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A NOTE ON THE AP-DENJOY INTEGRAL

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Abstract. In this paper, we define the ap-Denjoy integral and investigate some properties od the ap-Denjoy integral. In particular, we show that a function $f : [a, b] \to \mathbb{R}$ is ap-Denjoy integrable on $[a, b]$ if and only if there exists an ACG_s function F on $[a, b]$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$.

1. Introduction and preliminaries

For a measurable set E of real numbers we denote by $|E|$ its Lebesgue measure. Let E be a measurable set ant let c be a real number. The *density* of E at c is defined by

$$d_c E = \lim_{h \to 0^+} \frac{\big|E \cap (c-h, c+h)\big|}{2h} \quad ,$$

provided the limit exists. The point c is called a *point of density* of E if $d_c E = 1$ and a *point of dispersion* of E if $d_c E = 0$. The set E^d represents the set of all points $x \in E$ such that x is a point of density of E. A function $F : [a, b] \to \mathbb{R}$ is said to be *approximately differentiable* at $c \in [a, b]$ if there exists a measurable set $E \subseteq [a, b]$ such that $c \in E^d$ and

$$\lim_{\substack{x \to c \\ x \in E}} \frac{F(x) - F(c)}{x - c}$$

exists. The approximate derivative of F at c is denoted by $F'_{ap}(c)$.

An *approximate neighborhood*(or ap-nbd) of $x \in [a, b]$ is a measurable set $S_x \subseteq [a, b]$ containing x as a point of density. For every $x \in E \subseteq [a, b]$, choose an ap-nbd $S_x \subseteq [a, b]$ of x. Then we say that $S = \{S_x : x \in E\}$

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is a *choice* on E. A tagged interval $(x, [c, d])$ is said to be *subordinate* to the choice $S = \{S_x\}$ if $c, d \in S_x$. Let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \le i \le n\}$ be a finite collection of non-overlapping tagged intervals. If $(x_i, [c_i, d_i])$ is subordinate to a choice S for each i, then we say that $\mathcal{P}$ is subordinate to S. Let $E \subseteq [a, b]$. If $\mathcal{P}$ is subordinate to S and each $x_i \in E$, then $\mathcal{P}$ is called E-subordinate to S. If $\mathcal{P}$ is subordinate to S and $[a, b] = \bigcup_{i=1}^{n} [c_i, d_i]$, then we say that $\mathcal{P}$ is a tagged partition of $[a, b]$ that is subordinate to S.

2. The ap-Denjoy integral

We introduce the notion of the approximate Lusin function. This function is used to define the ap-Denjoy integral.

For a function $F : [a, b] \to \mathbb{R}$, F can be treated as a function of intervals by defining $F([c, d]) = F(d) - F(c)$.

DEFINITION 2.1. *Let $F : [a, b] \to \mathbb{R}$ be a function. The function F is an approximate Lusin function(or F is an AL function) on $[a, b]$ if for every measurable set $E \subseteq [a, b]$ of measure zero and for every $\varepsilon > 0$ there exists a choice S on E such that $|(\mathcal{P}) \sum F(I)| < \varepsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is E-subordinate to S.*

A function $F : [a, b] \to \mathbb{R}$ is AC_s on a measurable set $E \subseteq [a, b]$ if for each $\varepsilon > 0$ there exist a positive number δ and a choice S on E such that $|(\mathcal{P}) \sum F(I)| < \varepsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is E-subordinate to S and satisfies $(\mathcal{P}) \sum |I| < \delta$. The function F is ACG_s on E if E can be expressed as a countable union of measurable sets on each of which F is AC_s.

LEMMA 2.2. *If $F : [a, b] \to \mathbb{R}$ is ACG_s on $[a, b]$, then F is an AL function on $[a, b]$.*

Proof. Suppose that $E \subseteq [a, b]$ is a measurable set of measure zero. Let $E = \cup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a sequence of disjoint measurable sets and F is AC_s on each E_n. Let $\varepsilon > 0$. For each positive integer n, there exists a choice $S^n = \{S_x^n : x \in E_n\}$ on E_n and a positive number δ_n such that $|(\mathcal{P}) \sum F(I)| < \epsilon/2^n$ whenever $\mathcal{P}$ is E_n-subordinate to S^n and $(\mathcal{P}) \sum |I| < \delta_n$. For each positive integer n, choose an open set O_n such that $E_n \subseteq O_n$ and $|O_n| < \delta_n$. Let $S_x = S_x^n \cap \big(x - \rho(x, O_n^c), x + \rho(x, O_n^c)\big)$ for each $x \in E_n$, where $\rho(x, O_n^c)$ is the distance from x to $O_n^c = [a, b] - O_n$.

