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THE STABILITY OF THE GENERALIZED SINE FUNCTIONAL EQUATIONS III

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Abstract. The aim of this paper is to investigate the stability problem bounded by function for the generalized sine functional equations as follow:

$$f(x)g(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2$$
$$g(x)g(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x+\sigma y}{2}\right)^2.$$

As a consequence, we have generalized the superstability of the sine type functional equations.

1. Introduction

The stability problem of functional equation was raised by S. M. Ulam [15]. Most research follows the Hyers-Ulam stability which is to construct of a convergent sequence by an iteration process. In 1979, J. Baker, J. Lawrence and F. Zorzitto [3] postulated that if f satisfies the stability inequality $|E_1(f) - E_2(f)| \leq \varepsilon$, then either f is bounded or $E_1(f) = E_2(f)$. This is referred as the superstability.

Baker [2] showed the superstability of the cosine functional equation (also called the d'Alembert functional equation)

$$\text{(A)} \qquad f(x+y) + f(x-y) = 2f(x)f(y).$$

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This article's results are related with the following equations

$$(\widetilde{A}) \qquad f(x+y)+f(x+\sigma y)=2f(x)f(y)$$

$$(A_{fg}) \qquad f(x+y)+f(x-y)=2f(x)g(y)$$

$$(\widetilde{A}_{fg}) \qquad f(x+y)+f(x+\sigma y)=2f(x)g(y)$$

$$(T) \qquad f(x+y)-f(x-y)=2f(x)f(y)$$

$$(\widetilde{T}) \qquad f(x+y)-f(x+\sigma y)=2f(x)f(y)$$

$$(T_{fg}) \qquad f(x+y)-f(x-y)=2f(x)g(y)$$

$$(\widetilde{T}_{fg}) \qquad f(x+y)-f(x+\sigma y)=2f(x)g(y),$$

which stabilities have been researched in papers ([1], [5], [8], [10], [11]) in which (A_{fg}) is called the Wilson equation.

In this paper, let $(G,+)$ be a uniquely 2-divisible Abelian group, $\mathbb{C}$ the field of complex numbers, and $\mathbb{R}$ the field of real numbers, and let σ be an endomorphism of G with $\sigma(\sigma(x))=x$ for all $x\in G$ with a notation $\sigma(x)=\sigma x$. The properties $g(x)=g(\sigma x)$ and $g(\sigma x)=-g(x)$ with respect to σ will be represented as even and odd function, respectively. We assume that f and g are nonzero functions and ε is a nonnegative real constant, $\varphi: G\to\mathbb{R}$ be a mapping.

The superstability bounded by constant for the sine functional equation

$$(S) \qquad f(x)f(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2$$

is investigated by P.W. Cholewa [4], and is improved in R. Badora and R. Ger [1], G. H. Kim [9].

Let consider the generalized equations of the sine equation (S) as follow :

$$(\widetilde{S}_{fg}) \qquad f(x)g(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x+\sigma y}{2}\right)^2$$

$$(\widetilde{S}_{gg}) \qquad g(x)g(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x+\sigma y}{2}\right)^2.$$

For the cases $\sigma y=-y$ or $g=f$, they imply the following equations:

$$(S_{fg}) \qquad f(x)g(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2$$

$$(S_{gg}) \qquad g(x)g(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2$$

$$(\widetilde{S}) \qquad f(x)f(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x+\sigma y}{2}\right)^2.$$

Given mappings $f, g : G \to \mathbb{C}$, we define a difference operator $D\widetilde{S}_{fg} : G \to \mathbb{C}$ as

$$D\widetilde{S}_{fg}(x,y) := f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2,$$

which is called the approximate remainder of $(\widetilde{S}_{fg})$ and acts as a perturbation of the equation.

In [12], the author proved the superstability for the equation (S_{fg}) under the condition $|DS_{fg}(x,y)| \le \varepsilon$.

