JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **20**, No. 4, December 2007

WEAKLY LARGE SUBSYSTEMS OF S-SYSTEM

Jupil Kim* and Sungjin Kim **

Abstract. The purpose of this paper is to introduce and investigate the weakly large subsystem of S-system M_S. In this study, we consider injective algebraic system and large subsystem. We also characterize weakly nonsingular congruence on S-system.

1. Introduction

Let S be a semigroup. A right S*-system* M_S is a set M together with a map(written multiplicatively) from $M \times S$ into M satisfying $x(ab) = (xa)b$ for all $x \in M$ and $a, b \in S$. If one things of each element of S as inducing a unary operation on an S-system M_S, then M_S is a finitary algebra and all the notions of universal algebra are available. A nonempty subset N of an S-system M_S is S*-subsystem* if $NS \subset N$.

If subsystem N_S of M_S consists of a single element z, then $za = z$ for all $a \in S$; such an element z we call *fixed element* of M_S. If S-system M_S contains a unique fixed element z and S has a zero element 0, then $m0 = z$ for every $m \in M$. For $m0$ is clearly a fixed element of M_S. If M_S has a unique fixed element z over a semigroup S with 0, then every subsystem N_S of M_S contains z and the symbol z will be called *zero* of M_S([3]). Dually, we can define left S-system.

If S has an identity 1, the S-system M_S is *unital* when $m1 = m$ for each $m \in M$. For each semigroup S we shall define S^1 by $S^1 = S \cup \{1\}$ where 1 is a symbol not in S and where multiplication on S is extened to S^1 by defining $1x = x1 = x$ for all $x \in S^1$. With the operation so defined, S^1 is a semigroup. Note that this definition for S^1 differs from the standard one. However, with the definition given here each S-system M_S becomes a unital S^1-system by defining $m1 = m$ for each $m \in M$.

Received October 29, 2007.
2000 Mathematics Subject Classification: Primary 20M99; Secondary 16A52.
Key words and phrases: S-system, injective, large, weakly large, weakly nonsingular.
This work was supported by Hannam University Research Fund, 2006.

We shall use the term S-system, simply, to mean "right S-system". Unless otherwise stated, all algebraic notions will be in this category. We will assume throughout this paper S will always denote a semigroup and S^1 is a semigroup with identity 1.

Let M_S and N_S be two S-systems. A mapping $f : M_S \to N_S$ such that $f(ms) = f(m)s$ for all $m \in M$ and $s \in S$ is called an *homomorphism.* The usual definition for monomorphism, epimorphism and isomorphism hold. The set of all homomorphism from M_S to N_S is denoted by it $Hom_S(M, N)$.
An equivalence relation θ on M_S is a *congruence relation* if and only if $(a, b) \in \theta$ implies $(as, bs) \in \theta$ for all $s \in S$.

An algebra A in an equational class $\mathcal{C}$ is called *injective* in $\mathcal{C}$ if and only if for any homomorphism $f : B \to A$ and any monomorphism $h : B \to C$, where B and C belong to $\mathcal{C}$, there exist a homomorphism $g : C \to A$ such that $f = gh$. This situation does indeed occur, and examples are provided by some very familiar equational classes, such as the class of groups, all monoids, and ring with unit, among others.([1])

S-system M_S is called *weakly injective* if, for every right ideal K of S and every homomorphism $\phi : K \to M$, there exists an $m \in M$ such that $\phi(k) = mk$ for all $k \in K$. Injective S-system is weakly injective, but the converse does not hold in general. It was shown by Berthiaume([2]) that, unlike the unitary R-module situation, this notion does not coincide with that of injectivity of S-systems. Berthiaume's counterexample was a semilattice considered as an S-system over itself.

An algebra B in an equational class $\mathcal{C}$ is called an *essential extension* of an algebra $A \in \mathcal{C}$ if and only if B is an extension of A such that any homomorphism $h : B \to C$, where $C \in \mathcal{C}$, is a monomorphism whenever retraction of h in A is a monomorphism. Subalgebra N of M is *large* in M when M is an essential extension of N.

