of nonlinear integro-differential equations.

1. Introduction

Inequalities of Gronwall type allow one to estimate a function that is known to satisfy a certain differential or integral inequality. Also, they play a very important role in the study of differential and integral equations.

Growall [6] in 1919 proved the following :

Let $u(t)$ be a continuous real valued function on $J = [\alpha, \alpha + h]$ and let

$$0 \le u(t) \le \int_{\alpha}^{t} [a + bu(s)]ds \ \text{ on } J,$$

where a and b are nonnegative constants. Then

$$u(t) \le ahe^{bh} \ \text{ on } J.$$

Another inequality of this type was proved in 1943 by Bellman [2] : see Lemmma 2.1 in Section 2.

It is clear that Bellman's result contains that of Gronwall. Inequalities of this type were called "Gronwall inequalites" or "Inequalites of Gronwall type ", " Bellman's lemma or inequality" or sometimes "Bellman-Gronwall" or "Gronwall-Bellman" inequalites (see [8]).

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Due to various motivations, several generalizations and applications of this inequality have been obtained [1, 2, 7, 8]. Pachpatte [10, 11] obtained some general versions of this inequality. Oguntuase [9] established some generalizations of the inequalities obtained in [10]. However, there are some defects in the proofs of Theorems 2.1 and 2.7 in [9]. In this paper, we improve the results of [9] and give an application to boundedness of the solutions of nonlinear integro-differential equations.

2. Main results

Bellman [2] established the following basic integral inequality known as Gronwall-Bellman inequality.

LEMMA 2.1. *Let $u(t)$ and $g(t)$ be nonnegative continuous functions on $\mathbb{R}_+ = [0, \infty)$ for which the inequality*

$$u(t) \leq c + \int_a^t g(s)u(s)ds,\ t \in \mathbb{R}_+$$

holds, where c is a nonnegative constant. Then

$$u(t) \leq x \exp\left(\int_a^t g(s)ds\right)_,\ t \in \mathbb{R}_+.$$

We need to modify Theorem 2.1 in [9] into the following :

THEOREM 2.2. *Let $u(t), f(t)$ be nonnegative continuous functions in an interval $I = [a, b]$. Suppose that $k(t, s)$ and $k_t(t, s)$ exist and are nonnegative continuous functions for almost $t, s \in I$. If the inequality*

$$u(t) \leq c + \int_a^t f(s)u(s)ds + \int_a^t f(s)\left(\int_a^s k(s,\tau)u(\tau)d\tau\right)ds, t \in I \tag{2.1}$$

holds, where c is a nonnegative constant, then

$$u(t) \leq c\left[1 + \int_a^t f(s)\exp\left(\int_a^s (f(\tau) + k(\tau,\tau) + \int_a^\tau k_\tau(\tau,\sigma)d\sigma)d\tau\right)ds\right].$$

Proof. Define a function $v(t)$ by the right hand side of (2.1). Then

$$\begin{aligned} v'(t) &= f(t)u(t) + f(t)\int_a^t k(t,\tau)u(\tau)d\tau,\ v(a) = c \\ &\leq f(t)(v(t) + \int_a^t k(t,\tau)v(\tau)d\tau). \qquad (2.2) \end{aligned}$$

Letting

$$m(t) = v(t) + \int_a^t k(t,\tau)v(\tau)d\tau,\ m(a) = v(a) = c,$$

which implies

$$\begin{aligned} m'(t) &= v'(t) + k(t,t)v(t) + \int_a^t k_t(t,\tau)v(\tau)d\tau \\ &\le f(t)m(t) + k(t,t)m(t) + \int_a^t k_t(t,\tau)m(\tau)d\tau \\ &\le (f(t) + k(t,t) + \int_a^t k_t(t,\tau)d\tau)m(t). \end{aligned} \tag{2.3}$$

Integrate (2.3) from a to t, we obtain

$$m(t) \le c\exp\left(\int_a^t f(s) + k(s,s) + \int_a^s k_s(s,\sigma)d\sigma ds\right). \tag{2.4}$$

Substitute (2.4) into (2.2) and then integrate it from a to t, we have

$$v(t) \le c\left[1 + \int_a^t f(s)\exp\left(\int_a^s (f(\tau) + k(\tau,\tau) + \int_a^\tau k_\tau(\tau,\sigma)d\sigma)d\tau\right)ds\right].$$

Hence the proof is complete. □

Remark 2.3. Theorem 2.2 is useful to correct Theorem 2.1 and Theorem 2.2 in [9].

Letting $K(t,s) = h(t)g(s)$ in Theorem 2.2, we obtain the following corollary.