The aim of this paper is to investigate the superstability for the generalized sine functional equation $(\widetilde{S}_{fg})$ under the conditions $|D\widetilde{S}_{fg}(x,y)| \le \varphi(x)$ or $\varphi(y)$. From the obtained results, we also obtain the superstability for the equations (S), $(\widetilde{S})$, (S_{fg}), (S_{gg}), $(\widetilde{S}_{gg})$ as corollaries.

2. Stability of the Equation $(\widetilde{S}_{fg})$

We will investigate the stability of the generalized functional equation $(\widetilde{S}_{fg})$ of the sine functional equation (S).

Theorem 2.1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right| \le \varphi(y) \tag{1}$$

for all $x, y \in G$. Then either f is bounded or g satisfies $(\widetilde{S})$. If, additionally, f satisfies $(\widetilde{A})$, then f and g are solutions of $g(x+y) - g(x+\sigma y) = 2f(x)g(y)$.

Proof. Let f be unbounded. Then we can choose a sequence $\{x_n\}$ in G such that

$$0 \ne |f(2x_n)| \to \infty \qquad \text{as} \quad n \to \infty. \tag{2}$$

Inequality (1) may equivalently be written as

$$|f(2x)g(2y) - f(x+y)^2 + f(x+\sigma y)^2| \le \varphi(2y) \quad \forall\, x, y \in G. \tag{3}$$

Taking $x = x_n$ in (3) we obtain

$$\left| g(2y) - \frac{f(x_n+y)^2 - f(x_n+\sigma y)^2}{f(2x_n)} \right| \le \frac{\varphi(2y)}{|f(2x_n)|},$$

that is, using (2)

$$g(2y) = \lim_{n\to\infty} \frac{f(x_n+y)^2 - f(x_n+\sigma y)^2}{f(2x_n)} \quad \forall\, y \in G. \tag{4}$$

Using (1) we have

$$\begin{aligned}2\varphi(y) \geq & \left|f(2x_n+x)g(y) - f\left(x_n+\frac{x+y}{2}\right)^2 + f\left(x_n+\frac{x+\sigma y}{2}\right)^2\right| \\ & + \left|f(2x_n+\sigma x)g(y) - f\left(x_n+\frac{\sigma x+y}{2}\right)^2 + f\left(x_n+\frac{\sigma(x+y)}{2}\right)^2\right| \\ = & \left| \left(f(2x_n+x)+f(2x_n+\sigma x)\right)g(y)\right. \\ & - \left(f\left(x_n+\frac{x+y}{2}\right)^2 - f\left(x_n+\frac{\sigma(x+y)}{2}\right)^2\right) \\ & \left. + \left(f\left(x_n+\frac{x+\sigma y}{2}\right)^2 - f\left(x_n+\frac{\sigma(x+\sigma y)}{2}\right)^2\right)\right|\end{aligned}$$

for all $x, y \in G$ and every $n \in \mathbb{N}$. Consequently, that is

$$\begin{aligned}\frac{2\varphi(y)}{|f(2x_n)|} \geq & \left|\frac{f(2x_n+x)+f(2x_n+\sigma x)}{f(2x_n)}g(y)\right. \\ & - \frac{f\left(x_n+\frac{x+y}{2}\right)^2 - f\left(x_n+\frac{\sigma(x+y)}{2}\right)^2}{f(2x_n)} \\ & \left. + \frac{f\left(x_n+\frac{x+\sigma y}{2}\right)^2 - f\left(x_n+\frac{\sigma(x+\sigma y)}{2}\right)^2}{f(2x_n)}\right|\end{aligned}$$

for all $x, y \in G$ and every $n \in \mathbb{N}$. Taking the limit as $n \longrightarrow \infty$ with the use of (2) and (4), we conclude that, for every $x \in G$, there exists the limit