It is well known that submodule N_R is large in M_R if and only if it has nonzero intersection with every nonzero submodule of M_R. A subsystem N_S of M_S is *meet-large* in M_S if for each $0 \neq m \in M$ there exist $s \in S^1$ such that $0 \neq ms \in N_S$. Note that N_S is meet-large in M_S if and only if the intersection of N_S with any nonzero subsystem of M_S is always nonzero. This definition is a generalization of a corresponding concept in ring theory. The following result shows that the relationship between injectivity and essential extensions in modules.

THEOREM 1.1. *([8]). R-module M_R is injective if and only if M_R has no proper essential extension.*

The following theorem due to Berthiaume([2]) guarantees the existence of minimal injective extension which is unique up to isomorphism for any S-system M_S. If N_S is subsystem of M_S and if M_S is injective then M_S is called an *injective extension* of N_S.

THEOREM 1.2. *([5]). The S-system M_S is a maximal essential extension of N_S if and only if M_S is a minimal injective extension of N_S. Every S-system M_S has such an extension which is unique up to isomorphism over M_S.*

The minimal injective extension of M_S given in the above theorem 1.2 is called the *injective hull* of M_S and denoted by $I(M)$. Note that $I(M)$ is the injective hull of M_S if and only if M_S is large in $I(M)$ and $I(M)$ is injective. In [7], J.P. Kim and Y.S. Park characterize the $I(M)$ where M_S is S-system over Clifford semigroup. The following lemma characterizes large subsystems in terms of congruences. For any S-system M_S, 1_M means identity relation of M_S.

LEMMA 1.3. *([5]). For any S-system M_S, N_S is large in M_S if and only if every congruence ρ on M_S such that $\rho \neq 1_M$ we have $\rho|_N \neq 1_N$.*

2. Weakly large S-system

In general S-system M_S does not has zero, so we define the following; The definition is generalization of a corresponding concept in module. $|A|$ denotes the cardinal number of set A.

DEFINITION 2.1. A subsystem N_S of M_S is called *weakly large* if for every non-fixed subsystem A_S of M_S, $|A \cap N| \geq 2$.

1) If M_S has unique fixed element z and S is a semigroup without zero, then we adjoin 0 on S and let $S^0 = S \cup \{0\}$. We can make M into an S^0-system M_{S^0} by defining $m0 = z$ for all $m \in M$. Since every subsystem N_S of M_S is also subsystem of M_{S^0}, N_S contains z.

2) If M_S has more than two fixed elements, then for any fixed element $z \in M_S$ take another fixed element $x \in M_S$. Since $A = \{x, z\}$ is a subsystem of M_S, $2 = |A| \geq |A \cap N| \geq 2$ and so $A \cap N = N$. Thus $z \in A \subset N$.

Hence N_S contains all fixed elements of M_S.

In fact, above definition means a subsystem N_S of M_S is weakly large if

(1) N_S contains all fixed elements of M_S and

(2) for any non-fixed subsystem A_S of M_S, A_S meets N_S with more than

two elements.

S-System M_S called *weakly essential extension* of N_S when N_S is weakly large subsystem of M_S. We can see that above definition 2.1 is equivalent to meet-large(Lopez and Ludeman [9]) if S-system M_S has a zero element. By definition, weakly large subsystem N_S of any non zero S-system M_S has at least two elements.

EXAMPLE 2.2. If S is left zero semigroup, then S_S itself is S-system and every subset of S is subsystem of S_S. So that S_S has no proper weakly large subsystem.

The set of all fixed elements of M_S will be denoted by $\mathcal{F}(M)$. Following example shows $\mathcal{F}(M)$ may be empty. If S is semigroup with 0, then $\mathcal{F}(M) \neq \emptyset$.

EXAMPLE 2.3. Infinite chain $S = \{e_1, e_2, ...\}$ with order $e_1 > e_2 >, ..., .$ has no fixed element considering S itself as S-system S_S.

