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ON THE CONTINUITY OF THE ZADEH EXTENSIONS

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Abstract. In this paper, we prove the continuity of the Zadeh extensions for continuous surjections and for semiflows.

1. Introduction

The Zadeh extension is the way we produce a fuzzy transformation $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ for a given function $f : X \to X$. The Zadeh extension $\hat{f}$ of a function f has been studied and applied by many authors, including Barros et al. [1, 7, 8], Cabrelli et al. [2] in the study of fuzzy fractals and Nguyen [5] in set–representation of fuzzy sets.

We are interested in the study of Fuzzy Dynamical Systems applied to biological population dynamics, and it is our objective in the future, to model some biological phenomenal using difference equations playing an important role in the dynamics of the process. In this sense the Zadeh extension will appear as a fundamental tool relating the classical and fuzzy models.

The main result in [8] establishs that if a function $f : R^n \to R^n$ is continuous, then the Zadeh extension $\hat{f} : (\mathcal{F}(R^n), D) \to (\mathcal{F}(R^n), D)$ is also continuous and conversely, where D is a normal metric on $\mathcal{F}(R^n)$.

In this paper we investigate that the results [8] are true taking a locally compact metric space X instead of R^n, being the generalizations.

The structure of this paper is as follows. In Section 2, if a function $f : X \to X$ is a continuous surjection, then we prove the continuity of the Zadeh extension $\hat{f}$ in D-metric and finally, in Section 3, in the case a semiflow $f : X \times R^+ \to X$, we also establish the continuity of the Zadeh extension $\hat{f}$ in D-metric.

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2. Zadeh's extension of continuous surjections

Let (X, d) be a locally compact metric space. We denote the set of all nonempty compact subsets of X by $\Omega(X)$. For $A, B \in \Omega(X)$, we define

$$h(A, B) = \max\{\beta(A, B), \beta(B, A)\}$$

where $\beta(A, B) = \sup\{d(a, B) \mid a \in A\}$. Then h is a metric on $\Omega(X)$ which is called the Hausdorff metric induced by d.

For $u : X \to [0, 1]$ and $\alpha \in [0, 1]$, we define the α–level set $[u]^\alpha$ of u by

$$[u]^\alpha = \left\{ \begin{array}{ll} \{x \in X | u(x) \geq \alpha\}, & \text{if } \ 0 < \alpha \leq 1 \\ \overline{\{x \in X | u(x) > 0\}}, & \text{if } \ \alpha = 0. \end{array} \right\}$$

The following lemma shows that the α–level set $[u]^\alpha$ is well–defined.

Lemma 2.1. *Let $u, v : X \to [0, 1]$ be the fuzzy sets. Then*

$$u = v \text{ if and only if } [u]^\alpha = [v]^\alpha \text{ for all } \alpha \in [0, 1].$$

Proof. It is clear that if $u = v$ then $[u]^\alpha = [v]^\alpha$ for all $\alpha \in [0, 1]$. If $u \neq v$, then there exists $x \in X$ such that $u(x) \neq v(x)$. We may assume that $u(x) < v(x)$. Choose α with $u(x) < \alpha < v(x)$. Since $x \in [u]^\alpha - [v]^\alpha$, we have $[u]^\alpha \neq [v]^\alpha$. This completes the proof. □

We define the family of the fuzzy sets whose α–level sets are in $\Omega(X)$ and denote $\mathcal{F}(X)$. Let

$$\mathcal{F}(X) = \{u : X \to [0, 1] \mid [u]^\alpha \in \Omega(X) \text{ for all } \alpha \in [0, 1]\}.$$

In this family we define D-metric on $\mathcal{F}(X)$ that we call normal metric. For $u, v \in \mathcal{F}(X)$, we define

$$D(u, v) = \sup\{h([u]^\alpha, [v]^\alpha) \mid \alpha \in [0, 1]\}.$$

The next lemma establishes that D is a metric on $\mathcal{F}(X)$.

