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ALMOST PERIODIC HOMEOMORPHISMS AND CHAOTIC HOMEOMORPHISMS

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Abstract. Let $h : M \to M$ be an almost periodic homeomorphism of a compact metric space M onto itself. We prove that h is topologically transitive iff every element of M has a dense orbit. It follows as a corollary that an almost periodic homeomorphism of a compact metric space onto itself can not be chaotic. Some additional related observations on a Cantor set are made.

1. Introduction

The purpose of this paper is to study the relationship between almost periodic homeomorphisms and chaotic homeomorphisms. The study of almost periodic homeomorphisms is one of the classical fields in topological dynamics. Gottschalk[3] defined a regularly almost periodic homeomorphism of a metric space and the author characterized the Hilbert-Smith conjecture using almost periodic homeomorphisms and regularly almost periodic homeomorphisms[5]. Recently, the study of chaotic dynamical systems has come into prominence. In this paper, we use the definition of chaos introduced in [2].

One problem of interest to the author has been to classify the chaotic homeomorphisms of the Riemann sphere onto itself. While this is still an open question, there are some interesting partial results. Recently, the author constructed several chaotic maps of the Riemann sphere $\bar{C}$ induced by branched coverings of $\bar{C}$ and linear maps [6]. In particular, he classified those chaotic maps of the Riemann sphere $\bar{C}$ which are induced by regular branched coverings of $\bar{C}$ and the linear map $2z$.

The main results of this paper are Theorem 3.1, where we show that *an almost periodic homeomorphism of a compact metric space onto itself*

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has a dense orbit iff every orbit is dense, and its Corollary 3.2, which says that *an almost periodic homeomorphism of a compact metric space cannot be chaotic*. We then make some additional observations, using results in the literature about homeomorphisms on the Cantor dyadic set.

2. Background and Definitions

A homeomorphism h of a metric space (M, d) onto itself is said to be *almost periodic* iff, for every $\epsilon > 0$, there exists a relatively dense sequence $\{n_i\}$ of integers (i. e. the gaps are bounded) such that $d(x, h^{n_i}(x)) < \epsilon$ for all $x \in M$ and $i = \pm 1, \pm 2, \pm 3, \ldots$. In particular if, for every $\epsilon > 0$, there exists a positive integer n_ϵ such that $d(x, h^k(x)) < \epsilon$ for all $x \in M$ and for all $k \in n_\epsilon Z$, we say that the homeomorphism h is *regularly almost periodic*.

The two propositions below are well-known, and we will use them later.

Proposition 2.1. [4][pg. 341] Let M be a compact metric space. Then h is an almost periodic homeomorphism on M iff the set of powers of h is equicontinuous on M. □

Proposition 2.2. [3] [pg. 55] *Let h be an almost periodic homeomorphism on a compact metric space (M, d) and let $\epsilon > 0$. Then there exists a regularly almost periodic homeomorphism H on M such that $d(h(x), H(x)) < \epsilon$ for each $x \in M$. The homeomorphism H may be chosen as the uniform limit of a sequence of positive powers of h.* □

Let $f : M \to M$ be a map of metric space M. A map $f : M \to M$ is *chaotic* iff f has sensitive dependence on initial conditions, f is topologically transitive and periodic points are dense in M. Recall that f is *topologically transitive* iff for any pair of open sets $U, V \subset M$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$. We refer to the reader[2] for detailed definition and examples of chaotic map.

The following simple characterization of a chaotic map, which is proved by Touhey [10], is very useful to prove whether a map is chaotic or not. For example, we can easily check that the inverse of a chaotic homeomorphism is also chaotic by this characterization. For the proof of the following proposition, he applied [1] which showed that sensitive dependence on initial conditions is implied by the remaining two conditions.

