# STABILITY OF A GENERALIZED JENSEN TYPE QUADRATIC FUNCTIONAL EQUATIONS 

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#### Abstract

In this paper, we investigate the Hyers-Ulam-Rassias stability of generalized Jensen type quadratic functional equations in Banach spaces.


## 1. Introduction

In 1940, S. M. Ulam [1] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let $G$ be a group and let $G^{\prime}$ be a metric group with metric $\rho(\cdot, \cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if $f: G \rightarrow G^{\prime}$ satisfies $\rho(f(x y), f(x) f(y))<\delta$ for all $x, y \in G$, then a homomorphism $h: G \rightarrow$ $G^{\prime}$ exists with $\rho(f(x), h(x))<\epsilon$ for all $x \in G$ ?

In 1941, D. H. Hyers [2] considered the case of an approximately additive mapping $f: E \rightarrow E^{\prime}$, where $E$ and $E^{\prime}$ are Banach spaces and $f$ satisfies Hyers inequality

$$
\|f(x+y)-f(x)-f(y)\| \leq \epsilon
$$

for all $x, y \in E$. It was shown that the limit $L(x)=\lim _{n \rightarrow \infty} \frac{f\left(2^{n} x\right)}{2^{n}}$ exists for all $x \in E$ and that $L: E \rightarrow E^{\prime}$ is the unique additive mapping satisfying

$$
\|f(x)-L(x)\| \leq \epsilon .
$$

Let $E_{1}$ and $E_{2}$ be real vector spaces. A function $f: E_{1} \rightarrow E_{2}$, there exists a quadratic function if and only if $f$ is a solution function of the

[^0]quadratic functional equation
\[

$$
\begin{equation*}
f(x+y)-f(x-y)=2 f(x)+2 f(y) . \tag{1.1}
\end{equation*}
$$

\]

A stability problem for the quadratic functional equation (1.1) was solved by F. Skof [3] for a mapping $f: E_{1} \rightarrow E_{2}$, where $E_{1}$ is a normed space and $E_{2}$ is a Banach space.

In 1978, Th. M. Rassias [4] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded.
S. Czerwik [5] proved the Hyers-Ulam-Rassias stability of quadratic functional equation (1.1). Let $E_{1}$ and $E_{2}$ be a real normed space and a real Banach space, respectively, and let $p \neq 2$ be a positive constant. If a function $f: E_{1} \rightarrow E_{2}$ satisfies the inequality

$$
\|f(x+y)-f(x-y)-2 f(x)-2 f(y)\| \leq \epsilon\left(\|x\|^{p}+\|y\|^{p}\right)
$$

for some $\epsilon>0$ and for all $x, y \in E_{1}$, then there exists a unique quadratic function $q: E_{1} \rightarrow E_{2}$ such that

$$
\|f(x)-q(x)\| \leq \frac{2 \epsilon}{\left|4-2^{p}\right|}\|x\|^{p}
$$

for all $x \in G$.
A mapping $g: X \rightarrow Y$ is called a Jensen mapping if $g$ satisfies the functional equation

$$
2 g\left(\frac{x+y}{2}\right)=g(x)+g(y)
$$

for all $x, y \in X$.
Jun and Lee [6] proved the following: Let $X$ and $Y$ be Banach spaces. Denote by $\varphi: X \backslash\{0\} \times X \backslash\{0\} \rightarrow[0, \infty)$ function such that

$$
\psi(x, y)=\sum_{j=0}^{\infty} 3^{-k} \varphi\left(3^{k} x, 3^{k} y\right)<\infty
$$

for all $x, y \in X \backslash\{0\}$. Suppose that $f: X \rightarrow Y$ is a mapping satisfying

$$
\left\|2 f\left(\frac{x+y}{2}\right)-f(x)-f(y)\right\| \leq \varphi(x, y)
$$

for all $x, y \in X \backslash\{0\}$. Then there exists a unique additive mapping $T: X \rightarrow Y$ such that

$$
\|f(x)-f(0)-T(x)\| \leq \frac{1}{3}(\psi(x,-x)+\psi(-x, 3 x))
$$

for all $x \in X \backslash\{0\}$. Recently C. Park and W. Park [7] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a $C^{*}$-algebra.

Now, we consider the following functional equation

$$
\begin{align*}
f\left(\frac{x+y}{a}+b z\right) & +f\left(\frac{x+y}{a}-b z\right)+f\left(\frac{x-y}{a}+b z\right)+f\left(\frac{x-y}{a}-b z\right) \\
& =\frac{4}{a^{2}} f(x)+\frac{4}{a^{2}} f(y)+4 b^{2} f(z), \tag{1.2}
\end{align*}
$$

where $a, b \neq 0$ are real numbers.
In this paper, we will establish the general solution and the generalized Hyers-Ulam-Rassias stability problem for the equation (1.2) in Banach spaces.