Lemma 2.2. *The mapping D is a metric on $\mathcal{F}(X)$.*

Proof. It is clear that $D(u, v) \geq 0$ and $D(u, v) = D(v, u)$ for all $u, v \in \mathcal{F}(X)$. Let $D(u, v) = 0$. For all $\alpha \in [0, 1]$, since $h([u]^\alpha, [v]^\alpha) \leq D(u, v) = 0$, we have $h([u]^\alpha, [v]^\alpha) = 0$. Thus $[u]^\alpha = [v]^\alpha$. By Lemma 2.1, we get $u = v$. Clearly if $u = v$ then $D(u, v) = 0$. Let $u, v, w \in \mathcal{F}(X)$. Since

$$\begin{aligned} h([u]^\alpha, [v]^\alpha) &\leq h([u]^\alpha, [w]^\alpha) + h([w]^\alpha, [v]^\alpha) \\ &\leq D(u, w) + D(w, v) \ \text{ for all } \ \alpha \in [0, 1], \end{aligned}$$

we have $D(u,v) \le D(u,w) + D(w,v)$ and the proof is complete. $\square$

Let $f : X \to X$ be a continuous surjection. We define the Zadeh extension $\hat{f}$ of f by

$$\hat{f}(u)(x) = \sup\{u(y) \mid y \in f^{-1}(x)\}.$$

The following lemma proves that $\hat{f}$ is a well–defined function.

LEMMA 2.3. *Let $f : X \to X$ be a continuous surjection, $\hat{f}$ the Zadeh extension of f and $u \in \mathcal{F}(X)$. Then we have $[\hat{f}(u)]^\alpha = f([u]^\alpha)$ for all $\alpha \in [0,1]$.*

Proof. Let $0 < \alpha \le 1$. For any $x \in [\hat{f}(u)]^\alpha$, we have $\hat{f}(u)(x) \ge \alpha$. If $\hat{f}(u)(x) > \alpha$, then there exists $y \in f^{-1}(x)$ such that $u(y) > \alpha$. Since $y \in [u]^\alpha$, we have $x = f(y) \in f([u]^\alpha)$. If $\hat{f}(u)(x) = \alpha$, then there exists a sequence (y_n) in $f^{-1}(x)$ such that $u(y_n) \to \alpha$. Choose ζ with $0 < \zeta < \alpha$. Since $u(y_n) \to \alpha$, we may assume that $u(y_n) > \zeta$. Since (y_n) is a sequence in $[u]^\zeta$ and $[u]^\zeta$ is a compact set, (y_n) has a convergent subsequence. Let $y_n \to y$. Then $y \in \overline{f^{-1}(x)} = f^{-1}(x)$. We claim that $u(y) \ge \alpha$. If $u(y) < \alpha$, then we can choose ξ such that $u(y) < \xi < \alpha$. Since $u(y_n) \to \alpha$, we may assume that $u(y_n) > \xi$ for all n. Since $y_n \in [u]^\xi$ and $[u]^\xi$ is a compact set, we have $y_n \in [u]^\xi$. Thus $\alpha \le u(y) < \xi$. This is a contradiction. Hence $u(y) \ge \alpha$ and so $y \in [u]^\alpha$. Therefore we get $x = f(y) \in f([u]^\alpha)$.

Conversely, given any $x \in f([u]^\alpha)$, there exists $y \in [u]^\alpha$ such that $x = f(y)$. Since $\hat{f}(u)(x) \ge u(y) \ge \alpha$, we have $x \in [\hat{f}(u)]^\alpha$. Thus we have $[\hat{f}(u)]^\alpha = f([u]^\alpha)$.

Let $\alpha = 0$. Given any $x \in [\hat{f}(u)]^0$, there exists a sequence (x_n) such that $\hat{f}(u)(x_n) > 0$ and $x_n \to x$. By the definition of $\hat{f}(u)(x_n)$, there exists $y_n \in f^{-1}(x_n)$ such that $u(y_n) > 0$. Since (y_n) is a sequence in $[u]^0$ and $[u]^0$ is a compact set, (y_n) has a convergent subsequence. Let $y_n \to y \in [u]^0$. Since f is continuous, we have $f(y_n) \to f(y)$. Since $f(y_n) = x_n \to x$, we get $x = f(y) \in f([u]^0)$.