PROPOSITION 2.3. [10] *A map $f : M \to M$ is chaotic iff for every pair of non-empty open sets U and V of M, f has a periodic orbit Γ such that $\Gamma \cap U \neq \emptyset$ and $\Gamma \cap V \neq \emptyset$.* □

The following, known as the generalized Hilbert-Smith conjecture, which originated from Hilbert's 5-th problem, is one of the classic unsolved problems in topological transformation groups:

If G is a compact group and acts effectively on a manifold, then G is a Lie group.

The author proved that there are several equivalent conditions to the generalized Hilbert-Smith conjecture using almost periodic homeomorphisms and regularly almost periodic homeomorphisms[5]. Therefore we can approach the Hilbert-Smith Conjecture using such homeomorphisms. Recall that one of the equivalent conditions in [5] to the generalized Hilbert-Smith conjecture is the following: *If h is a regularly almost periodic homeomorphism of a compact manifold M onto itself then h is periodic.* Therefore, by Proposition 2.2, we have:

PROPOSITION 2.4. *If the Hilbert-Smith conjecture is true on a compact manifold then the set of all periodic homeomorphisms are dense in the set of all almost periodic homeomorphisms.*□

Since the Hilbert-Smith conjecture is known to be true for all compact one- and two-manifolds, we know that the set of periodic homeomorphisms is dense in the set of all almost periodic homeomorphisms of such manifolds.

3. Main Theorems

THEOREM 3.1. *Let $h : M \to M$ be an almost periodic homeomorphism of a compact metric space onto itself. Then h is topologically transitive iff every element of M has a dense orbit.*

Proof. Suppose that h is topologically transitive. Let $x_0 \in M$ and let $\epsilon > 0$. Now let $U_1, U_2, \ldots, U_m$ be a finite open cover such that U_i is an open ball $B_\epsilon(x)$ centered at x with radius ϵ. We may suppose that x_0 is the center of U_1. We will show that there exists n such that $h^n(x_0) \in U_k$ for $k = 1, 2, \ldots, m$.

Let U'_k be an open ball such that $U'_k \subset U_k$ and $\bar{U'_k} \cap Bd(U_k) = \emptyset$. Let $\gamma = d(\bar{U'_k}, Bd(U_k))$. Note that the family of powers of h is equicontinuous [Proposition 2.1]. Therefore, for $\frac{\gamma}{3}$, there exists $\delta_1 > 0$ such that $d(x, y) < \delta_1$ implies $d(h^n(x), h^n(y)) < \frac{\gamma}{3}$ for $n \in Z$.

Let $V \subset U_1$ be a small open ball centered at x_0 whose radius is less than δ_1. Since h is topologically transitive, there exists $n \in N$ such that $h^n(V) \cap U'_k \neq \emptyset$. Then the diameter of $h^n(V)$ is less than $\frac{2}{3}\gamma$ by the choice of V and the equicontinuity of the powers of h. Therefore $h^n(V)$ is contained in U_k. Consequently U_k contains an orbit point $h^n(x_0)$ of x_0.

Now let W be an open set in M. We choose $\epsilon_1 > 0$ such that W contains an ϵ_1 neighborhood of some $x \in W$. We apply the above argument to ϵ_1. Then W must contain an orbit point of x_0. Since x_0 is an arbitrary point of M, we conclude that every element of M has a dense orbit.

The converse is clear from the definition of topological transitivity. □

COROLLARY 3.2. *Let $h : M \to M$ be an almost periodic homeomorphism of a compact metric space onto itself. Then h is not a chaotic.*

Proof. Let $h : M \to M$ be an almost periodic homeomorphism. Suppose that h is chaotic. Then h is topologically transitive and the set of periodic points is dense in M by the definition of chaotic map. Since h is topologically transitive, every element of M has a dense orbit by Theorem 3.1. Therefore h has no periodic points. This contradiction shows that h cannot be chaotic. □

Recall that h is an almost periodic homeomorphism on a compact metric space M iff the set of powers of h is equicontinuous on M (Proposition 2.1). Therefore we also have the following corollary.