## 2. Jesen type quadratic mapping in Banach spaces

Lemma 2.1. Let $X$ and $Y$ be vector spaces. If a mapping $f: X \rightarrow Y$ satisfies $f(0)=0$ and

$$
f\left(\frac{x+y}{a}+b z\right)+f\left(\frac{x+y}{a}-b z\right)+f\left(\frac{x-y}{a}+b z\right)+f\left(\frac{x-y}{a}-b z\right)
$$

$$
\begin{equation*}
=\frac{4}{a^{2}} f(x)+\frac{4}{a^{2}} f(y)+4 b^{2} f(z) \tag{2.1}
\end{equation*}
$$

for all $x, y, z \in X$, then the mapping $f$ is quadratic.
Proof. Letting $x=y$ in (2.1), we get

$$
\begin{align*}
f\left(\frac{2 x}{a}+b z\right)+ & f\left(\frac{2 x}{a}-b z\right)+f(b z)+f(-b z)  \tag{2.2}\\
& =\frac{8}{a^{2}} f(x)+4 b^{2} f(z)
\end{align*}
$$

for all $x, z \in X$. Letting $x=0$ in (2.2), we get $2 f(b z)+2 f(-b z)=$ $4 b^{2} f(z)$. Setting $y=-x$ in (2.1), we obtain

$$
\begin{align*}
f\left(\frac{2 x}{a}+b z\right) & +f\left(\frac{2 x}{a}-b z\right)+f(b z)+f(-b z)  \tag{2.3}\\
& =\frac{4}{a^{2}} f(x)+\frac{4}{a^{2}} f(-x)+4 b^{2} f(z)
\end{align*}
$$

By (2.2) and (2.3), we conclude that $f$ is even. And by setting $z=0$ in (2.2), we get $f\left(\frac{2 x}{a}\right)=\frac{4}{a^{2}} f(x)$ for all $x \in X$. So, we get

$$
f\left(\frac{2 x}{a}+b z\right)+f\left(\frac{2 x}{a}-b z\right)=2 f\left(\frac{2 x}{a}\right)+2 f(b z)
$$

for all $x, z \in X$. Hence $f$ is quadratic.

The mapping $f: X \rightarrow Y$ given in the statement of Lemma 2.1 is called a generalized Jensen type quadratic mapping. Putting $z=0$ in (2.1) with $a=2$, we get the Jensen type quadratic mapping $2 f\left(\frac{x+y}{2}\right)+$ $2 f\left(\frac{x-y}{2}\right)=f(x)+f(y)$, and putting $x=y$ in (2.1) with $a=2$ and $b=1$, we get the quadratic mapping $f(x+z)+f(x-z)=2 f(x)+2 f(z)$.

From now on, let $X$ and $Y$ be a normed vector space and a Banach space ,respectively.

For a given mapping $f: X \rightarrow Y$, we define

$$
\begin{aligned}
D f(x, y, z):= & f\left(\frac{x+y}{a}+b z\right)+f\left(\frac{x+y}{a}-b z\right)+f\left(\frac{x-y}{a}+b z\right) \\
& +f\left(\frac{x-y}{a}-b z\right)-\frac{4}{a^{2}} f(x)-\frac{4}{a^{2}} f(y)-4 b^{2} f(z)
\end{aligned}
$$

for all $x, y, z \in X$
Theorem 2.2. Let $f: X \rightarrow Y$ be a mapping satisfying $f(0)=0$ for which there exists a function $\phi: X^{3} \rightarrow[0, \infty)$ such that

$$
\begin{gather*}
\Phi(x, y, z):=\sum_{j=1}^{\infty} \frac{1}{2}\left(\frac{4}{a^{2}}\right)^{j} \phi\left(\left(\frac{a}{2}\right)^{j} x,\left(\frac{a}{2}\right)^{j} y,\left(\frac{a}{2}\right)^{j} z\right)<\infty  \tag{2.4}\\
\|D f(x, y, z)\| \leq \phi(x, y, z) \tag{2.5}
\end{gather*}
$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q$ : $X \rightarrow Y$ such that $D Q(x, y, z)=0$ and

$$
\begin{equation*}
\|Q(x)-f(x)\| \leq \frac{a^{2}}{4} \Phi(x, x, 0) \tag{2.6}
\end{equation*}
$$

