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HYERS–ULAM STABILITY OF FUNCTIONAL INEQUALITIES ASSOCIATED WITH CAUCHY MAPPINGS

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Abstract. In this paper, we investigate the generalized Hyers–Ulam stability of the functional inequality

$$\|af(x) + bf(y) + cf(z)\| \le \|f(ax + by + cz))\| + \phi(x, y, z)$$

associated with Cauchy additive mappings. As a result, we obtain that if a mapping satisfies the functional inequality with perturbing term which satisfies certain conditions then there exists a Cauchy additive mapping near the mapping.

1. Introduction

In 1940, S. M. Ulam [18] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

In 1941, D. H. Hyers [7] considered the case of approximately additive mappings $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon$$

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for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

In 1978, Th. M. Rassias [15] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$.

In 1991, Z. Gajda [3] following the same approach as in Th. M. Rassias [15], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [3], as well as by Th. M. Rassias and P. Šemrl [16] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. The inequality (1.1) that was introduced for the first time by Th. M. Rassias [15] provided a lot of influence in the development of a generalization of the Hyers–Ulam stability concept. This new concept of stability is known as *generalized Hyers–Ulam stability* or *Hyers–Ulam–Rassias stability* of functional equations (cf. the books of P. Czerwik [1], D. H. Hyers, G. Isac and Th. M. Rassias [8]).

P. Găvruta [6] provided a further generalization of Th. M. Rassias' Theorem. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam stability to a number of functional equations and mappings (see [9]–[14]).

Gilányi[4] and Rätz[17] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \tag{1.3}$$

then f satisfies the Jordan–von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

Gilányi [5] and Fechner [2] proved the generalized Hyers–Ulam stability of the functional inequality (1.3).

Now, we consider the following functional inequality

$$(1.4)\quad \|af(x)+bf(y)+cf(z)\| \;\le\; \|f(ax+by+cz)\|+\phi(x,y,z)$$

which is associated with Jordan–von Neumann type Cauchy additive functional equation, where the function ϕ is a perturbing term of the functional inequality

$$\|af(x)+bf(y)+cf(z)\| \;\le\; \|f(ax+by+cz)\|.$$

The purpose of this paper is to prove that if f satisfies one of the inequality (1.4) which satisfies certain conditions, then we can find a Cauchy additive mapping near f and thus we prove the generalized Hyers–Ulam stability of the functional inequality (1.4).

2. Main results

Throughout this paper, let G be a normed vector space and Y a Banach space. First, we consider solutions of the functional inequality (1.4) with perturbing term zero.

LEMMA 2.1. *Let $f : G \to Y$ be a mapping with $f(0) = 0$ such that*

$$(2.1)\qquad \|af(x)+bf(y)+cf(z)\| \le \|f(ax+by+cz)\|$$

for all $x, y, z \in G$, where $abc \neq 0$. Then f is Cauchy additive.

Proof. By setting $z := \frac{-ax}{c}$ and $y := 0$ in (2.1), we get

$$(2.2)\qquad \|af(x)+cf(\frac{-ax}{c})\| \le \|f(0)\| = 0$$

for all $x \in G$. Also by letting $x := 0$, $z := \frac{-by}{c}$ in (2.1), we get

$$(2.3)\qquad \|bf(y)+cf(\frac{-by}{c})\| \le \|f(0)\| = 0$$

for all $x \in G$. Letting $z = \frac{-ax-by}{c}$ in (2.1), we get

$$\|af(x)+bf(y)+cf(\frac{-ax-by}{c})\| \le \|f(0)\| = 0$$

for all $x, y \in G$. It follows from the equalities (2.2) and (2.3) that

$$-cf(\frac{-ax}{c}) - cf(\frac{-by}{c}) + cf(\frac{-ax-by}{c}) = 0,$$

that is, $-f(u) - f(v) + f(u+v) = 0$ for all $u, v \in G$, as desired. □

We recall that a subadditive function is a function $\phi : A \to B$, having a domain A and a codomain $(B, \leq)$ that are both closed under addition, with the following property:

$$\phi(x+y) \leq \phi(x) + \phi(y), \ \forall x, y \in A.$$

Now we say that a function $\phi : A \to B$ is *contractively subadditive* if there exists a constant L with $0 < L < 1$ such that