Conversely, given any $x \in f([u]^0)$, there exists $y \in [u]^0$ such that $x = f(y)$. For each neighborhood U of x, there exists a neighborhood V of y such that $f(V) \subset U$. Since $y \in [u]^0$, we can choose $z \in V$ with $u(z) > 0$. Let $w = f(z)$. Since $w = f(z) \in f(V) \subset U$ and $\hat{f}(u)(w) \ge u(z) > 0$, we have $w \in U \cap \{p \in X \mid u(p) > 0\}$. Thus $x \in [\hat{f}(u)]^0$. Hence we have $[\hat{f}(u)]^0 = f([u]^0)$ and so the proof is complete. $\square$

For our main result we need the following lemma.

LEMMA 2.4. *Let K be a compact subset of X and f a continuous surjection. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that $x \in K$ and $d(x,y) < \delta$ implies $d(f(x), f(y)) < \epsilon$.*

Proof. For each $x \in K$, there exists $\delta_x > 0$ such that

$$d(x,y) < \delta_x \text{ implies } d(f(x), f(y)) < \frac{\epsilon}{2}.$$

$\left\{B(x, \frac{\delta_x}{2}) \mid x \in K\right\}$ is an open cover of K. Since K is compact, there exist finitely many $x_1, \cdots, x_n \in K$ such that $K \subset \bigcup_{i=1}^{n} B(x_i, \frac{\delta_{x_i}}{2})$. Let $\delta = \frac{1}{2}\min\{\delta_{x_1}, \cdots, \delta_{x_n}\}$. If $x \in K$ and $d(x,y) < \delta$, then $x \in B(x_i, \frac{\delta_{x_i}}{2})$ for some i. Since $d(x_i, x) < \frac{\delta_{x_i}}{2} < \delta_{x_i}$, we have $d(f(x_i), f(x)) < \frac{\epsilon}{2}$. Since

$$d(x_i, y) \leq d(x_i, x) + d(x, y) < \frac{\delta_{x_i}}{2} + \delta \leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i},$$

we have $d(f(x_i), f(y)) < \frac{\epsilon}{2}$. Thus we get

$$d(f(x), f(y)) \leq d(f(x), f(x_i)) + d(f(x_i), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof. $\square$

We are now going to prove the continuity for the Zadeh extension of a continuous surjection in D-metric.

THEOREM 2.5. *Let $f : X \to X$ be a continuous surjection. Then the Zadeh extension $\hat{f}$ of f is continuous.*

Proof. Let $u \in \mathcal{F}(X)$. Since $[u]^0 \in \Omega(X)$ and X is a locally compact metric space, there exists $\zeta > 0$ such that $K \equiv \overline{B([u]^0, \zeta)}$ is compact. For any $\epsilon > 0$, by Lemma 2.4, there exists $\delta \in (0, \zeta)$ such that $x \in K$ and $d(x,y) < \delta$ implies $d(f(x), f(y)) < \frac{2}{3}\epsilon$. Let $D(u,v) < \delta$. Since $h([u]^0, [v]^0) \leq D(u,v) < \delta$, we have

$$[v]^0 \subset B([u]^0, \delta) \subset B([u]^0, \zeta) \subset K.$$

It is clear that $[u]^\alpha \subset [u]^0 \subset K$ and $[v]^\alpha \subset [v]^0 \subset K$ for all $\alpha \in [0,1]$. Let $\alpha \in [0,1]$. Since $h([u]^\alpha, [v]^\alpha) \leq D(u,v) < \delta$, we have $[v]^\alpha \subset B([u]^\alpha, \delta)$ and $[u]^\alpha \subset B([v]^\alpha, \delta)$. For any $x \in [v]^\alpha$, there exists $y \in [u]^\alpha$ such that $d(x,y) < \delta$. Thus $d(f(x), f(y)) < \frac{2}{3}\epsilon$ and so $f(x) \in B(f([u]^\alpha), \frac{2}{3}\epsilon)$.