Corollary 2.4. *Let $u(t), f(t), h(t)$ and $g(t)$ be nonnegative continuous functions in $I = [a,b]$. Suppose that $h'(t)$ exists and is nonnegative continuous function. If the following inequality*

$$u(t) \le c + \int_a^t f(s)u(s)ds + \int_a^t f(s)h(s)(\int_a^s g(\tau)u(\tau)d\tau)ds, t \in I$$

holds, where c is a nonnegative constant, then

$$u(t) \le c\left[1 + \int_a^t f(s)\exp(\int_a^s (f(\tau) + h(\tau)g(\tau) + h'(\tau)\int_a^\tau g(\sigma)d\sigma)d\tau)ds\right].$$

We will prove the following theorem by comparison with a differential equation of Bernoulli type. To this end we need the following lemma.

LEMMA 2.5. [1, Lemma 4.1.] *Let $v(t)$ be a positive differentiable function satisfying the inequality*

$$v'(t) \leq b(t)v(t) + k(t)v^p(t),\ t \in [a,b], \tag{2.5}$$

where the functions b and k are continuous in $[a,b]$, and $0 \leq p \neq 1$ is a constant. Then

$$v(t) \leq \exp\left(\int_a^t b(s)ds\right)\left[v^q(a) + q\int_a^t k(s)\exp\left(-q\int_a^s b(\tau)d\tau\right)ds\right]^{\frac{1}{q}}$$

for $t \in [a, b_1)$, where $q = 1-p$ and b_1 is chosen so that the following expression

$$v^q(a) + q\int_a^t k(s)\exp\left(-q\int_a^s b(\tau)d\tau\right)ds$$

is positive in the subinterval $[a, b_1)$.

REMARK 2.6. Note that (2.5) with $p = 1$ in Lemma 2.5 implies $v(t) \leq v_0 \exp(\int_a^t (b(s) + k(s))ds)$ for $t \geq a$. We obtain a comparison result for linear differential inequalities.

THEOREM 2.7. *Let $u(t), f(t)$ be nonnegative continuous functions in an interval $I = [a,b]$. Suppose that $k(t,s)$ and $k_t(t,s)$ exist and are nonnegative continuous functions for almost $t, s \in I$. If the inequality*

$$u(t) \leq c + \int_a^t f(s)u(s)ds + \int_a^t f(s)(\int_a^s k(s,\tau)u^p(\tau)d\tau)ds, t \in I \tag{2.6}$$

holds, where $0 \leq p \neq 1$ and c is a nonnegative constant, then

$$\begin{aligned} u(t) \leq c + \int_a^t &\left\{f(s)\exp\left(\int_a^s (f(\tau)d\tau\right)\times\right. \\ &\left.\left[c^q + q\int_a^s k(\tau,\tau)\exp\left(-q\int_a^\tau k_\tau(\tau,\sigma)d\sigma)d\tau\right)\right]^{\frac{1}{q}}\right\}ds, \end{aligned} \tag{2.7}$$

where $q = 1-p$.

Proof. Define $v(t)$ by the right member of (2.6). Then

$$\begin{aligned} v'(t) &= f(t)u(t) + f(t)\int_a^t k(t,\tau)u^p(\tau)d\tau,\ v(a) = c \\ &\leq f(t)(v(t) + \int_a^t k(t,\tau)v^p(\tau)d\tau), \end{aligned} \tag{2.8}$$

by $u(t) \leq v(t)$ and $u^p(t) \leq v^p(t)$ for $0 \leq p \neq 1$. Letting

$$m(t) = v(t) + \int_a^t k(t,\tau)v^p(\tau)d\tau,\ m(a) = v(a) = c,$$

which implies

$$\begin{aligned} m'(t) &= v'(t) + k(t,t)v^p(t) + \int_a^t k_t(t,\tau)v^p(\tau)d\tau \\ &\leq f(t)m(t) + k(t,t)m^p(t) + \int_a^t k_t(t,\tau)m^p(\tau)d\tau \\ &\leq f(t)m(t) + \left[k(t,t) + \int_a^t k_t(t,\tau)d\tau\right] m^p(t). \end{aligned} \tag{2.9}$$

By Lemma 2.5 we have

$$\begin{aligned} &m(t) \leq \exp\left(\int_a^t f(s)ds\right)[m^q(a) + q\int_a^t ((k(s,s) \\ &+ \int_a^s k_s(s,\sigma)d\sigma)\exp(-q\int_a^s f(\tau)d\tau))ds]^{\frac{1}{q}}, \end{aligned} \tag{2.10}$$

where $q = 1-p$. Substitute (2.10) into (2.8) and then integrate it from a to t, we have

$$\begin{aligned} v(t) &\leq c + \int_a^t \Big\{f(s)\exp\left(\int_a^s f(\tau)d\tau\right)\Big[c^q \\ &+ q\int_a^s \left((k(\tau,\tau) + \int_a^\tau k_\tau(\tau,\sigma)d\sigma)\exp[-q\int_a^\tau f(\sigma)d\sigma]\right)d\tau\Big]^{\frac{1}{q}}\Big\}ds, \end{aligned}$$