$$\phi(x+y) \leq L[\phi(x) + \phi(y)], \ \forall x, y \in A.$$

Then ϕ satisfies the following properties $\phi(2x) \leq 2L\phi(x)$ and so $\phi(2^n x) \leq (2L)^n \phi(x)$. Similarly, we say that a function $\phi : A \to B$ is *expansively superadditive* if there exists a constant L with $0 < L < 1$ such that

$$\phi(x+y) \geq \frac{1}{L}[\phi(x) + \phi(y)], \ \forall x, y \in A.$$

Then ϕ satisfies the following properties $\phi(x) \leq \frac{L}{2}\phi(2x)$ and so $\phi(\frac{x}{2^n}) \leq (\frac{L}{2})^n \phi(x)$.

Now we prove the generalized Hyers–Ulam stability of a functional inequality (1.4) associated with a Jordan–von Neumann type 3-variable Cauchy additive functional equation.

Theorem 2.2. *Assume that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|af(x) + bf(y) + cf(z)\| \leq \|f(ax + by + cz)\| + \phi(x, y, z) \tag{2.4}$$

and that the map $\phi : G \times G \times G \to [0, \infty)$ is expansively superadditive with a constant L. Then there exists a unique Cauchy additive mapping $A : G \to Y$, defined by $A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$, such that

$$\begin{aligned} &\|aA(x) + bA(y) + cA(z)\| \leq \|A(ax + by + cz)\|, \\ &\|f(x) - A(x)\| \leq \frac{L}{2|c|(1-L)} \cdot \\ &\qquad \left[\phi\Big(\frac{-cx}{a}, \frac{-cx}{b}, 2x\Big) + \phi\Big(\frac{-cx}{a}, 0, x\Big) + \phi\Big(0, \frac{-cx}{b}, x\Big)\right] \end{aligned} \tag{2.5}$$

for all $x, y, z \in G$.

Proof. We observe by the expansively superadditive condition that for any $x, y, z \in G$ $\phi(\frac{(x,y,z)}{2^n}) \leq (\frac{L}{2})^n \phi(x, y, z)$.

Letting $y := 0$ and $z := \frac{-ax}{c}$ in (2.4), we get

$$\left\|af(x) + cf(\frac{-ax}{c})\right\| \leq \phi(x, 0, \frac{-ax}{c}) \tag{2.6}$$

for all $x \in G$. By letting $x := 0$ and $z := \frac{-by}{c}$ in (2.4), one obtains

$$\left\| bf(y) + cf(\frac{-by}{c}) \right\| \leq \phi(0, y, \frac{-by}{c}) \tag{2.7}$$

for all $x \in G$. Replacing z by $\frac{-ax-by}{c}$ in (2.4), we get

$$\left\| af(x) + bf(y) + cf(\frac{-ax - by}{c}) \right\| \leq \phi(x, y, \frac{-ax - by}{c}). \tag{2.8}$$

It follows from (2.6), (2.7) and (2.8) that

$$\begin{aligned} |c| \left\| f(\frac{-ax - by}{c}) - f(\frac{-ax}{c}) - f(\frac{-by}{c}) \right\| \\ \leq \phi(x, y, \frac{-ax - by}{c}) + \phi(x, 0, \frac{-ax}{c}) + \phi(0, y, \frac{-by}{c}), \end{aligned}$$

which yields the Cauchy difference

$$\begin{aligned} &\|f(x + y) - f(x) - f(y)\| \\ &\quad \leq \frac{1}{|c|} \left[\phi(\frac{-cx}{a}, \frac{-cy}{b}, x + y) + \phi(\frac{-cx}{a}, 0, x) + \phi(0, \frac{-cy}{b}, y) \right], \\ &\|f(2x) - 2f(x)\| \\ &\quad \leq \frac{1}{|c|} \left[\phi(\frac{-cx}{a}, \frac{-cx}{b}, 2x) + \phi(\frac{-cx}{a}, 0, x) + \phi(0, \frac{-cx}{b}, x) \right] \end{aligned} \tag{2.9}$$