Hence $f([v]^\alpha) \subset B(f([u]^\alpha), \frac{2}{3}\epsilon)$. Similarly we can show that $f([u]^\alpha) \subset B(f([v]^\alpha), \frac{2}{3}\epsilon)$. By Lemma 2.3, we have

$$[\hat{f}(v)]^\alpha \subset B([\hat{f}(u)]^\alpha, \frac{2}{3}\epsilon) \ \text{ and } \ [\hat{f}(u)]^\alpha \subset B([\hat{f}(v)]^\alpha, \frac{2}{3}\epsilon).$$

Thus $h([\hat{f}(u)]^\alpha, [\hat{f}(v)]^\alpha) \leq \frac{2}{3}\epsilon$. Hence we have $D(\hat{f}(u), \hat{f}(v)) \leq \frac{2}{3}\epsilon \leq \epsilon$. Therefore $\hat{f}$ is continuous at u and hence the proof is complete. □

Let $f : X \to X$ be a continuous surjection. A subset A of X is said to be invariant for f if $f(A) = A$.

The next theorem shows that the Zadeh extension has a fixed point.

THEOREM 2.6. *Let $\hat{f}$ be the Zadeh extension of a continuous surjection f. Then a subset A of X is invariant if and only if $\hat{f}(\chi_A) = \chi_A$ where χ_A is the characteristic function of A.*

Proof. Let A be an invariant set. For any $x \in X$, if $x \in A = f(A)$, then there exists $y \in A$ such that $x = f(y)$. Since $1 \geq \hat{f}(\chi_A)(x) \geq \chi_A(y) = 1$, we have $\hat{f}(\chi_A)(x) = 1$. Let $x \in X - A$. For each $y \in f^{-1}(x)$, since $y \in X - A$, we have $\chi_A(y) = 0$. Thus we get $\hat{f}(\chi_A)(x) = 0$. Hence $\hat{f}(\chi_A) = \chi_A$.

Assume that $\hat{f}(\chi_A) = \chi_A$. For any $x \in f(A)$, there exists $y \in A$ such that $x = f(y)$. Since $1 \geq \hat{f}(\chi_A)(x) \geq \chi_A(y) = 1$, we have $\hat{f}(\chi_A)(x) = 1$. Thus $\chi_A(x) = \hat{f}(\chi_A)(x) = 1$ and so $x \in A$.

Let $x \in A$. If $f^{-1}(x) \cap A = \emptyset$, then $\chi_A(y) = 0$ for all $y \in f^{-1}(x)$. Thus $\hat{f}(\chi_A)(x) = 0$. Hence $\chi_A(x) = \hat{f}(\chi_A)(x) = 0$ and so we have a contradiction. Therefore we can choose $y \in f^{-1}(x) \cap A$. Then we have $x = f(y) \in f(A)$. This completes the proof. □

3. Zadeh's extension for semiflows

In this section, we study some properties of the Zadeh extension for semiflows.

A continuous map $f : X \times R^+ \to X$ is called a semiflow if

(3.1) $\quad f(x, 0) = x$ for all $x \in X$ and

(3.2) $\quad f(f(x, s), t) = f(x, s + t)$ for all $x \in X$ and $s, t \in R^+$.

Let f be a semiflow on X. For each $t \in R^+$, define $f_t : X \to X$ by $f_t(x) = f(x,t)$. Then f_t is continuous for all $t \in R^+$. Assume that f_t is a surjection for all $t \in R^+$.

If $f : X \times R^+ \to X$ is a semiflow, we define Zadeh's extension $\hat{f}$ of a semiflow f by

$$\hat{f} : \mathcal{F}(X) \times R^+ \to \mathcal{F}(X),\ \ \hat{f}(u,t)(x) = \sup\{u(y) \mid y \in f_t^{-1}(x)\}.$$

The following result shows that Zadeh's extension $\hat{f}$ of a semiflow f is well–defined.