$$h(x) := \lim_{n\to\infty} \frac{f(2x_n+x)+f(2x_n+\sigma x)}{f(2x_n)},$$

where the function $h : G \to \mathbb{C}$ satisfies the equation

$$h(x)g(y) = g(x+y) - g(x+\sigma y) \quad \forall x, y \in G. \tag{5}$$

From the definition of h, we get the equality $h(0) = 2$, which jointly with (5) implies that g is an odd w.r.t. σ, namely, $g(y) = -g(\sigma y)$. Keeping this in mind, by means of (5), we infer the equality

$$\begin{aligned}g(x+y)^2 - g(x+\sigma y)^2 &= [g(x+y)+g(x+\sigma y)][g(x+y)-g(x+\sigma y)] \\ &= [g(x+y)+g(x+\sigma y)]h(x)g(y) \\ &= \left[g(2x+y)+g(2x+\sigma y)\right]g(y) \\ &= \left[g(y+2x)-g(y+\sigma(2x))\right]g(y) \\ &= h(y)g(2x)g(y).\end{aligned}$$

The oddness of g forces $g(x+\sigma x)=0$ for all $x\in G$. Hence, putting $x=y$ in (5), we get

$$g(2y)=g(y)h(y).$$

This, in return, leads to the equation

$$g(x+y)^2-g(x+\sigma y)^2=g(2x)g(2y)$$

valid for all $x,y\in G$, which, in the light of the unique 2-divisibility of G, states that g satisfies $(\widetilde{S})$.

Assume that f satisfies $(\widetilde{A})$. Then, since the limited function h becomes $2f$, the equation (5) implies $g(x+y)-g(x+\sigma y)=2f(x)g(y)$. □

THEOREM 2.2. *Suppose that $f,g:G\to\mathbb{C}$ satisfy the inequality*

$$\left|f(x)g(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x+\sigma y}{2}\right)^2\right|\le\varphi(x), \tag{6}$$

which satisfies the cases $f(0)=0$ or $f(x)^2=f(\sigma x)^2$ for all $x,y\in G$.

Then either g is bounded or f satisfies $(\widetilde{S})$. If, additionally, g satisfies $(\widetilde{A})$, then f and g satisfy the equation $(\widetilde{A}_{fg})$.

Proof. Suppose that g is unbounded, then we can choose a sequence $\{y_n\}$ in G such that $0\neq|g(2y_n)|\to\infty$ as $n\to\infty$.

An obvious slight change in the proof steps applied in Theorem 2.1 allows one to state the existence of a limit function

$$p(y):=\lim_{n\to\infty}\frac{g(y+2y_n)+g(\sigma y+2y_n)}{g(2y_n)},$$

where $p:G\to\mathbb{C}$ satisfies the equation

$$f(x)p(y)=f(x+y)+f(x+\sigma y)\qquad\forall x,y\in G. \tag{7}$$

From the definition of p, we get the equality $p(y)=p(\sigma y)$. Clearly, this applies also to the function $\widetilde{p}:=\frac{1}{2}p$. Moreover, $\widetilde{p}(0)=\frac{1}{2}p(0)=1$ and

$$f(x+y)+f(x+\sigma y)=2f(x)\widetilde{p}(y)\qquad\forall x,y\in G. \tag{8}$$

Consider the case $f(0)=0$, then we know under (8) that

$$f(0)=0\Longrightarrow f(x)=-f(\sigma x)\Longrightarrow f(x+\sigma x)=0\Longrightarrow f(0)=0. \tag{9}$$

Putting $y=x$ in (8), we get by (9) a duplication formula

$$f(2x)=2f(x)\widetilde{p}(x).$$

Using (9) and a dupplication of f, we obtain, by means of (8), the equation

$$\begin{aligned} f(x+y)^2 - f(x+\sigma y)^2 &= [f(x+y)+f(x+\sigma y)][f(x+y)-f(x+\sigma y)] \\ &= 2f(x)\widetilde{p}(y)[f(x+y)-f(x+\sigma y)] \\ &= f(x)[f(x+2y)-f(x+2\sigma y)] \\ &= f(x)[f(x+2y)+f(\sigma x+2y)] \\ &= 2f(x)f(2y)\widetilde{p}(x) = f(2x)f(2y) \end{aligned}$$

holds true for all $x, y \in G$, which, in the light of the unique 2-divisibility of G, states that f satisfies nothing else but $(\widetilde{S})$.