where $q = 1-p$. Hence the proof is complete. □

Corollary 2.8. *Let $u(t), f(t), h(t)$ and $g(t)$ be nonnegative continuous functions in $I = [a,b]$. Suppose that $h'(t)$ exists and is nonnegative continuous function. If the following inequality*

$$u(t) \leq c + \int_a^t f(s)u(s)ds + \int_a^t f(s)h(s)(\int_a^s g(\tau)u^p(\tau)d\tau)ds, t \in I$$

holds, where $0 \leq p \neq 1$ and c is a positive constant, then

$$\begin{aligned} &u(t) \leq c + \int_a^t f(s)\exp\left(\int_a^s (f(\tau)d\tau\right)[c^q + q\int_a^s [h(\tau)g(\tau) \\ &+h'(\tau)\int_a^\tau g(\sigma)d\sigma]\exp\left(-q\int_a^\tau k_\tau(\tau,\sigma)d\sigma)d\tau\right)ds]^{\frac{1}{q}}, t \in I \end{aligned}$$

where $q = 1-p$.

3. Application

We give an application of our results. We consider the linear integro-differential equation

$$x'(t) = A(t)x(t) + \int_{t_0}^{t} B(t,s)x(s)ds, \tag{3.1}$$

where $A(t)$ and $B(t,s)$ are continuous $n \times n$ matrices on $\mathbb{R}_+$ and $\mathbb{R}_+^2$, respectively, and its perturbed equation

$$u'(t) = A(t)u(t) + \int_{t_0}^{t} B(t,s)u(s)ds + h(t, u(t), \int_{t_0}^{t} k(t,s,u(s))ds), \tag{3.2}$$

where $k \in C(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $h \in C(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$.

Let $x(t) = x(t, t_0, x_0)$ and $u(t) = u(t, t_0, x_0)$ be denoted by the unique solutions of (3.1) and (3.2), respectively with $x(t_0) = u(t_0) = x_0$. Then the unique solution $u(t)$ of (3.2) is given by

$$u(t) = R(t,t_0)x_0 + \int_{t_0}^{t} R(t,s)h(s, u(s), \int_{t_0}^{s} k(s,\sigma,u(\sigma))d\sigma)ds, \ t \geq t_0,$$

where $R(t,s)$ is the solution of the initial value problem

$$\frac{\partial R}{\partial s}(t,s) + R(t,s)A(s) + \int_{s}^{t} R(t,\sigma)B(\sigma,s)d\sigma, \tag{3.3}$$

$R(t,t) = I$ for $0 \leq s \leq t < \infty$, I being the identity matrix (see [7]).

Theorem 3.1. *Suppose that the following inequalities hold:*

(i) $|R(t,s)| \leq Me^{-\alpha(t-s)}$,
(ii) $|R(t,s)h(s,u,z)| \leq p(s)(|u| + |z|)$,
(iii) $|k(t,s,u)| \leq q(t,s)|u|, \ t \geq s \geq 0$,

where $M \geq 1$ and $\alpha > 0$ are constants. If $p(t), q(t,t)$ and $q_t(t,\tau)$ are continuous and nonnegative, and satisfies

$$\int_0^\infty p(s)ds < \infty, \ \int_0^\infty (q(\tau,\tau) + \int_{t_0}^{\tau} q_\tau(\tau,\sigma)d\sigma)d\tau < \infty,$$

then all solutions of (3.2) are bounded in $\mathbb{R}_+$.

Proof. By the assumptions, we obtain

$$\begin{aligned}|u(t)| &\le |R(t,t_0)||x_0| + \int_{t_0}^{t} |R(t,s)h(s,u(s),\int_{t_0}^{s} k(s,\sigma,u(\sigma))d\sigma)|ds \\ &\le M|x_0| + \int_{t_0}^{t} p(s)(|u(s)| + \int_{t_0}^{s} q(s,\sigma)|u(\sigma)|d\sigma)ds \\ &\le M|x_0| + \int_{t_0}^{t} p(s)|u(s)|ds + \int_{t_0}^{t} p(s)\int_{t_0}^{s} q(s,\sigma)|u(\sigma)|d\sigma ds\end{aligned}$$

for all $t \ge t_0$. By Theorem 2.2, we have

$$\begin{aligned}|u(t)| &\le M|x_0|[1 + \int_{t_0}^{t} p(s)\exp(\int_{t_0}^{s}(p(\tau) \\ &+ q(\tau,\tau) + \int_{t_0}^{\tau} q_\tau(\tau,\sigma)d\sigma)d\tau)ds] \\ &\le d|x_0|,\end{aligned}$$

where $d = M[1 + \int_{t_0}^{\infty} p(s)\exp(\int_{t_0}^{s}(p(\tau) + q(\tau,\tau) + \int_{t_0}^{\tau} q_\tau(\tau,\sigma)d\sigma)d\tau)ds]$. Hence the solution $u(t)$ of (3.2) is bounded in $\mathbb{R}_+$ and the proof is complete. □

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