LEMMA 3.1. *Let $u \in \mathcal{F}(X)$ and $t \in R^+$. Then we have $[\hat{f}(u,t)]^\alpha = f([u]^\alpha \times \{t\})$ for all $\alpha \in [0,1]$.*

Proof. Since f_t is a continuous surjection for all $t \in R^+$, by Lemma 2.3, we have

$$[\hat{f}(u,t)]^\alpha = [\hat{f}_t(u)]^\alpha = f_t([u]^\alpha) = f([u]^\alpha \times \{t\})$$

and so the proof is complete. □

We need to state one observation before the main result.

LEMMA 3.2. *Let K be a compact subset of X, f a semiflow and $a \in R^+$. For any $\epsilon > 0$, there exists $\delta > 0$ such that if $x \in K$, $d(x,y) < \delta$, and $s,t \in (a-\delta, a+\delta) \cap R^+$, then $d(f(x,s), f(y,t)) < \epsilon$.*

Proof. Let $\epsilon > 0$. For each $x \in K$, since f is continuous at (x,a), there is $\delta_x > 0$ such that $d(x,y) < \delta_x$ and $|a-t| < \delta_x$ implies $d(f(x,a), f(y,t)) < \frac{\epsilon}{2}$. Thus we see that $\left\{B(x, \frac{\delta_x}{2}) \mid x \in K\right\}$ is an open covering of K. Since K is a compact set, there are finitely many $x_1, \cdots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n B(x_i, \frac{\delta_{x_i}}{2})$. Let $\delta = \frac{1}{2}\min\{\delta_{x_1}, \cdots, \delta_{x_n}\}$. If $x \in K$, $d(x,y) < \delta$, and $s,t \in (a-\delta, a+\delta) \cap R^+$, then $x \in B(x_i, \frac{\delta_{x_i}}{2})$ for some i. Since

$$d(x_i, x) < \frac{\delta_{x_i}}{2} < \delta_{x_i} \text{ and } \ |a-s| < \delta \leq \frac{\delta_{x_i}}{2} < \delta_{x_i},$$

we have $d(f(x_i,a), f(x,s)) < \frac{\epsilon}{2}$. Since

$$d(x_i, y) \leq d(x_i, x) + d(x,y) < \frac{\delta_{x_i}}{2} + \delta \leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i} \ \text{ and}$$

$$|a-t| < \delta \leq \frac{\delta_{x_i}}{2} < \delta_{x_i},$$

we have $d(f(x_i,a),f(y,t)) < \frac{\epsilon}{2}$. Thus we obtain

$$\begin{aligned} d(f(x,s),f(y,t)) &\le d(f(x,s),f(x_i,a)) + d(f(x_i,a),f(y,t)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This completes the proof. □

Next we state the main result that Zadeh's extension of a semiflow is continuous.

THEOREM 3.3. *If $f : X \times R^+ \to X$ is a semiflow, then Zadeh's extension $\hat{f}$ of a semiflow f is continuous.*

Proof. Let $u \in \mathcal{F}(X)$ and $a \in R^+$. Since $[u]^0$ is a compact set and X is a locally compact metric space, there exists $\zeta > 0$ such that $K \equiv \overline{B([u]^0,\zeta)}$ is compact. Given any $\epsilon > 0$, by Lemma 3.2, there exists $\delta > 0$ such that

$$\begin{aligned} &\text{if } x \in K,\ d(x,y) < \delta, \ \text{ and } s,t \in (a-\delta, a+\delta) \cap R^+, \\ &\text{then } \ d(f(x,s),f(y,t)) < \frac{2}{3}\epsilon. \end{aligned}$$

