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DECOMPOSITIONS AND EXPANSIONS OF FILTERS IN TARSKI ALGEBRAS

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Abstract. We show that any filter of Tarski algebra can be decomposed into the union of some sets. Moreover, we introduce the notion of expansions of filters in Tarski algebras, and discuss the notion of σ-primary filters in Tarski algebras. Finally, we show that there is no non-trivial quadratic Tarski algebras on a field X with $|X| \geq 3$.

1. Introduction

The variety of Tarski algebras was introduced by J. C. Abbott in [2]. These algebras are an algebraic counterpart of the $\{\vee, \rightarrow\}$-fragment of the propositional classical calculus. S. A. Celani ([4]) introduced Tarski algebras with a modal operator as a generalization of the concept of Boolean algebra with a modal operator which he researched into these fragments of the algebraic viewpoint. In this paper, we shall do the research of the algebraic viewpoint. First, we show that any filter of Tarski algebra can be decomposed into the union of some sets. Next, we introduce the notion of expansions of filters in Tarski algebras, and discuss the notion of σ-primary filters in Tarski algebras. Finally, we show that there is no non-trivial quadratic Tarski algebras on a field X with $|X| \geq 3$.

Let us review some definitions and results. By a *Tarski algebra* we mean an algebra $(X; \rightarrow, 1)$ of type $(2, 0)$ satisfying the following conditions:
(T1) $(\forall a \in X)(1 \rightarrow a = a)$.

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(T2) $(\forall a \in X)(a \to a = 1)$.
(T3) $(\forall a, b, c \in X)(a \to (b \to c) = (a \to b) \to (a \to c)$.
(T4) $(\forall a, b \in X)((a \to b) \to b = (b \to a) \to a)$.

In a Tarski algebra X we can define an order relation $\leq$ by setting $a \leq b$ if and only if $a \to b = 1$ for any $a, b \in X$. It is well known that $(X; \leq)$ is an ordered set and that X is a join-semilattice where the supremum of two elements $a, b \in X$ is defined by $a \vee b = (a \to b) \to b$ ([3]).

In any Tarski algebra X, the following hold:

(p1) $(\forall a \in X)(a \to 1 = 1)$.
(p2) $(\forall a, b \in X)(a \to (b \to a) = 1)$.

A non-empty subset F of a Tarski algebra X is said to be a *filter* if $1 \in F$, and $a \in F$ and $a \to b \in F$ imply $b \in F$ ([4]).

Lemma 1.1. ([4]) *Let P be a proper filter of a Tarski algebra X. Then the following conditions are equivalent:*

(i) *P is maximal, i.e., for any filter F in X such that $P \subseteq F$, we have $P = F$ or $F = X$.*
(ii) *P is prime, i.e., for all $a, b \in X$ if $a \vee b \in P$, then $a \in P$ or $b \in P$.*
(iii) *For any filters F, G in X, if $P = F \cap G$ then $P = F$ or $P = G$.*

2. Decompositions of filters

Now, we consider any filter decomposed into the union of some sets in Tarski algebra. Let us start following definitions. For any Tarski algebra X and $x, y \in X$, we denote

$$A(x, y) := \{z \in X | x \to (y \to z) = 1\} \text{ and } A[x) := \{z \in X | x \to z = 1\}.$$

Obviously, $A(x, y)$ and $A[x)$ are nonempty sets.

Theorem 2.1. *If F is a filter of a Tarski algebra X, then*

$$F = \bigcup_{x \in F} A[x).$$

Proof. Let F be a filter of a Tarski algebra X. If $a \in F$, then we have $a \in A[a)$ by (T2). It follows that

$$F \subseteq \bigcup_{x \in F} A[x).$$

If $a \in \bigcup_{x \in F} A[x)$, then there exists $b \in F$ such that $a \in A[b)$, so that $b \to a = 1$. Since F is a filter, it follows that $a \in F$. Thus $\bigcup_{x \in F} A[x) \subseteq F$, and consequently $F = \bigcup_{x \in F} A[x)$. □