for all $x \in X$.
Proof. Letting $x=y$ and $z=0$ in (2.5), we get

$$
\left\|f\left(\frac{2}{a} x\right)-\frac{4}{a^{2}} f(x)\right\| \leq \frac{1}{2} \phi(x, x, 0)
$$

for all $x \in X$. So

$$
\left\|f(x)-\frac{4}{a^{2}} f\left(\frac{a}{2} x\right)\right\| \leq \frac{1}{2} \phi\left(\frac{a}{2} x, \frac{a}{2} x, 0\right)
$$

for all $x \in X$. Hence

$$
\begin{aligned}
& \left\|\left(\frac{4}{a^{2}}\right)^{l} f\left(\left(\frac{a}{2}\right)^{l} x\right)-\left(\frac{4}{a^{2}}\right)^{m} f\left(\left(\frac{a}{2}\right)^{m} x\right)\right\| \\
& \quad \leq \sum_{j=l+1}^{m}\left\|\left(\frac{4}{a^{2}}\right)^{j-1} f\left(\left(\frac{a}{2}\right)^{j-1} x\right)-\left(\frac{4}{a^{2}}\right)^{j} f\left(\left(\frac{a}{2}\right)^{j} x\right)\right\| \\
& \quad \leq \sum_{j=l+1}^{m}\left(\frac{4}{a^{2}}\right)^{j-1} \frac{1}{2} \phi\left(\left(\frac{a}{2}\right)^{j} x,\left(\frac{a}{2}\right)^{j} x, 0\right)
\end{aligned}
$$

for all $x \in X$. It means that a sequence $\left\{\left(\frac{4}{a^{2}}\right)^{n} f\left(\left(\frac{a}{2}\right)^{n} x\right)\right\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\left\{\left(\frac{4}{a^{2}}\right)^{n} f\left(\left(\frac{a}{2}\right)^{n} x\right)\right\}$ converges. So one can define a mapping $Q: X \rightarrow Y$ by $Q(x):=\lim _{n \rightarrow \infty}\left(\frac{4}{a^{2}}\right)^{n} f\left(\left(\frac{a}{2}\right)^{n} x\right)$ for all $x \in X$.

By (2.4) and (2.5),

$$
\begin{align*}
\|D Q(x, y, z)\| & =\lim _{n \rightarrow \infty}\left(\frac{4}{a^{2}}\right)^{n}\left\|D f\left(\left(\frac{a}{2}\right)^{n} x,\left(\frac{a}{2}\right)^{n} y,\left(\frac{a}{2}\right)^{n} z\right)\right\|  \tag{2.7}\\
& \leq \lim _{n \rightarrow \infty}\left(\frac{4}{a^{2}}\right)^{n} \phi\left(\left(\frac{a}{2}\right)^{n} x,\left(\frac{a}{2}\right)^{n} y,\left(\frac{a}{2}\right)^{n} z\right)=0 \tag{2.8}
\end{align*}
$$

for all $x, y, z \in X$. So $D Q(x, y, z)=0$. By Lemma 2.1, the mapping $Q: X \rightarrow Y$ is a quadratic mapping.

Moreover, letting $l=0$ and passing the limit $m \rightarrow \infty$ in (2.7), we get the approximation (2.6) of $f$ by $Q$.

Now, let $Q^{\prime}: X \longrightarrow Y$ be another quadratic mapping satisfying (2.6). Then we obtain

$$
\begin{aligned}
& \left\|Q(x)-Q^{\prime}(x)\right\|=\left(\frac{4}{a^{2}}\right)^{n}\left\|Q\left(\left(\frac{a}{2}\right)^{n} x\right)-Q\left(\left(\frac{a}{2}\right)^{n} x\right)\right\| \\
& \left.\leq\left(\frac{4}{a^{2}}\right)^{n}\left[\left\|Q\left(\left(\frac{a}{2}\right)^{n} x\right)-f\left(\left(\frac{a}{2}\right)^{n} x\right)\right\|+\| Q^{\prime}\left(\left(\frac{a}{2}\right)^{n} x\right)-f\left(\left(\frac{a}{2}\right)^{n} x\right)\right) \|\right] \\
& \leq 2\left(\frac{4}{a^{2}}\right)^{n-1} \Phi\left(\left(\frac{a}{2}\right)^{n} x,\left(\frac{a}{2}\right)^{n} x, 0\right),
\end{aligned}
$$

which tends to zero as $n \rightarrow \infty$. So we can conclude that $Q(x)=Q^{\prime}(x)$ for all $x \in X$. This proves the uniqueness of $Q$. Hence the mapping $Q: X \rightarrow Y$ is a unique quadratic mapping satisfying (2.6).