for all $x, y \in G$. Thus it follows from (2.9) that

$$\begin{aligned} &\|2^m f(\frac{x}{2^m}) - 2^n f(\frac{x}{2^n})\| \leq \sum_{j=m}^{n-1} \|2^j f(\frac{x}{2^j}) - 2^{j+1} f(\frac{x}{2^{j+1}})\| \\ &\leq \frac{1}{2|c|} \sum_{j=m}^{n-1} 2^{j+1} \Big[\phi\Big(\frac{-cx}{2^{j+1}a}, \frac{-cx}{2^{j+1}b}, \frac{2x}{2^{j+1}}\Big) + \phi\Big(\frac{-cx}{2^{j+1}a}, 0, \frac{x}{2^{j+1}}\Big) \\ &\qquad\qquad + \phi\Big(0, \frac{-cx}{2^{j+1}b}, \frac{x}{2^{j+1}}\Big) \Big] \\ &\leq \frac{1}{2|c|} \sum_{j=m}^{m} L^{j+1} \left[\phi\Big(\frac{-cx}{a}, \frac{-cx}{b}, 2x\Big) + \phi\Big(\frac{-cx}{a}, 0, x\Big) + \phi\Big(0, \frac{-cx}{b}, x\Big) \right] \end{aligned}$$

for all $x \in G$ and for all nonnegative integers n and m with $n > m$. It means that a sequence $\{2^n f(\frac{x}{2^n})\}$ is Cauchy sequence for all $x \in G$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define a mapping $A : G \to Y$ by $A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ for all $x \in G$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in the last inequality, we get the approximation (2.5) of f by A.

Next, we claim that the mapping $A : G \longrightarrow Y$ is Cauchy additive satisfying the functional inequality (2.1). In fact, it follows easily from (2.4) and the condition of ϕ that

$$\begin{aligned}
&\|aA(x) + bA(y) + cA(z)\| \\
&= \lim_{n\to\infty} 2^n \left\| af\Big(\frac{x}{2^n}\Big) + bf\Big(\frac{y}{2^n}\Big) + cf\Big(\frac{z}{2^n}\Big)\right\| \\
&\le \lim_{n\to\infty} 2^n \left[\Big\| f\big(\frac{ax+by+cz}{2^n}\big)\Big\| + \phi\big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\big)\right] \\
&\le \lim_{n\to\infty} \left[2^n \Big\| f\big(\frac{ax+by+cz}{2^n}\big)\Big\| + L^n\phi(x,y,z)\right] \\
&= \|A(ax+by+cz)\|.
\end{aligned}$$

Thus the mapping $A : G \longrightarrow Y$ is Cauchy additive by Lemma 2.1.

Now, let $T : G \longrightarrow Y$ be another Cauchy additive mapping satisfying (2.5). Then we obtain

$$\begin{aligned}
&\|2^n f(\frac{x}{2^n}) - T(x)\| = 2^n \|f(\frac{x}{2^n}) - T(\frac{x}{2^n})\| \\
&\le \frac{L2^n}{2|c|(1-L)}\Big[\phi\Big(\frac{-cx}{2^n a}, \frac{-cx}{2^n b}, \frac{2x}{2^n}\Big) \\
&\qquad\qquad + \phi\Big(\frac{-cx}{2^n a}, 0, \frac{x}{2^n}\Big) + \phi\Big(0, \frac{-cx}{2^n b}, \frac{x}{2^n}\Big)\Big] \\
&\le \frac{L^{n+1}}{2|c|(1-L)}\left[\phi\Big(\frac{-cx}{a}, \frac{-cx}{b}, 2x\Big) + \phi\Big(\frac{-cx}{a}, 0, x\Big) + \phi\Big(0, \frac{-cx}{b}, x\Big)\right],
\end{aligned}$$

which tends to zero as $n \to \infty$. So we can conclude that $A(x) = T(x)$ for all $x \in G$. This proves the uniqueness of A. $\square$

Corollary 2.3. *Assume that there exist a nonnegative numbers θ and a real $p > 1$ such that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|af(x) + bf(y) + cf(z)\| \le \|f(ax+by+cz)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in G$. Then there exists a unique Cauchy additive mapping $A : G \to Y$ such that

$$\begin{aligned}
&\|aA(x) + bA(y) + cA(z)\| \le \|A(ax+by+cz)\|, \\
&\|f(x) - A(x)\| \le \frac{\theta\|x\|^p}{|c|(2^p-2)}\left[\frac{2|c|^p}{|a|^p} + \frac{2|c|^p}{|b|^p} + 2^p + 2\right]
\end{aligned}$$

for all $x, y, z \in G$.