COROLLARY 3.3. *Let $h : M \to M$ be a homeomorphism of a compact metric space onto itself such that the set of powers of h is equicontinuous on M. Then h is not a chaotic map.*□

Let h be a homeomorphism of a metric space (X, d) onto itself. h is said to be *nearly periodic* iff there exists a complete system $\{\Theta_i\}_{i=1}^{\infty}$ of finite covers which are invariant under h [9]. The sequence $\{\Theta_i\}_{i=1}^{\infty}$ is called a *complete system* iff $\{mesh(\Theta_i)\}$ has limit 0.

P. A. Smith [9] showed how to construct a compact 0-dimensional group acting on a compact metric space M generated by a given nearly periodic homeomorphism h of M onto itself. He then stated, without proof, that every element of a p-adic transformation group acting on a compact metric space is nearly periodic. The author provided a proof of this statement [5]. Since regularly almost periodic is equivalent to nearly periodic for homeomorphisms on compact metric spaces citele1, we have the following proposition:

Proposition 3.4. [5] *Let M be a compact metric space and let G be a p-adic transformation group acting on M. Then every element of G is regularly almost periodic on M.* $\square$

Now let G be a p-adic transformation group acting on a compact metric space M. Then every element of G is almost periodic homeomorphism by Proposition 3.4 and the definition of regularly almost periodic homeomorphism. Therefore we have the following theorem. We will discuss more on such homeomorphisms in Section 4.

Theorem 3.5. *Let M be a compact metric space and let G be a p-adic transformation group acting on M. Then no element of G can be chaotic.*

Proof. By Proposition 3.4, every element of G is regularly almost periodic, and thus almost periodic. If an element g of G has a dense orbit, then every orbit would be dense, by Theorem 3.1. But a chaotic homeomorphism has both dense orbits and periodic points. Thus no element of G can be chaotic. $\square$

We remark that Theorem 3.5 shows that we can *not* approach the Hilbert-Smith conjecture using chaotic homeomorphisms.

4. Homeomorphisms on the Cantor dyadic set

The Cantor dyadic set(some mathematician call it sequence space on two symbols[2]) is the one of the most interesting topics in symbolic dynamics. In particular, the shift map on the Cantor dyadic set is widely used to show that some maps are chaotic using topological conjugacy and semi-conjugacy. In this section, as an example of the previous section, we construct almost periodic homeomorphisms(in fact, regularly almost periodic) which is not periodic on the Cantor dyadic set.

We briefly review how we construct the Cantor dyadic set. Let $\Sigma_2 = \{s = (s_0, s_1, s_2, ...)|s_j = 0 \,\text{or}\, 1\}$. We define a metric on Σ_2 as follows. For $s = (s_0, s_1, ...)$ and $t = (t_0, t_1, ...)$ define $d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$. Then the shift map $\sigma : \Sigma_2 \to \Sigma_2$ with $\sigma(s_0, s_1, s_2, ...) = (s_1, s_2, s_3, ...)$ is a chaotic homeomorphism [2][pg. 39-42].

The following is a typical construction of p-adic group, which is very useful to study topological transformation group theory.

Let p be a prime number and let D_p be the set of all formal series in powers of p:

$$g = a_0 + a_1 p + \ldots + a_n p^n + \ldots, \text{ each } a_n = 0, \ldots, p-1.$$

If we add elements with infinite carry-over(some mathematicians call it adding machine operation) , then D_p forms an abelian group and the topology is determined by the following choice of neighborhoods of the identity:

$$U_m = \{g \in D_p | a_i = 0 \,\text{if}\, i < m\},\, m = 1, 2, \dots .$$

We call D_p the *p-adic topological group* or simply a *Cantor group.*

Recall that the infinite carry over operation is the following: Let $g = a_0 + a_1 p + a_2 p^2 ...$ and $h = b_0 + b_1 p + b_2 p^2 ...$ be elements of D_p.