LEMMA 2.4. *If A_S and B_S are weakly large subsystems of M_S, then $A \cap B$ is weakly large subsystem of M_S.*

Proof. Evidently $\mathcal{F}(M) \subset A \cap B$. For any subsystem N_S of M_S such that $|N| \geq 2$, $|B \cap N| \geq 2$. Since $B \cap N$ is also subsystem of M_S having more than two elements and since A_S is weakly large subsystem of $M, |A \cap (B \cap N)| \geq 2$.

THEOREM 2.5. *If N_S is large subsystem of M_S, then N_S is weakly large.*

Proof. Let N_S be a large subsystem of M_S and A_S be any non-fixed subsystem of M_S. Take any two distinct elements a and b in A_S and if θ is the smallest congruence of M_S containing (a, b), then by lemma 5.8([6]), the relation θ defined on M_S is

$$\theta = 1_M \cup \{(a, b), (b, a)\} \cup \{(x, y) | x = as_1, bs_1 = as_2, bs_2, ..., as_n, bs_n = y\}.$$

Since $\theta \neq 1_M$, there exist $(c, d) \in \theta$ such that $c \neq d, c, d \in N$. Since A_S is subsystem of $M_S, c, d \in A$. Thus $|A \cap N| \geq 2$ □

COROLLARY 2.6. *([4]) If M_S is a S-system with zero, then every large subsystem of M_S is meet-large in M_S.*

COROLLARY 2.7. *Let $I(M)$ be injective hull of M_S, then M_S is weakly large in $I(M)$.*

Following example shows there is a weakly large subsystem which is not large.

Example 2.8. Let $S = \{e_1, e_2, ...\}$ be infinite chain with order $e_1 > e_2 >, ...,$. the map $f : S \to S$ by $f(e_1) = f(e_2) = e_1, f(e_i) = e_{i-1}$ for $i = 3, 4, ...$ is homomorphism and $f|_{(e_2S)}$ is one to one. but f is not one to one. So $N = e_2S = \{e_2, e_3, ...\}$ is not large right ideal of S_S. For any subsystem A_S of S_S such that $|A| \geq 2, e_iS \subset A$ if $e_i \in A$. Thus $|A \cap N| \geq 2$. This means N_S is weakly large subsystem of S_S.

Lemma 2.9. *If $A_S \subset B_S \subset M_S$ are S-systems, then A_S is weakly large in M_S if and only if A_S is weakly large in B_S and B_S is weakly large in M_S.*

Proof. Sufficiency. Let N_S be any non-fixed subsystem of B_S. Since N_S is also subsystem of M_S, $|N \cap A| \geq 2$. Hence A_S is weakly large in B_S. Again let N_S be any subsystem of M_S. Then

$$|N \cap B| \geq |N \cap A| \geq 2$$

and so B_S is weakly large in M_S.

Necessity. Let N_S be any subsystem of M_S such that $|N| \geq 2$. Then $|B \cap N| \geq 2$ and since A_S is weakly large in B_S,

$$|(A \cap N)| \geq |(A \cap N) \cap B| \geq 2$$

$\mathcal{F}(M) \subset A$ is trivial. Thus A_S is weakly large in M_S. □

The following theorem ensures that there are plenty of weakly large subsystems.

Theorem 2.10. *For any subsystem A_S of S-system M_S, let $N_S = \max \{B_S | B_S$ is a subsystem of M_S such that $A \cap B = \mathcal{F}(A)\}$. Then $A \cup N$ is weakly large subsystem of M_S. In case there are no subsystem B_S of M_S such that $A \cap B = \mathcal{F}(A)$, N_S is empty set.*