In case $f(x)^2 = f(\sigma x)^2$, it is sufficient to show that $f(0) = 0$. Suppose that this is not the case. Then, without loss of generality, we may assume that $f(0) = 1$.

Putting $x = 0$ in (6) with $f(x)^2 = f(\sigma x)^2$ and the 2-divisibility of group G, we obtain the inequality

$$|g(y)| \le \varphi(0) \qquad \forall\, y \in G.$$

This inequality means that f is globally bounded – a contradiction. Thus the claimed $f(0) = 0$ holds, it is completed that f satisfies $(\widetilde{S})$.

For the additive case, assume that g satisfies $(\widetilde{A})$. Then, since the limited function p becomes $2g$, the equation (7) implies $(\widetilde{A}_{fg})$. The proof of the theorem is completed. □

Theorem 2.3. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$(10) \qquad \left| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right| \le \min\{\varphi(x), \varphi(y)\}$$

for all $x, y \in G$.

(a) either f is bounded or g satisfies $(\widetilde{S})$. If, additionally, f satisfies $(\widetilde{A})$, then f and g satisfy the equation $g(x+y) - g(x+\sigma y) = 2f(x)g(y)$.

(b) either g is bounded or g satisfies $(\widetilde{S})$. If, additionally, f satisfies $(\widetilde{A})$, then f and g are solutions of $g(x+y) - g(x+\sigma y) = 2f(x)g(y)$.

(c) if the inequality (10) satisfies the cases $f(0) = 0$ or $f(x)^2 = f(\sigma x)^2$ for all $x \in G$, then either g is bounded or f satisfies $(\widetilde{S})$. If, additionally, g satisfies $(\widetilde{A})$, then f and g satisfy the equation $(\widetilde{A}_{fg})$.

Proof. (a) and (c) are trivial from Theorem 2.1 and Theorem 2.2, respectively.

For (b), it is enough from (a) to show that the unboundedness of g implies that of f.

The inequality (10) can be represented by the equation

$$(11) \qquad |f(2x)g(2y) - f(x+y)^2 + f(x+\sigma y)^2| \le \min\{\varphi(2x), \varphi(2y)\}.$$

If f is bounded, choose $x_0 \in G$ such that $f(2x_0) \neq 0$, and then by (11) we obtain

$$\begin{aligned} |g(2y)| - \left| \frac{f(x_0+y)^2 - f(x_0+\sigma y)^2}{f(2x_0)} \right| &\leq \left| \frac{f(x_0+y)^2 - f(x_0+\sigma y)^2}{f(2x_0)} - g(2y) \right| \\ &\leq \frac{\min\{\varphi(2x_0), \varphi(2y)\}}{|f(2x_0)|} \leq \frac{\varphi(2x_0)}{|f(2x_0)|} \end{aligned}$$

and it follows that g is also bounded. Hence the required condition is satisfied. □

By putting $\varphi(x) = \varphi(y) = \varepsilon$ in the above Theorems 2.1, 2.2, 2.3, we obtain the following results.

Corollary 2.4. ([12] Theorem 3.1, Theorem 3.2) *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right| \leq \varepsilon \tag{12}$$

for all $x, y \in G$.

(a) either f is bounded or g satisfies $(\widetilde{S})$. If, additionally, f satisfies $(\widetilde{A})$, then f and g satisfy the equation $g(x+y) - g(x+\sigma y) = 2f(x)g(y)$.

(b) either g is bounded or g satisfies $(\widetilde{S})$. If, additionally, f satisfies $(\widetilde{A})$, then f and g are solutions of $g(x+y) - g(x+\sigma y) = 2f(x)g(y)$.