Then $S = \{S_x : x \in E\}$ is a choice on E. Suppose that $\mathcal{P}$ is E-subordinate to S. Let $\mathcal{P}_n$ be a subset of $\mathcal{P}$ that has tags in E_n and note that $(\mathcal{P}_n)\sum |I| < |O_n| < \delta_n$. Hence, we have

$$|(\mathcal{P})\sum F(I)| \leq \sum_{n=1}^{\infty} |(\mathcal{P}_n)\sum F(I)| < \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon \ .$$

□

Definition 2.3. *A function $f : [a,b] \to \mathbb{R}$ is ap-Denjoy integrable on $[a,b]$ if there exists an AL function F on $[a,b]$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$. The function f is ap-Denjoy integrable on a measurable set $E \subseteq [a,b]$ if $f\chi_E$ is ap-Denjoy integrable on $[a,b]$.*

If we add the condition $F(a) = 0$, then the function F is unique. We will denote this function $F(x)$ by $(AD)\int_a^x f$.

It is easy to show that if $f : [a,b] \to \mathbb{R}$ is ap-Denjoy integrable on $[a,b]$, then f is ap-Denjoy integrable on every subinterval of $[a,b]$. This gives rise to an interval function F such that $F(I) = (AD)\int_I f$ for every subinterval $I \subseteq [a,b]$. The function F is called the primitive of f.

Recall that the function $F : [a,b] \to \mathbb{R}$ is AC_* on a measurable set $E \subseteq [a,b]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $(\mathcal{P})\sum \omega(F,I) < \varepsilon$ for every finite collection $\mathcal{P}$ of non-overlapping intervals that have endpoints in E and satisfy $(\mathcal{P})\sum |I| < \delta$, where $\omega(F,I) = \sup\{|F(y) - F(x)| : x, y \in I\}$. The function F is ACG_* on E if $F|_E$ is continuous on E, $E = \cup_{n=1}^{\infty} E_n$ and F is AC_* on each E_n. It is easy to show that if F is ACG_* on $[a,b]$, then F is ACG_s on $[a,b]$. A function $f : [a,b] \to \mathbb{R}$ is *Denjoy integrable* on $[a,b]$ if there exists an ACG_* function $F : [a,b] \to \mathbb{R}$ such that $F' = f$ almost everywhere on $[a,b]$.

The following theorem shows that the ap-Denjoy integral is an extension of the Denjoy integral.

Theorem 2.4. *If $f : [a,b] \to \mathbb{R}$ is Denjoy integrable on $[a,b]$, then f is ap-Denjoy integrable on $[a,b]$.*

Proof. Suppose that $f : [a,b] \to \mathbb{R}$ is Denjoy integrable on $[a,b]$. Then there exists an ACG_* function $F : [a,b] \to \mathbb{R}$ such that $F' = f$ almost everywhere on $[a,b]$. Since F is ACG_s on $[a,b]$, by Lemma 2.2 F is an AL function on $[a,b]$ and $F'_{ap} = F' = f$ almost everywhere on $[a,b]$. Hence, f is ap-Denjoy integrable on $[a,b]$. □

There exists a function that is ap-Denjoy integrable on $[a,b]$, but not Denjoy integrable on $[a,b]$.

EXAMPLE 2.5. *Let $\{(a_n, b_n)\}$ be a sequence of disjoint open intervals in (a, b) with the following properties ;*

(1) $b_1 < b$ *and* $b_{n+1} < b_n$ *for all* n*;*
(2) $\{a_n\}$ *converges to* a*;*
(3) a *is a point of dispersion of* $O = \cup_{n=1}^{\infty}(a_n, b_n)$.

Define $F : [a, b] \to \mathbb{R}$ by $F(x) = 0$ for all $x \in [a, b] - O$ and

$$F(x) = sin^2\Big(\frac{x - a_n}{b_n - a_n}\pi\Big)$$

for $x \in (a_n, b_n)$. Then it is easy to show that the function F is differentiable on $(a, b]$ and approximately differentiable at a, but F is not continuous at a. Hence $F' = F'_{ap}$ almost everywhere on $[a, b]$, but F'_{ap} is not Denjoy integrable on $[a, b]$, since F is not continuous on $[a, b]$.

To show that F'_{ap} is ap-Denjoy integrable on $[a, b]$, it is sufficient to show that F is an AL function on $[a, b]$. Let E be a measurable set in $[a, b]$ of measure zero and let $\varepsilon > 0$. For each positive integer n, choose an open set O_n such that $E \cap [a_n, b_n] \subseteq O_n$ and $|O_n| < (b_n - a_n)\varepsilon/\pi 2^{n+1}$.