Proof. Let C_S be any subsystem of M_S such that $|C| \geq 2$,
1) If $(A \cup N) \cap C = \emptyset$, then $C \cup N = N$ by maximality of N_S. So $C \subset N$ and leads contradiction.
2) Assume $(A \cup N) \cap C = \{a\}$, then a is fixed element of M_S.
2)-1 If $a \in A$, then $a \in A \cap N$. Hence $A \cap (C \cup N) = A \cap N = \mathcal{F}(A)$ and $C \subset N$ also contradict
2)-2 If $a \in N - A$, then $A \cap (N \cup C) = \mathcal{F}(A)$ and also $C \subset N$ leads contradiction.
Evidently N_S contains all fixed elements of M_S. □

Theorem 2.11. *Let M_S and N_S are S-systems and $f \in Hom_S(M, N)$. If A_S is weakly large subsystem of N_S, then $f^{-1}(A)$ is weakly large subsystem of M_S.*

Proof. We can prove easily that if $z \in \mathcal{F}(M)$, then $f(z)$ is also fixed element of N_S. By definition of weakly large, $f(z) \in A$ and so $z \in f^{-1}(A)$. Let B_S be any subsystem of M_S more than two elements. If $f^{-1}(A) \cap B = \emptyset$, then $f(B) \cap A = \emptyset$.
1) If $f(B) = \{a\}$ is singleton, since $f(B)$ is subsystem of N_S, a is fixed element of N_S. And so from $f(B) \subset A, B \subset f^{-1}(A)$, $|f^{-1}(A) \cap B| = |B| \geq 2$
2) If $f(B)$ is not singleton, then from $|f(B) \cap A| \geq 2$ we can take $c, d \in f(B) \cap A, c \neq d$. Let $f(a) = c, f(b) = d$, then $a \neq b$ and $a, b \in f^{-1}(A) \cap B$. Thus $|f^{-1}(A) \cap B| \geq 2$ □

Corollary 2.12. *If N_S is a weakly large subsystem of M_S, then $m^{-1}N = \{s \in S | ms \in N\}$ is weakly large subsystem of S_S for all $m \in M$.*

Proof. Let $f : S \to M$ by $f(s) = ms$ then f is homomorphism and $f^{-1}(N) = \{s \in S | ms \in N\} = m^{-1}N$.

A subsystem N_S of M_S is *dense* in M_S if and only if $a \neq b, m \in M$, there exist $s \in S^1$ such that $as \neq bs, ms \in N$. We can prove easily that dense subsystem of M_S is meet-large. However, following example shows the converse is false. □

Example 2.13. $S = \{0, a, b, 1\}$ with order $0 < a < b < 1$ is semilattice. The ideal $N_S = \{0, a, b\}$ is clearly weakly large. but $b \neq 1, 1 \in S$ and for any non identity x of $S, bx = x$. So N_S is not dense subsystem of S_S.
But for the map $f : S \to S$ by $f(1) = f(b) = b, f(a) = a, f(0) = 0, f|_N$ is one to one. Therefore N_S is not large subsystem of S_S.

Theorem 2.14. *If N_S is dense subsystem of M_S, then N_S is weakly large subsystem of M_S*

Proof. If z is fixed element of M_S, then $z = zs$ for all $s \in S$. So $z \in N$. Let A_S be any non-fixed subsystem of M_S. Take $a \neq b$ in $A, a \in M$, then there exist $s_1 \in S^1$ such that $as_1 \neq bs_1, as_1 \in N$. Again for $as_1 \neq bs_1, bs_1 \in M$, there exist $s_2 \in S^1$ such that $as_1s_2 \neq bs_1s_2, bs_1s_2 \in N$. Since $as_1 \in N, as_1s_2 \in N$, and since $a, b \in A, as_1s_2, bs_1s_2 \in A$. We have $|N \cap A| \geq 2$.

□

The class of all weakly large subsystem of S-system M_S will be denoted by $\mathcal{W}(M)$. The class is closed under finite intersections

Definition 2.15. The relation
$\psi_S(M) = \{(a,b) \in M \times M|$ there exist $I \in \mathcal{W}(S)$ such that $ax = bx$ for all $x \in I\}$
is called *weakly singular congruence* of M_S

It is easily seen from the properties noted above that $\psi_S(M)$ is a congruence on M_S. If S is left zero semigroup with 1, then weakly singular congruence of any S-system M_S is 1_M. We call M_S is *weakly nonsingular* if $\psi_S(M) = 1_M$.