Corollary 2.3. Let $p$ and $\theta$ be positive real numbers with $p>2$ and $0<|a|<2$. Let $f: X \rightarrow Y$ be a mapping satisfying $f(0)=0$ and

$$
\begin{equation*}
\|D f(x, y, z)\| \leq \theta\left(\|x\|^{p}+\|y\|^{p}+\|z\|^{p}\right) \tag{2.9}
\end{equation*}
$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q$ : $X \rightarrow Y$ such that

$$
\begin{equation*}
\|f(x)-Q(x)\| \leq \frac{|a|^{p} \cdot \theta}{4\left(2^{p-2}-|a|^{p-2}\right)}\|x\|^{p} \tag{2.10}
\end{equation*}
$$

for all $x \in X$.
Proof. Define $\phi(x, y, z)=\theta\left(\|x\|^{p}+\|y\|^{p}+\|z\|^{p}\right)$, and apply Theorem 2.2.

Corollary 2.4. Let $p$ and $\theta$ be positive real numbers with $p<2$ and $|a|>2$. Let $f: X \rightarrow Y$ be a mapping satisfying $f(0)=0$ and

$$
\begin{equation*}
\|D f(x, y, z)\| \leq \theta\left(\|x\|^{p}+\|y\|^{p}+\|z\|^{p}\right) \tag{2.11}
\end{equation*}
$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q$ : $X \rightarrow Y$ such that

$$
\begin{equation*}
\|f(x)-Q(x)\| \leq \frac{|a|^{2} \cdot \theta}{2^{p}\left(|a|^{2-p}-2^{2-p}\right)}\|x\|^{p} \tag{2.12}
\end{equation*}
$$

for all $x \in X$.
Proof. Define $\phi(x, y, z)=\theta\left(\|x\|^{p}+\|y\|^{p}+\|z\|^{p}\right)$, and apply Theorem 2.2.

Theorem 2.5. Let $f: X \rightarrow Y$ be a mapping satisfying $f(0)=0$ for which there exists a function $\phi: X^{3} \rightarrow[0, \infty)$ such that

$$
\begin{gather*}
\Phi(x, y, z):=\sum_{j=1}^{\infty} \frac{1}{2}\left(\frac{a^{2}}{4}\right)^{j} \phi\left(\left(\frac{2}{a}\right)^{j-1} x,\left(\frac{2}{a}\right)^{j-1} y,\left(\frac{2}{a}\right)^{j-1} z\right)  \tag{2.13}\\
<\infty, \\
\|D f(x, y, z)\| \leq \phi(x, y, z) \tag{2.14}
\end{gather*}
$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q$ : $X \rightarrow Y$ such that $D Q(x, y, z)=0$ and

$$
\begin{equation*}
\|Q(x)-f(x)\| \leq \Phi(x, x, 0) \tag{2.15}
\end{equation*}
$$

for all $x \in X$.
Proof. Letting $x=y$ and $z=0$ in (2.5), we get

$$
\begin{equation*}
\left\|f\left(\frac{2}{a} x\right)-\frac{4}{a^{2}} f(x)\right\| \leq \frac{1}{2} \phi(x, x, 0) \tag{2.16}
\end{equation*}
$$

for all $x \in X$. So

$$
\begin{equation*}
\left\|f(x)-\frac{a^{2}}{4} f\left(\frac{2}{a} x\right)\right\| \leq\left(\frac{a^{2}}{4}\right) \frac{1}{2} \phi(x, x, 0) \tag{2.17}
\end{equation*}
$$

for all $x \in X$.
Hence

$$
\begin{align*}
& \left\|\left(\frac{a^{2}}{4}\right)^{l} f\left(\left(\frac{2}{a}\right)^{l} x\right)-\left(\frac{a^{2}}{4}\right)^{m} f\left(\left(\frac{2}{a}\right)^{m} x\right)\right\| \\
& \quad \leq \sum_{j=l+1}^{m}\left(\frac{a^{2}}{4}\right)^{j} \frac{1}{2} \phi\left(\left(\frac{2}{a}\right)^{j-1} x,\left(\frac{2}{a}\right)^{j-1} x, 0\right) \tag{2.18}
\end{align*}
$$

for all $x \in X$. It means that a sequence $\left\{\left(\frac{a^{2}}{4}\right)^{n} f\left(\left(\frac{2}{a}\right)^{n} x\right)\right\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\left\{\left(\frac{a^{2}}{4}\right)^{n} f\left(\left(\frac{2}{a}\right)^{n} x\right)\right\}$ converges. So one can define a mapping $Q: X \rightarrow Y$ by $Q(x):=\lim _{n \rightarrow \infty}\left(\frac{a^{2}}{4}\right)^{n} f\left(\left(\frac{2}{a}\right)^{n} x\right)$ for all $x \in X$.