For each $x \in E$, define

$$S_x = \begin{cases} [a, b] - \cup_{n=1}^{\infty}(a_n, b_n) & \text{if } x = a; \\ (b_{n+1}, a_n) & \text{if } b_{n+1} < x < a_n, \ \ n = 1, 2, \cdots; \\ \big(x - \rho(x, O_n^c), x + \rho(x, O_n^c)\big) & \text{if } a_n \le x \le b_n, \ \ n = 1, 2, \cdots; \\ (b_1, b] & \text{if } b_1 < x \le b. \end{cases}$$

Then $S = \{S_x : x \in E\}$ is a choice on E. Let $\mathcal{P}$ be a finite collection of non-overlapping tagged intervals that is E-subordinate to S. Then we have

$$\begin{aligned} (\mathcal{P})\sum |F([c,d])| &= \sum_{n=1}^{\infty} \sum_{x \in (b_{n+1}, a_n)} |F([c, d])| + \sum_{n=1}^{\infty} \sum_{x \in [a_n, b_n]} |F([c, d])| \\ &\le \sum_{n=1}^{\infty} \sum_{x \in [a_n, b_n]} \frac{2\pi(d - c)}{b_n - a_n} \le \sum_{n=1}^{\infty} \frac{2\pi}{b_n - a_n}|O_n| < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon. \end{aligned}$$

Hence, F is an AL function on $[a, b]$.

THEOREM 2.6. *Let $f : [a, b] \to \mathbb{R}$ be ap-Denjoy integrable on $[a, b]$ and let $F(x) = (AD) \int_a^x f$ for each $x \in [a, b]$. Then*

(a) the function F is approximately differentiable almost everywhere on $[a, b]$ and $F'_{ap} = f$ almost everywhere on $[a, b]$; and

(b) the functions F and f are measurable.

Proof. (a) follows from the definition of the ap-Denjoy integral. Since F is approximately continuous almost everywhere on $[a, b]$, F is measurable by [[4], Theorem 14.7]. It follows from [[4], Theorem 14.12] that f is measurable. □

THEOREM 2.7. *Let $f : [a, b] \to \mathbb{R}$ and let $c \in (a, b)$.*

(a) If f is ap-Denjoy integrable on $[a, b]$, then f is ap-Denjoy integrable on every subinterval of $[a, b]$.

(b) If f is ap-Denjoy integrable on each of the intervals $[a, c]$ and $[c, b]$, then f is ap-Denjoy integrable on $[a, b]$ and

$$(AD)\int_a^b f = (AD)\int_a^c f + (AD)\int_c^b f.$$

Proof. (a) Let $[c, d]$ be any subinterval of $[a, b]$. Let $F(x) = (AD)\int_a^x f$. Since F is an AL function on $[a, b]$ and $F'_{ap} = f$ almost everywhere on $[a, b]$, F is an AL function on $[c, d]$ and $F'_{ap} = f$ almost everywhere on $[c, d]$. Hence, f is ap-Denjoy integrable on $[c, d]$.

(b) Since f is ap-Denjoy integrable on each of intervals $[a, c]$ and $[c, b]$, there exist AL functions F and G such that $F'_{ap} = f$ almost everywhere on $[a, c]$ and $G'_{ap} = f$ almost everywhere on $[c, b]$ respectively. Define $H : [a, b] \to \mathbb{R}$ by

$$H(x) = \begin{cases} F(x), & \text{if } x \in [a, c] \\ F(c) + G(x), & \text{if } x \in (c, b]. \end{cases}$$

Then H is an AL function on $[a, b]$ and $H'_{ap} = f$ almost everywhere on $[a, b]$. Hence f is ap-Denjoy integrable on $[a, b]$ and $H(b) = F(c) + G(b)$, i.e.,

$$(AD)\int_a^b f = (AD)\int_a^c f + (AD)\int_c^b f.$$

□

We can easily get the following theorem.

THEOREM 2.8. *Suppose that f and g are ap-Denjoy integrable on $[a, b]$. Then*

(a) kf is ap-Denjoy integrable on $[a, b]$ and $(AD)\int_a^b kf = k(AD)\int_a^b f$ for each $k \in \mathbb{R}$

(b) $f + g$ is ap-Denjoy integrable on $[a, b]$ and $(AD)\int_a^b (f + g) = (AD)\int_a^b f + (AD)\int_a^b g$

(c) if $f \le g$ almost everywhere on $[a, b]$, then $(AD)\int_a^b f \le (AD)\int_a^b g$

(d) if $f = g$ almost everywhere on $[a, b]$, then $(AD)\int_a^b f = (AD)\int_a^b g$.