Theorem 2.16. *M_S is weakly nonsingular if and only if for any relation R of M_S*
$r(R) = \{x \in S|mx = nx$ for all $(m,n) \in R\}$
has no proper weakly essential extension on S^1_S.

Proof. "only if" ; Let R be any relation of $M_S, r(R)$ be weakly large subsystem of some right ideal I of S^1. For any $u \in I$ if we define $f \in Hom(S,I)$ by $f(x) = ux$, then $f^{-1}(r(R)) \in \mathcal{W}(S)$ by theorem 2.3. For any $(m,n) \in R, t \in f^{-1}(r(R)), f(t) \in r(R)$ and so $mut = nut$. Thus $(mu,nu) \in \psi_S(M)$. By hypothesis, we have $mu = nu$. Therefore $u \in r(R)$.
"if"; Let $(a,b) \in \psi_S(M)$. Then there exist some $I \in \mathcal{W}(S)$ such that $ax = bx$ for all $x \in I$. For singleton relation $R = \{(a,b)\}$ of $M, I \subset r(R) = \{x \in S|ax = bx\}$. Since $I \in \mathcal{W}(S)$, by lemma 2.2 $r(R) \in \mathcal{W}(S)$. Since $\mathcal{W}(S) \subset \mathcal{W}(S^1)$ and $r(R)$ is weakly large subsystem of S^1_S, $r(R) = S^1$. Thus $ax = bx$ for all $x \in S^1$ and so $a = b$. □

It is well known that S-system M_S is nonsingular if and only if every meet-large subsystem of M_S is dense.

Theorem 2.17. *If M_S is weakly non-singular S-systems, then every weakly large subsystem of M_S is large in M_S.*

Proof. Let A be weakly large subsystem of M_S. For any subsystem of B_S, and any $f \in Hom_S(M,B)$ such that $f|_A$ is one to one. Suppose $f(a) = f(b), a,b \in M$. Let $N = a^{-1}A \cap b^{-1}A = \{x \in S|ax \in A$ and $bx \in A\}$. Then by lemma 2.1, N_S is weakly large right ideal of S_S and $f(as) = f(bs)$ for all $s \in N$. Since $as, bs \in A$ for all $s \in N$ and $f|_A$ is one to one, $as = bs$ and $(a,b) \in \psi_S(M)$. From M_S is weakly non-singular we have $a = b$, □

THEOREM 2.18. *If M_S is weakly non-singular, weakly injective, then M_S is injective.*

Proof. Let B_S be any S-system and A_S be a subsystem of B_S. Let $f \in Hom_S(A, M), I(M)$ be injective hull of M_S. Since M_S is large in $I(M)$, by theorem 2.5. M_S is weakly large in $I(M)$. Since $I(M)$ is injective, there exist $h \in Hom_S(B, I(M))$ such that $h|_A = f$. We claim that $h(B) \subset M$. Let $b \in B$ and let $h(b) = c$, then $c^{-1}M = \{s \in S | cs \in M\}$ is weakly large. the function $g : c^{-1}M \to M$ by $g(x) = cx$ is homomorphism and since M_S is weakly injective, there exist $m \in M$ such that $g(x) = mx$ for all $x \in c^{-1}M$. Thus $cx = mx$ for all $x \in c^{-1}M$. And since $c^{-1}M \in \mathcal{W}(S)$, we have $(c, m) \in \psi_S(M) = 1_M$. Thus $c = m \in M$. □

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Department of Mathematics
Hannam University
Daejeon 306-791, Republic of Korea
E-mail: jpkim@hnu.ac.kr

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Department of Mathematics
Hannam University
Daejeon 306-791, Republic of Korea