$$g \odot h = c_0 + c_1 p + c_2 p^2 \dots, \text{ where } c_o \equiv a_0 + b_0 (mod\, p),$$
$$c_i \equiv a_i + b_i (mod\, p) \text{ if } a_{i-1} + b_{i-1} < p$$
$$c_i \equiv a_i + b_i + 1 (mod\, p) \text{ if } a_{i-1} + b_{i-1} \geq p.$$

Another important construction of the *p-adic* group is the following which is useful to study covering transformations: Let D_p be the p-adic group which we already constructed. Then

$$U_m = \{g \in D_p | a_i = 0 \,\text{if}\, i < m\},\, m = 1, 2, \dots$$

form open subgroups and hence closed subgroups, since the cosets of U_m are open in D_p. We consider the sequence of quotient groups

$$D_p/U_1, D_p/U_2, \dots, D_p/U_n, \dots .$$

For $j > i$, let

$$h_{i,j} : D_p/U_j \longrightarrow D_p/U_i$$

be the continuous homomorphisms defined by $gU_j \longrightarrow gU_i$. Then we have

$$D_p \simeq \varprojlim \{D_p/U_j\} \text{ with bonding map } h_{i,j}.$$

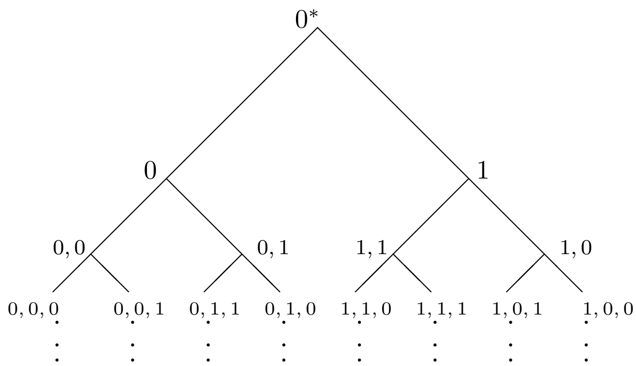
We notice that D_p/U_i is a cyclic group of order p^i. Therefore we can also define the *p-adic* group as the inverse limit of cyclic groups of order p^i for $i = 1, 2, \dots$. See for details [7] and [8][pg.42-56]. We will denote an element of D_p/U_i with $(a_0, a_1, \dots, a_{i-1})$ instead of $(a_0, a_1, \dots, a_{i-1})U_i$.

Construction of nearly periodic homeomorphism on Cantor dyadic set

Let D_2 be the Cantor dyadic set . If we consider Cantor dyadic tree as shown in Figure 4.1, then we can easily identify an element of Cantor dyadic set with an infinite path whose end point is 0^* and $aU_i = (a_0, a_1, \dots, a_{i-1})$ is the unique coset in D_2/U_i such that the path, corresponding to a, pass through.

We define the period 2 homeomorphism $T_2 : D_p/U_1 \to D_2/U_1$, which is the π rotation fixing 0^*, i.e., we rotate it so that the subtree below 0

goes to the subtree below 1. We now define a periodic homeomorphism $T_{2^i} : D_2/U_i \to D_2/U_i$ with period 2^i depending on $T_{2^{i-1}}$.
For example, $T_{2^2} : D_2/U_2 \to D_2/U_2$ is the composition of T_2 and the π rotation of subtree below 1 so that the subtree below (1,0) goes to the subtree below (1,1), fixing the complement of this subtree. Then (0,0), (1,1), (0,1), and (1,0) maps to (1,1), (0,1) ,(1,0) and (0,0) by T_{2^2} resp. Consequently $T_{2^2} : D_2/U_2 \to D_2/U_2$ is periodic with period 4.
Define $T = \lim_{i\to\infty} T_{2^i}$. Recall that $D_2 \simeq \lim_{\leftarrow}\{D_2/U_i\}$. Therefore $T : D_2 \to D_2$ is nearly periodic and therefore regularly almost periodic on D_2. Clearly, T can not be periodic from the above construction.



Cantor Dyadic Tree

Figure 4.1

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