LEMMA 2.2. *In a Tarski algebra X and $x \in X$, we have*

$$A(x, x) = A(x, 1) = A[x).$$

Proof. The proof is straightforward and omitted. □

COROLLARY 2.3. *If F is a filter of a Tarski algebra X, then*

$$F = \bigcup_{x,y \in F} A(x, y).$$

Proof. By Theorem 2.1 and Lemma 2.2, we have $F = \bigcup_{x \in F} A(x, x) \subseteq \bigcup_{x,y \in F} A(x, y)$. If $a \in \bigcup_{x,y \in F} A(x, y)$. Then there exist $b, c \in F$ such that $a \in A(b, c)$, so that $b \to (c \to a) = 1$. Since F is a filter, it follows that $a \in F$. Thus $\bigcup_{x,y \in F} A(x, y) \subseteq F$, and consequently $F = \bigcup_{x,y \in F} A(x, y)$. □

We give an example satisfying Theorem 2.1. See the following example.

EXAMPLE 2.1. Let $X := \{a, b, c, d, 1\}$ be a set with the following Cayley table:

$\to$	a	b	c	d	1
a	1	1	1	d	1
b	a	1	c	d	1
c	a	b	1	d	1
d	a	b	c	1	1
1	a	b	c	d	1

Then $(X; \to, 1)$ is a Tarski algebra, and $F := \{c, d, 1\}$ and $G := \{a, b, c, 1\}$ are filters of X ([4]). It is routine to check that $F = A[c) \cup A[d)$ and $G = A[a) \cup A[b) \cup A[c)$.

THEOREM 2.4. *Let F be a subset of a Tarski algebra X such that $1 \in F$ and $F = \bigcup_{x,y \in F} A(x, y)$. Then F is a filter of X.*

Proof. Let $a, b \in X$ such that $a, a \to b \in F$. Since $(a \to b) \to (a \to b) = 1$, it follows that $b \in A(a \to b, a) \subseteq F$. Hence F is a filter of X. □

Combining Corollary 2.3 and Theorem 2.4, we have the following theorem.

THEOREM 2.5. *Let X be a Tarski algebra and F be a subset of X such that $1 \in F$. Then F is a filter of X if and only if $F = \bigcup_{x,y \in F} A(x, y)$.*

3. Expansions of filters

Now, we consider the expansions of filters in Tarski algebras, and discuss the notion of σ-primary filters in Tarski algebras. In what follows let $\mathfrak{F}(X)$ be the set of filters in a Tarski algebra X.

DEFINITION 3.1. By an *expansion of filters* in a Tarski algebra X we shall mean a function $\sigma : \mathfrak{F}(X) \to \mathfrak{F}(X)$ such that

(e1) $(\forall F \in (F)(X))(F \subseteq \sigma(F))$.
(e2) $(\forall F, G \in \mathfrak{F}(X))(F \subseteq G \Rightarrow \sigma(F) \subseteq \sigma(G))$.

EXAMPLE 3.1. (1) The function $\sigma_0 : \mathfrak{F}(X) \to \mathfrak{F}(X)$ defined by $\sigma_0(F) = F$ for all $F \in \mathfrak{F}(X)$ is an expansion of filters in a Tarski algebra X.

(2) For each filter F of a Tarski algebra X, let

$$\mathfrak{M}(F) := \bigcap \{M | F \subseteq M, M \text{ is a maximal filter of } X\}.$$

Then $\mathfrak{M}$ is an expansion of filters in X.

DEFINITION 3.2. Let σ be an expansion of filters in a Tarski algebra X. Then a filter F of X is said to be *σ-primary* if it satisfies:

$$(\forall a, b \in X)(a \vee b \in F, a \notin F \Rightarrow b \in \sigma(F)).$$

Note that a filter F of a Tarski algebra X is σ_0-primary if and only if it is a prime filter of X, where σ_0 is the function in Example 3.1 (1).