We can easily get the following theorem.

Theorem 2.9. *Let $f : [a, b] \to \mathbb{R}$ be ap-Denjoy integrable on $[a, b]$.*
(a) If f is bounded on $[a, b]$, then f is Lebesgue integrable on $[a, b]$.
(b) If f is nonnegative on $[a, b]$, then f is Lebesgue integrable on $[a, b]$.
(c) If f is ap-Denjoy integrable on every measurable subset of $[a, b]$, then f is Lebesgue integrable on $[a, b]$.

The ap-Denjoy integral is a nonabsolute integral.

Remark 2.10. *Let f be ap-Denjoy integrable on $[a, b]$, but not Lebesgue integrable on $[a, b]$. Then $|f|$ is not ap-Denjoy integrable. If $|f|$ is ap-Denjoy integrable on $[a, b]$, then $|f|$ is Lebesgue integrable on $[a, b]$ since $|f|$ is nonnegative on $[a, b]$. It follows that f is also Lebesgue integrable on $[a, b]$.*

Theorem 2.11. *A function $f : [a, b] \to \mathbb{R}$ is ap-Denjoy integrable on $[a, b]$ if and only if there exists an ACG_s function F on $[a, b]$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$.*

Proof. Suppose that there exists an ACG_s function F on $[a, b]$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$. Then F is an AL function by Lemma 2.2. Hence, f is ap-Denjoy integrable on $[a, b]$.

Conversely, suppose that f is ap-Denjoy integrable on $[a, b]$ and let $F(x) = (AD) \int_a^x f$ for each $x \in [a, b]$. Then F is an AL function such that $F'_{ap} = f$ almost everywhere on $[a, b]$. Let $E = \{x \in [a, b] : F'_{ap}(x) \neq f(x)\}$. Then $\mu(E) = 0$. Since F is an AL function, F is AC_s on E. For each positive integer n, let

$$E_n = \{x \in [a, b] - E \mid n - 1 \leq |f(x)| < n\}.$$

Note that each E_n is measurable. Fix n and let $\varepsilon > 0$. Since F is approximately differentiable for each $x \in E_n$, there exists a measurable set A_x containing x as a point of density and a positive number δ_x such that

$$\left| \frac{F(y) - F(x)}{y - x} - f(x) \right| < \varepsilon$$

i.e.,

$$|F(y) - F(x) - f(x)(y - x)| < \varepsilon |y - x|,$$

if $y \in A_x \cap (x - \delta_x, x + \delta_x)$. For each $x \in E_n$, let

$$S_x = A_x \cap (x - \delta_x, x + \delta_x).$$

Then $S = \{S_x : x \in E_n\}$ is a choice on E_n. Suppose that $\mathcal{P}$ is a finite collection of non-overlapping tagged intervals that is E_n-subordinate to S and satisfies $\mu(\mathcal{P}) < \frac{\varepsilon}{n}$. Then since $|F(\mathcal{P}) - f(\mathcal{P})| < \varepsilon\mu(\mathcal{P})$, we have

$$\begin{aligned}|F(\mathcal{P})| &\leq |F(\mathcal{P}) - f(\mathcal{P})| + |f(\mathcal{P})| \\ &< \varepsilon\mu(\mathcal{P}) + n\mu(\mathcal{P}) \\ &< (b - a + 1)\varepsilon.\end{aligned}$$

Hence, F is AC_s on E_n. Since $[a, b] = [\cup_{n=1}^{\infty} E_n] \cup E$, F is ACG_s on $[a, b]$. $\square$

THEOREM 2.12. *Let $f : [a, b] \to \mathbb{R}$ be ap-Denjoy integrable on $[a, b]$ and let $F(x) = (AD) \int_a^x f$ for each $x \in [a, b]$. Then F is approximately continuous on $[a, b]$.*

Proof. From the definition of the ap-Denjoy integral, F is approximately differentiable almost everywhere on $[a, b]$. Let E be the set of all non-differentiable points in $[a, b]$. Then E is a measurable set of measure zero. Since F is approximately continuous on $[a, b] - E$, it is sufficient to show that F is approximately continuous on E. Let $c \in E$ and let $\varepsilon > 0$. Since F is an AL function, there exists a choice $S = \{S_x : x \in E\}$ such that $|(\mathcal{P})\Sigma F(I)| < \varepsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is E-subordinate to S. If $x \in S_c \cap (c - \eta, c + \eta)$ for some $\eta > 0$, then the tagged interval $(c, [c, x])$ (or $(c, [x, c])$) is E-subordinate to S. Hence, $|F(x) - F(c)| = |F([c, x])| < \varepsilon$. This shows that F is approximately continuous on E. $\square$

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