THEOREM 2.4. *Assume that a mapping* $f : G \to Y$ *with* $f(0) = 0$ *satisfies the inequality* (2.4) *and that the map* $\phi : G \times G \times G \to [0, \infty)$ *is contractively subadditive with a constant* L. *Then there exists a unique Cauchy additive mapping* $A : G \to Y$, *defined by* $A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$, *such that*

$$\|aA(x) + bA(y) + cA(z)\| \le \|A(ax + by + cz)\|, \tag{2.10}$$
$$\|f(x) - A(x)\| \le \frac{1}{2|c|(1-L)}\Big[\phi\Big(\frac{-cx}{a}, \frac{-cx}{b}, 2x\Big) + \phi\Big(\frac{-cx}{a}, 0, x\Big) + \phi\Big(0, \frac{-cx}{b}, x\Big)\Big]$$

for all $x, y, z \in G$.

Proof. We get by (2.9)

$$\begin{aligned}\Big\|\frac{1}{2^m} f(2^m x) - \frac{1}{2^n} f(2^n x)\Big\| &\le \sum_{j=m}^{n-1}\Big\|\frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x)\Big\| \\ &\le \frac{1}{|c|}\sum_{j=l}^{m-1}\frac{1}{2^{j+1}}\Big[\phi\Big(\frac{-2^j cx}{a}, \frac{-2^j cx}{b}, 2^{j+1}x\Big) \\ &\quad + \phi\Big(\frac{-2^j cx}{a}, 0, 2^j x\Big) + \phi\Big(0, \frac{-2^j cx}{b}, 2^j x\Big)\Big] \\ &\le \frac{1}{2|c|}\sum_{j=m}^{n-1} L^j \Big[\phi\Big(\frac{-cx}{a}, \frac{-cx}{b}, 2x\Big) + \phi\Big(\frac{-cx}{a}, 0, x\Big) + \phi\Big(0, \frac{-cx}{b}, x\Big)\Big]\end{aligned} \tag{2.11}$$

for all nonnegative integers n, m with $n > m$ and all $x \in G$. It means that a sequence $\{\frac{1}{2^n} f(2^n x)\}$ is Cauchy sequence for all $x \in G$. So one can define a mapping $A : G \to Y$ by $A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ for all $x \in G$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.11), we get (2.10).

The remaining proof goes through by the similar argument to Theorem 2.2. □

COROLLARY 2.5. *Assume that there exist a nonnegative numbers* θ, δ *and a real* $p < 1$ *such that a mapping* $f : G \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|af(x) + bf(y) + cf(z)\| \le \|f(ax + by + cz)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in G$. Then there exists a unique Cauchy additive mapping $A : G \to Y$ such that

$$\|aA(x) + bA(y) + cA(z)\| \le \|A(ax + by + cz)\|,$$
$$\|f(x) - A(x)\| \le \frac{\theta\|x\|^p}{|c|(2-2^p)}\left[\frac{2|c|^p}{|a|^p} + \frac{2|c|^p}{|b|^p} + 2^p + 2\right]$$

for all $x, y, z \in G$.

THEOREM 2.6. *Assume that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the functional inequality (2.4) and that the map $\phi : G\times G\times G \to [0,\infty)$ satisfies the condition*

$$\Phi(x, y, z) := \sum_{j=1}^{\infty} 2^j \phi(\frac{x}{2^j}, \frac{y}{2^j}, \frac{z}{2^j}) < \infty$$

for all $x, y, z \in G$. Then there exists a unique Cauchy additive mapping $A : G \to Y$, defined by $A(x) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$, such that

$$\begin{aligned} &\|aA(x) + bA(y) + cA(z)\| \le \|A(ax + by + cz)\|, \\ &\|A(x) - f(x)\| \le \frac{1}{2|c|}\Big[\Phi\Big(\frac{-cx}{a}, \frac{-cx}{b}, 2x\Big) \\ &\qquad\qquad + \Phi\Big(\frac{-cx}{a}, 0, x\Big) + \Phi\Big(0, \frac{-cx}{b}, x\Big)\Big] \end{aligned} \tag{2.12}$$

for all $x, y, z \in G$.