Let $D(u,v) < \delta$ and $|a-t| < \delta$. Since $h([u]^0,[v]^0) \le D(u,v) < \delta$, we have

$$[v]^0 \subset B([u]^0,\delta) \subset B([u]^0,\zeta) \subset K.$$

It is clear that $[u]^\alpha \subset [u]^0 \subset K$ and $[v]^\alpha \subset [v]^0 \subset K$ for all $\alpha \in [0,1]$. Let $\alpha \in [0,1]$. Since $h([u]^\alpha,[v]^\alpha) \le D(u,v) < \delta$, we have $[v]^\alpha \subset B([u]^\alpha,\delta)$ and $[v]^\alpha \subset B([u]^\alpha,\delta)$. For any $x \in [v]^\alpha$, there exists $y \in [u]^\alpha$ such that $d(x,y) < \delta$. Thus $d(f(x,t),f(y,a)) < \frac{2}{3}\epsilon$ and so $f(x,t) \in B(f([u]^\alpha,a),\frac{2}{3}\epsilon)$. Hence $f([v]^\alpha,t) \subset B(f([u]^\alpha,a),\frac{2}{3}\epsilon)$. Similarly we can show that $f([u]^\alpha,a) \subset B(f([v]^\alpha,t),\frac{2}{3}\epsilon)$. By Lemma 3.1, we have

$$\begin{aligned} &[\hat{f}(v,t)]^\alpha \subset B([\hat{f}(u,a)]^\alpha, \frac{2}{3}\epsilon) \ \text{ and} \\ &[\hat{f}(u,a)]^\alpha \subset B([\hat{f}(v,t)]^\alpha, \frac{2}{3}\epsilon). \end{aligned}$$

Thus $h([\hat{f}(u,a)]^\alpha,[\hat{f}(v,t)]^\alpha) \le \frac{2}{3}\epsilon$. Hence we have $D(\hat{f}(u,a),\hat{f}(v,t)) \le \frac{2}{3}\epsilon < \epsilon$. Thus $\hat{f}$ is continuous at (u,a) and so the proof is complete. □

The following theorem gives a result about Zadeh's extension of a semiflow.

THEOREM 3.4. *Let f be a semiflow on X. Then so is Zadeh's extension $\hat{f}$ of a semiflow f.*

Proof. It is clear that $f(u, 0) = u$ for all $u \in \mathcal{F}(X)$. We will show that

$$\hat{f}(\hat{f}(u, s), t) = \hat{f}(u, s + t) \quad \text{for} \quad \text{all } u \in \mathcal{F}(X) \quad \text{and } s, t \in R^+.$$

For any $y \in f_{s+t}^{-1}(x)$, put $f(y, s) = z$. Since $x = f(y, s+t) = f(f(y, s), t) = f(z, t)$, we have

$$\hat{f}(\hat{f}(u, s), t)(x) \geq \hat{f}(u, s)(z) \geq u(y).$$

Thus we get $\hat{f}(u, s + t)(x) \leq \hat{f}(\hat{f}(u, s), t)(x)$.

Assume that $\hat{f}(\hat{f}(u, s), t)(x) > \hat{f}(u, s + t)(x)$. Choose ζ such that

$$\hat{f}(\hat{f}(u, s), t)(x) > \zeta > \hat{f}(u, s + t)(x).$$

Then there exists $z \in f_t^{-1}(x)$ such that $\hat{f}(u, s)(z) > \zeta$. There exists $y \in f_s^{-1}(z)$ such that $u(y) > \zeta$. Since $f(y, s+t) = f(f(y, s), t) = f(z, t) = x$, we have

$$\hat{f}(u, s + t)(x) \geq u(y) > \zeta > \hat{f}(u, s + t)(x).$$

This is a contradiction. Thus $\hat{f}(u, s+t)(x) = \hat{f}(\hat{f}(u, s), t)(x)$. Hence we have

$$\hat{f}(u, s + t) = \hat{f}(\hat{f}(u, s), t).$$

This completes the proof. □

Let f be a semiflow on X. A subset A of X is said to be invariant if $f_t(A) = A$ for all $t \in R^+$.

From Theorem 2.6, we obtain the following theorem.

THEOREM 3.5. *Suppose $\hat{f}_t$ is Zadeh's extension of a surjection f_t. Then a subset A of X is invariant if and only if $\hat{f}_t(\chi_A) = \chi_A$ for all $t \in R^+$.*

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