THEOREM 3.1. *Let σ and α be expansions of filters in a Tarski algebra X such that $\sigma(F) \subseteq \alpha(F)$ for every $F \in \mathfrak{F}(X)$, then every σ-primary filter is α-primary.*

Proof. Let F be a σ-primary filter of X and let $a, b \in X$ such that $a \vee b \in F$ and $a \notin F$. Then we have

$$b \in \sigma(F) \subseteq \alpha(F).$$

Thus F is a α-primary filter of X, proving the theorem. □

THEOREM 3.2. *Let α and β be expansions of filters in a Tarski algebra X. Let $\sigma : \mathfrak{F}(X) \to \mathfrak{F}(X)$ be a function defined by $\sigma(G) = \alpha(G) \cap \beta(G)$ for all $G \in \mathfrak{F}(X)$. Then σ is an expansion of filters in X.*

Proof. For every $G \in \mathfrak{F}(X)$, we have $G \subseteq \alpha(G)$ and $G \subseteq \beta(G)$ by (e1), and so $G \subseteq \alpha(G) \cap \beta(G) = \sigma(G)$. Let $G, H \in \mathfrak{F}(X)$ such that $G \subseteq H$. Than $\alpha(G) \subseteq \alpha(H)$ and $\beta(G) \subseteq \beta(H)$ by (e2), which imply that

$$\sigma(G) = \alpha(G) \cap \beta(G) \subseteq \alpha(H) \cap \beta(H) = \sigma(H).$$

Therefore σ is an expansion of filters in X, proving the theorem. □

THEOREM 3.3. *If σ is an expansion of filters in a Tarski algebra X, then the function $E_\sigma : \mathfrak{F}(X) \to \mathfrak{F}(X)$ defined by*

$$E_\sigma(F) := \bigcap\{H \in \mathfrak{F}(X) | F \subseteq H, \text{ and } H \text{ is } \sigma\text{-primary } \}$$

for all $F \in \mathfrak{F}(X)$ is an expansion of filters in X.

Proof. Clearly, $F \subseteq E_\sigma(F)$ for all $F \in \mathfrak{F}(X)$. Let $F, G \in \mathfrak{F}(X)$ such that $F \subseteq G$. Then

$$\begin{aligned} E_\sigma(F) &= \bigcap\{H \in \mathfrak{F}(X) | F \subseteq H, \text{ and } H \text{ is } \sigma\text{-primary } \} \\ &\subseteq \bigcap\{H \in \mathfrak{F}(X) | G \subseteq H, \text{ and } H \text{ is } \sigma\text{-primary } \} \\ &= E_\sigma(G). \end{aligned}$$

□

4. Quadratic Tarski algebras

Now, we consider a quadratic Tarski algebra. Let X be a field with $|X| \geq 3$. An algebra $(X; \to)$ is said to be *quadratic* if $x \to y$ is defined by

$$x \to y := a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6,$$

where $a_1, a_2, \cdots, a_6 \in X$, for any $x, y \in X$. A Tarski algebra X is said to be a *quadratic Tarski algebra* if it is quadratic.

THEOREM 4.1. *Let X be a field with $|X| \geq 3$. Then there is no non-trivial quadratic Tarski algebras.*

Proof. Let $x \to y := Ax^2 + Bxy + Cy^2 + Dx + Ey + F$, where $A, B, C, D, E, F \in X$. Consider (T1). Given $x \in X$, we have

$$x = 1 \to x = Cx^2 + (B + E)x + A + D + F.$$

Then we have $C = 0, B + E = 1$ and $A + D + F = 0$. If we consider (T2), then

$$1 = x \to x = (A + B + C)x^2 + (D + E)x + F,$$

i.e., $A + B + C = 0, D + E = 0, F = 1$. Consider (p1). Then we get

$$1 = x \to 1 = Ax^2 + (B + D)x + C + E + F,$$

i.e., $A = 0, B + D = 0, E = 0$. This is a contradiction by $1 = B + E = B + 0 = B = 0 + B + 0 = A + B + C = 0$. This means that there is no non-trivial quadratic polynomials satisfying the conditions (T1), (T2), (T3) and (T4) in X, proving the theorem. □

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