Proof. Now it follows from (2.9) that

$$\begin{aligned} \Big\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})\Big\| &\le \sum_{j=l+1}^{m} \Big\|2^j f(\frac{x}{2^j}) - 2^{j-1} f(\frac{x}{2^{j-1}})\Big\| \\ &\le \frac{1}{2|c|}\sum_{j=l+1}^{m} 2^j\Big[\phi(\frac{-cx}{2^j a}, \frac{-cx}{2^j b}, \frac{2x}{2^j}) + \phi(\frac{-cx}{2^j a}, 0, \frac{x}{2^j}) + \phi(0, \frac{-cx}{2^j b}, \frac{x}{2^j})\Big] \end{aligned} \tag{2.13}$$

for all $x \in G$ and for all nonnegative integers m and l with $m > l$. It means that for any $x \in G$ a sequence $\{2^m f(\frac{x}{2^m})\}$ is Cauchy in Y. Since Y is complete, the sequence $\{2^m f(\frac{x}{2^m})\}$ converges. So one can define a mapping $A : G \to Y$ by $A(x) := \lim_{m\to\infty} 2^m f(\frac{x}{2^m})$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.13), we get the approximation (2.12) of f by A.

The remaining proof goes through by the similar argument to Theorem 2.2. □

THEOREM 2.7. *Assume that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the inequality* (2.4) *and that the map $\phi : G \times G \times G \to [0, \infty)$ satisfies the condition*

$$\Phi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \phi(2^j x, 2^j y, 2^j z) < \infty$$

for all $x, y, z \in G$. Then there exists a unique Cauchy additive mapping $A : G \to Y$, defined by $A(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$, such that

$$\begin{aligned} &\|aA(x) + bA(y) + cA(z)\| \le \|A(ax + by + cz)\|, \\ &\|A(x) - f(x)\| \le \frac{1}{2|c|}\Big[\Phi\Big(\frac{-cx}{a}, \frac{-cx}{b}, 2x\Big) \\ &\qquad\qquad + \Phi\Big(\frac{-cx}{a}, 0, x\Big) + \Phi\Big(0, \frac{-cx}{b}, x\Big)\Big] \end{aligned} \tag{2.14}$$

for all $x, y, z \in G$.

Proof. We get by (2.9)

$$\begin{aligned} \Big\|\frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x)\Big\| &\le \sum_{j=l}^{m-1} \Big\|\frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x)\Big\| \\ \le \frac{1}{2|c|} \sum_{j=l}^{m-1} \frac{1}{2^j} \Big[\phi\Big(\frac{-2^j cx}{a}, \frac{-2^j cx}{b}, 2^{j+1} x\Big) \\ + \phi\Big(\frac{-2^j cx}{a}, 0, 2^j x\Big) + \phi\Big(0, \frac{-2^j cx}{b}, 2^j x\Big)\Big] \end{aligned} \tag{2.15}$$

for all nonnegative integers m and l with $m > l$ and all $x \in G$. It means that a sequence $\{\frac{1}{2^m} f(2^m x)\}$ is Cauchy sequence in Y for all $x \in G$. Since Y is complete, the sequence $\{\frac{1}{2^m} f(2^m x)\}$ converges. So one can define a mapping $A : G \to Y$ by $A(x) := \lim_{m\to\infty} \frac{1}{2^m} f(2^m x)$ for all $x \in G$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.15), we get the functional inequality (2.14).

The remaining proof goes through by the similar argument to Theorem 2.6. □

COROLLARY 2.8. *Assume that there exists a nonnegative numbers δ such that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|af(x) + bf(y) + cf(z)\| \le \|f(ax + by + cz)\| + \delta$$

for all $x, y, z \in G$. Then there exists a unique Cauchy additive mapping $A : G \to Y$ such that

$$\|aA(x) + bA(y) + cA(z)\| \leq \|A(ax + by + cz)\|,$$

$$\|f(x) - A(x)\| \leq \frac{3\delta}{|c|} \tag{2.16}$$

for all $x, y, z \in G$.