(c) if the inequality (12) satisfies the cases $f(0) = 0$ or $f(x)^2 = f(\sigma x)^2$ for all $x \in G$, then either g is bounded or f satisfies $(\widetilde{S})$. If, additionally, g satisfies $(\widetilde{A})$, then f and g satisfy the equation $(\widetilde{A}_{fg})$.

3. Applications to the sine type equations

The stability results for $(\widetilde{S}_{fg})$ treated in Section 2 will be applied to the functional equations (S), $(\widetilde{S})$, (S_{fg}), (S_{gg}), $(\widetilde{S}_{gg})$.

3.1. Stability of the Equation $(\widetilde{S}_{gg})$

The proof for the stability of the equation $(\widetilde{S}_{gg})$ runs along those of Section 2 by replacing $f(x)$ to $g(x)$. So we will cancel their proofs.

Theorem 3.1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| g(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right| \leq \varphi(y)$$

for all $x, y \in G$. Then either g is bounded or g satisfies $(\widetilde{S})$.

THEOREM 3.2. *Suppose that $f,g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| g(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right| \le \varphi(x),$$

which satisfies the cases $g(0) = 0$ or $f(x)^2 = f(\sigma x)^2$ for all $x, y \in G$. Then either g is bounded or g satisfies $(\widetilde{S})$.

COROLLARY 3.3. *Suppose that $f,g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| g(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right| \le \varepsilon$$

for all $x, y \in G$.

Then either g is bounded or g satisfies $(\widetilde{S})$.

REMARK 3.1. Putting $\sigma y = -y$ in above results : Theorem 3.1, Theorem 3.2, Corollary 3.3, we can obtain the same type's results for the equation (S_{gg}), in which Corollary 3.3 is founded in ([9]).

3.2. Applications to the Equations (S_{fg}) and (S)

Let $\sigma y = -y$ in Theorem 2.1, Theorem 2.2, Theorem 2.3, then we obtain the superstability for the equation (S_{fg}) and (S).

THEOREM 3.4. *Suppose that $f,g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \le \varphi(y)$$

for all $x, y \in G$. Then either f is bounded or g satisfies (S). If, additionally, f satisfies (A), then f and g are solutions of $g(x+y) - g(x-y) = 2f(x)g(y)$.

THEOREM 3.5. *Suppose that $f,g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \le \varphi(x),$$

which satisfies the cases $f(0) = 0$ or $f(x)^2 = f(-x)^2$ for all $x, y \in G$.

Then either g is bounded or f satisfies (S). If, additionally, g satisfies (A), then f and g satisfy the equation (A_{fg}).

THEOREM 3.6. *Suppose that $f,g : G \to \mathbb{C}$ satisfy the inequality*

$$\left| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \le \min\{\varphi(x), \varphi(y)\} \tag{13}$$

for all $x, y \in G$.

(a) either f is bounded or g satisfies (S). If, additionally, f satisfies (A), then f and g satisfy the equation $g(x+y) - g(x-y) = 2f(x)g(y)$.

(b) either g is bounded or g satisfies (S). If, additionally, f satisfies (A), then f and g are solutions of $g(x+y)-g(x-y)=2f(x)g(y)$.

(c) if the inequality (13) satisfies the cases $f(0)=0$ or $f(x)^2=f(-x)^2$ for all $x\in G$, then either g is bounded or f satisfies (S). If, additionally, g satisfies (A), then f and g satisfy the equation (A_{fg}).

Corollary 3.7. ([9] Corollary 4.) *Suppose that $f,g:G\to\mathbb{C}$ satisfy the inequality*

$$\left|f(x)g(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2\right|\le\varepsilon \tag{14}$$

for all $x,y\in G$.

(a) either f is bounded or g satisfies (S). If, additionally, f satisfies (A), then f and g satisfy the equation $g(x+y)-g(x-y)=2f(x)g(y)$.

(b) either g is bounded or g satisfies (S). If, additionally, f satisfies (A), then f and g are solutions of $g(x+y)-g(x-y)=2f(x)g(y)$.