By (2.13) and (2.14),

$$
\begin{align*}
\|D Q(x, y, z)\| & =\lim _{n \rightarrow \infty}\left(\frac{a^{2}}{4}\right)^{n} \| D f\left(\left(\frac{2}{a}\right)^{n} x,\left(\frac{2}{a}\right)^{n} y,\left(\frac{2}{a}\right)^{n} z\right)  \tag{2.19}\\
& \leq \lim _{n \rightarrow \infty}\left(\frac{a^{2}}{4}\right)^{n} \phi\left(\left(\frac{2}{a}\right)^{n} x,\left(\frac{2}{a}\right)^{n} y,\left(\frac{2}{a}\right)^{n} z\right)=0 \tag{2.20}
\end{align*}
$$

for all $x, y, z \in X$. So $D Q(x, y, z)=0$. By Lemma 2.1, the mapping $Q: X \rightarrow Y$ is a quadratic mapping.

Moreover, letting $l=0$ and passing the limit $m \rightarrow \infty$ in (2.18), we get the approximation (2.15) of $f$ by $Q$.

Now, let $Q^{\prime}: X \longrightarrow Y$ be another quadratic mapping satisfying (2.15) . Then we obtain

$$
\begin{aligned}
& \left\|Q(x)-Q^{\prime}(x)\right\|=\left(\frac{a^{2}}{4}\right)^{n}\left\|Q\left(\left(\frac{2}{a}\right)^{n} x\right)-Q\left(\left(\frac{2}{a}\right)^{n} x\right)\right\| \\
& \left.\leq\left(\frac{a^{2}}{4}\right)^{n}\left[\left\|Q\left(\left(\frac{2}{a}\right)^{n} x\right)-f\left(\left(\frac{2}{a}\right)^{n} x\right)\right\|+\| Q^{\prime}\left(\left(\frac{2}{a}\right)^{n} x\right)-f\left(\left(\frac{2}{a}\right)^{n} x\right)\right) \|\right] \\
& \leq 2\left(\frac{a^{2}}{4}\right)^{n} \Phi\left(\left(\frac{2}{a}\right)^{n} x,\left(\frac{2}{a}\right)^{n} x, 0\right)
\end{aligned}
$$

which tends to zero as $n \rightarrow \infty$. So we can conclude that $Q(x)=Q^{\prime}(x)$ for all $x \in X$. This proves the uniqueness of $Q$. Hence the mapping $Q: X \rightarrow Y$ is a unique quadratic mapping satisfying (2.15).

Corollary 2.6. Let $p$ and $\theta$ be positive real numbers with $p>2$ and $|a|>2$. Let $f: X \rightarrow Y$ be a mapping satisfying $f(0)=0$ and

$$
\begin{equation*}
\|D f(x, y, z)\| \leq \theta\left(\|x\|^{p}+\|y\|^{p}+\|z\|^{p}\right) \tag{2.21}
\end{equation*}
$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q$ : $X \rightarrow Y$ such that

$$
\begin{equation*}
\|f(x)-Q(x)\| \leq \frac{|a|^{p} \cdot \theta}{4\left(|a|^{p-2}-2^{p-2}\right)}\|x\|^{p} \tag{2.22}
\end{equation*}
$$

for all $x \in X$.
Proof. Define $\phi(x, y, z)=\theta\left(\|x\|^{p}+\|y\|^{p}+\|z\|^{p}\right)$, and apply Theorem 2.5.

Corollary 2.7. Let $p$ and $\theta$ be positive real numbers with $p<2$ and $|a|<2$. Let $f: X \rightarrow Y$ be a mapping satisfying $f(0)=0$ and

$$
\begin{equation*}
\|D f(x, y, z)\| \leq \theta\left(\|x\|^{p}+\|y\|^{p}+\|z\|^{p}\right) \tag{2.23}
\end{equation*}
$$

for all $x, y, z \in X$. Then there exists a unique Jensen type quadraticquadratic mapping $Q: X \rightarrow Y$ such that

$$
\begin{equation*}
\|f(x)-Q(x)\| \leq \frac{|a|^{2} \cdot \theta}{2^{p}\left(2^{2-p}-|a|^{2-p}\right)}\|x\|^{p} \tag{2.24}
\end{equation*}
$$

for all $x \in X$.
Proof. Define $\phi(x, y, z)=\theta\left(\|x\|^{p}+\|y\|^{p}+\|z\|^{p}\right)$, and apply Theorem 2.5.

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