The following approximation of f by A has much simpler upper bound than that of (2.12).

Theorem 2.9. *Assume that a mapping $f : G \to Y$ with $f(0) = 0$ satisfies the functional inequality* (2.4) *and that the map $\phi : G \times G \times G \to [0, \infty)$ satisfies the condition*

$$\Phi(x, y, z) := \sum_{j=1}^{\infty} |\lambda|^j \phi(\frac{x}{\lambda^j}, \frac{y}{\lambda^j}, \frac{z}{\lambda^j}) < \infty$$

for all $x, y, z \in G$, where $\lambda := \frac{-a-b}{c} \neq 0$. Then there exists a unique Cauchy additive mapping $A : G \to Y$, defined by $A(x) = \lim_{n\to\infty} \lambda^n f(\frac{x}{\lambda^n})$ such that

$$\|aA(x) + bA(y) + cA(z)\| \leq \|A(ax + by + cz)\|,$$

$$\|A(x) - f(x)\| \leq \frac{1}{|a + b|} \Phi\left(x, x, \lambda x\right)$$

for all $x, y, z \in G$.

Proof. Replacing (x, y, z) by $\left(x, x, \frac{-a-b}{c} x\right)$ in (2.4), we get

$$\left\| f(x) - \frac{f(\lambda x)}{\lambda} \right\| \leq \frac{1}{|a + b|} \phi\left(x, x, \lambda x\right). \tag{2.17}$$

Now it follows from (2.17) that

$$\begin{aligned} \left\|\lambda^l f(\frac{x}{\lambda^l}) - \lambda^m f(\frac{x}{\lambda^m})\right\| &\leq \sum_{j=l}^{m-1} \left\|\lambda^j f(\frac{x}{\lambda^j}) - \lambda^{j+1} f(\frac{x}{\lambda^{j+1}})\right\| \\ &\leq \frac{1}{|a + b|} \sum_{j=l}^{m-1} |\lambda|^{j+1} \phi\left(\frac{x}{\lambda^{j+1}}, \frac{x}{\lambda^{j+1}}, \frac{\lambda x}{\lambda^{j+1}}\right) \end{aligned}$$

for all $x \in G$ and for all nonnegative integers m and l with $m > l$.

The rest of proof is similar to the corresponding part of Theorem 2.6. $\square$

Theorem 2.10. *Assume that a mapping* $f : G \to Y$ *with* $f(0) = 0$ *satisfies the inequality* (2.4) *and that the map* $\phi : G \times G \times G \to [0, \infty)$ *satisfies the condition*

$$\Phi(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{|\lambda|^j} \phi(\lambda^j x, \lambda^j y, \lambda^j z) < \infty$$

for all $x, y, z \in G$, *where* $\lambda := \frac{-a-b}{c} \neq 0$. *Then there exists a unique Cauchy additive mapping* $A : G \to Y$, *defined by* $A(x) := \lim_{n\to\infty} \frac{f(\lambda^n x)}{\lambda^n}$ *such that*

$$\|aA(x) + bA(y) + cA(z)\| \le \|A(ax + by + cz)\|,$$
$$\|A(x) - f(x)\| \le \frac{1}{|a+b|} \Phi(x, x, \lambda x)$$

for all $x, y, z \in G$.

Corollary 2.11. *Assume that there exists a nonnegative numbers* δ *such that a mapping* $f : G \to Y$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|af(x) + bf(y) + cf(z)\| \le \|f(ax + by + cz)\| + \delta$$

for all $x, y, z \in G$, *where* $0 < \left|\lambda := \frac{-a-b}{c}\right| \neq 1$ *Then there exists a unique Cauchy additive mapping* $A : G \to Y$ *such that*

$$\|aA(x) + bA(y) + cA(z)\| \le \|A(ax + by + cz)\|,$$
$$\|f(x) - A(x)\| \le \frac{\delta}{\Big||a+b| - |c|\Big|} \tag{2.18}$$

for all $x, y, z \in G$.

We observe that the best approximation between (2.16) and (2.18) of f by A is determined by constants a, b, c.

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