(c) if the inequality (14) satisfies the cases $f(0)=0$ or $f(x)^2=f(-x)^2$ for all $x\in G$, then either g is bounded or f satisfies (S). If, additionally, g satisfies (A), then f and g satisfy the equation (A_{fg}).

Corollary 3.8. ([1] Theorem 6, [4] Theorem, [9] Corollary 10) *Suppose that $f:G\to\mathbb{C}$ satisfies the inequality*

$$\left|f(x)f(y)-f\left(\frac{x+y}{2}\right)^2+f\left(\frac{x-y}{2}\right)^2\right|\le\begin{cases}(i) & \varphi(y)\\ (ii) & \varphi(x)\\ (iii) & \varepsilon\end{cases}$$

for all $x,y\in G$. Then either f is bounded or f satisfies (S).

Proof. The cases (i) and (iii) are trivial.

In the proof of Theorem 5 in [1], whenever f is unbounded we find that $f(0)=0$. Hence we can eliminate the assumption $f(0)=0$. □

4. Applications on the Banach algebra

In this section, we will extend the range of the function from the field of complex numbers to the Banach algebra. For simplicity, we will combine them into one theorem.

THEOREM 4.1. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ and $\varphi : G \to \mathbb{R}$ satisfy the inequalities*

$$\left\| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right\| \le \begin{cases} (i) & \varphi(y) \\ (ii) & \varphi(x) \\ (iii) & \min\{\varphi(x), \varphi(y)\} \end{cases}$$

with the cases $f(0) = 0$ or $f(x)^2 = f(\sigma x)^2$ in the cases (ii) and (iii).

For an arbitrary linear multiplicative functional $x^* \in E^*$,

Case(i). *either the superposition $x^* \circ f$ is bounded or g satisfies $(\widetilde{S})$. If, additionally, f satisfies $(\widetilde{A})$, then f and g are solutions of $g(x+y) - g(x+\sigma y) = 2f(x)g(y)$.*

Case(ii). *either the superposition $x^* \circ g$ is bounded or f satisfies $(\widetilde{S})$. If, additionally, g satisfies $(\widetilde{A})$, then f and g are solutions of the equation $(\widetilde{A}_{fg})$.*

Case(iii). *(a) either the superposition $x^* \circ f$ is bounded or g satisfies $(\widetilde{S})$. If, additionally, f satisfies $(\widetilde{A})$, then f and g are solutions of the equation $g(x+y) - g(x+\sigma y) = 2f(x)g(y)$.*

(b) either the superposition $x^* \circ g$ is bounded or f and g satisfy $(\widetilde{S})$, respectively. If, additionally, f satisfies $(\widetilde{A})$, then f and g are solutions of $g(x+y) - g(x+\sigma y) = 2f(x)g(y)$, and also if, additionally, g satisfies $(\widetilde{A})$, then f and g satisfy the equation $(\widetilde{A}_{fg})$.

Proof. Assume that (i) holds and fix arbitrarily a linear multiplicative functional $x^* \in E$. As is well known we have $\|x^*\| = 1$.

In (i), we have

$$\begin{aligned}
\varphi(y) &\ge \left\| f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right\| \\
&= \sup_{\|y^*\|=1} \left| y^*\left(f(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right) \right| \\
&\ge \left| x^*(f(x)) \cdot x^*(g(y)) - x^*\left(f\left(\frac{x+y}{2}\right)^2 \right) + x^*\left(f\left(\frac{x+\sigma y}{2}\right)^2 \right) \right|,
\end{aligned}$$

which states that the superpositions $x^* \circ f$ and $x^* \circ g$ yield solutions of the inequality (1) of Theorem 2.1. Assume that the superposition $x^* \circ f$ is unbounded, then Theorem 2.1 forces that the function $x^* \circ g$ solves the equation $(\widetilde{S})$. In other words, keeping the linear multiplicativity of x^* in mind, this means that the difference $D\widetilde{S}_g(x,y)$ for the function g falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$D\widetilde{S}_g(x,y) = g(x)g(y) - g\left(\frac{x+y}{2}\right)^2 + g\left(\frac{x+\sigma y}{2}\right)^2 \in \bigcap\{\ker x^* : x^* \in E^*\}$$

for all $x, y \in G$. Since the algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.

$$g(x)g(y) - g\left(\frac{x+y}{2}\right)^2 + g\left(\frac{x+\sigma y}{2}\right)^2 = 0 \quad \text{for all} \quad x, y \in G,$$

as claimed. Each other cases runs away similar proceeding. □

Due to Theorem 3.1 and Theorem 3.2, the following theorem runs along that of Theorem 4.1.

THEOREM 4.2. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ and $\varphi : G \to \mathbb{R}$ satisfy the inequalities*

$$\left\| g(x)g(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x+\sigma y}{2}\right)^2 \right\| \leq \begin{cases} (i) & \varphi(y) \\ (ii) & \varphi(x) \end{cases}$$

with the cases $g(0) = 0$ or $f(x)^2 = f(\sigma x)^2$ in the cases (ii).

For an arbitrary linear multiplicative functional $x^* \in E^*$, either the superposition $x^* \circ g$ is bounded or g satisfies $(\widetilde{S})$.

REMARK 4.1. Putting $\sigma y = -y, g(x) = f(x)$ and $\varphi(x) = \varphi(y) = \varepsilon$ in Theorem 4.1 and Theorem 4.2, we obtain the same type's results for the equations $(\widetilde{S})$, (S), (S_{fg}), and (S_{gg}) in which last three cases are founded in ([4]) and ([9]), respectively.

References

[1] R. Badora and R. Ger, *On some trigonometric functional inequalities*, Functional Equations- Results and Advances, (2002), 3–15.

[2] J. A. Baker, *The stability of the cosine equation*, Proc. Amer. Math. Soc. **80** (1980), 411–416.

[3] J. Baker, J. Lawrence and F. Zorzitto, *The stability of the equation $f(x+y) = f(x)f(y)$*, Proc. Amer. Math. Soc. **74** (1979), 242–246.

[4] P.W. Cholewa, *The stability of the sine equation*, Proc. Amer. Math. Soc. **88** (1983), 631–634.

[5] P. Găvruta, On the stability of some functional equations, in : "Stability of mappings of Hyers-Ulam type" (Eds. Th. M. Rassias and J. Tabor), Hadronic Press, (1994), 93–98.

[6] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. **27** (1941), 222–224.

[7] Pl. Kannappan, *The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for groups*, Proc. Amer. Math. Soc. **19** (1968), 69–74.

[8] Pl. Kannappan and G. H. Kim, *On the stability of the generalized cosine functional equations*, Annales Acadedmiae Paedagogicae Cracoviensis - Studia mathematica, **1** (2001), 49–58.

[9] G. H. Kim, *A Stability of the generalized sine functional equations*, Jour. Math. Anal & Appl. **331** (2007), 886–894.

[10] G. H. Kim, *On the Stability of mixed trigonometric functional equations*, Banach J. Math. Anal. **1** (2007), no. 2, 227–236.

[11] G. H. Kim, *The Stability of the d'Alembert and Jensen type functional equations*, Jour. Math. Anal & Appl. **325** (2007), 237–248.
[12] G. H. Kim, *A Stability of the generalized sine functional equations II*, Jour. of Ineq. Pure & Appl. Math. **7** (2006), no. 5, Article 181.
[13] G. H. Kim & Sever S. Dragomir, *On the Stability of generalized d'Alembert and Jensen functional equation*, Intern. Jour. Math. & Math. Sci. Vol 2006 (2006), Article ID 43185.
[14] P. Sinopoulos, *Generalized sine equations. III.*, Aequationes Math. **51** (1996), 311–327.
[15] S. M. Ulam, *"Problems in Modern Mathematics"* Chap. VI, Science Editions, Wiley, New York, (1964)

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