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A NOTE ON CYCLOTOMIC UNITS IN FUNCTION FIELDS

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Abstract. Let $\mathbb{A} = \mathbb{F}_q[T]$ and $k = \mathbb{F}_q(T)$. Assume q is odd, and fix a prime divisor ℓ of $q-1$. Let P be a monic irreducible polynomial in $\mathbb{A}$ whose degree d is divisible by ℓ. In this paper we define a subgroup $\widetilde{C}_F$ of $\mathcal{O}_F^*$ which is generated by $\mathbb{F}_q^*$ and $\{\eta^{\tau^i} : 0 \leq i \leq \ell-1\}$ in $F = k(\sqrt[\ell]{P})$ and calculate the unit-index $[\mathcal{O}_F^* : \widetilde{C}_F] = \ell^{\ell-2}h(\mathcal{O}_F)$. This is a generalization of [3, Theorem 16.15].

1. Introduction and statement of result

Let $\mathbb{A} = \mathbb{F}_q[T]$ and $k = \mathbb{F}_q(T)$. Assume q is odd, and fix a prime divisor ℓ of $q-1$. Let P be a monic irreducible polynomial in $\mathbb{A}$ whose degree d is divisible by ℓ. Let K_P be the P-th cyclotomic function field, which is a finite cyclic extension of k. The Galois group $G_P := Gal(K_P/k)$ is isomorphic to $(\mathbb{A}/P\mathbb{A})^*$. For any $A \in (\mathbb{A}/P\mathbb{A})^*$, write σ_A for the automorphism in G_P characterized by $\sigma_A(\lambda) = \rho_A(\lambda)$ for any P-torsion point λ of the Carlitz module ρ. We write $\lambda^A := \rho_A(\lambda)$ for simplicity. Let K_P^+ be the maximal real subfield of K_P, so that $Gal(K_P/K_P^+) \cong \mathbb{F}_q^*$. Set

$$\mathcal{M}_P := \{0 \neq A \in \mathbb{A} : \deg(A) < \deg(P)\}$$

and

$$\mathcal{M}_P^+ := \{A \in \mathcal{M}_P : A \text{ is a monic}\}.$$

Then, as in the proof of Lemma 16.13 in [3], we have

$$P = (-1)^{\frac{q^d-1}{q-1}}\Big(\prod_{A \in \mathcal{M}_P^+} \lambda_P^A\Big)^{q-1},$$

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where λ_P is a primitive P-torsion point of the Carlitz module ρ. We fix

$$\sqrt[\ell]{P} := (-1)^{\frac{q^d-1}{\ell(q-1)}} \Big(\prod_{A \in \mathcal{M}_P^+} \lambda_P^A \Big)^{\frac{q-1}{\ell}}.$$

Let $F = k(\sqrt[\ell]{P})$, which is the unique cyclic extension of k of degree ℓ inside K_P^+ (see Lemma 2.1), and $\mathcal{O}_F$ the integral closure of $\mathbb{A}$ in F. For any polynomial $A \in \mathbb{A}$ with $P \nmid A$, write $\bar{A}$ for the unique element of $\mathcal{M}_P$ such that $A \equiv \bar{A} \bmod P$ and $\mathrm{sgn}_P(A)$ for the leading coefficient of $\bar{A}$. Also, $(A/P)_\ell$ denotes the unique element of $\mathbb{F}_q^*$ such that

$$A^{\frac{q^d-1}{\ell}} \equiv (A/P)_\ell \bmod P.$$

Let $\mathcal{R}$ be any complete set of representatives of $(\mathbb{A}/P\mathbb{A})^*/\mathbb{F}_q^*$. Define

$$\eta := \frac{\big(\prod_{A \in \mathcal{R}_0} \lambda_P^A/sgn_P(A)\lambda_P\big)^\ell}{\prod_{A \in \mathcal{R}} \lambda_P^A/sgn_P(A)\lambda_P},$$

where $\mathcal{R}_0 = \{A \in \mathcal{R} : (A/P)_\ell = 1\}$. This unit η is independent of the choice of $\mathcal{R}$ and is an element of F (see Lemma 2.3). Let $\widetilde{\mathcal{C}}_F$ be the subgroup of $\mathcal{O}_F^*$ generated by $\mathbb{F}_q^*$ and $\{\eta^{\tau^i} : 0 \le i \le \ell - 1\}$, where τ is a generator of $Gal(F/k)$.

The main result of this paper is the following theorem.

THEOREM 1.1. *Let ℓ be a prime divisor of $q-1$. Let P be a monic irreducible polynomial in $\mathbb{A}$ whose degree d is divisible by ℓ. Then $\{\eta^{\tau^i} : 0 \le i \le \ell - 2\}$ forms a basis for the non-torsion part of $\widetilde{\mathcal{C}}_F$ and the index of $\widetilde{\mathcal{C}}_F$ in the full unit group $\mathcal{O}_F^*$ is given by*

$$[\mathcal{O}_F^* : \widetilde{\mathcal{C}}_F] = \ell^{\ell-2} \cdot h(\mathcal{O}_F),$$

where $h(\mathcal{O}_F)$ is the ideal class number of $\mathcal{O}_F$.

REMARK 1.2. Write R_η for the regulator of the set $\{\eta^{\tau^i} : 0 \le i \le \ell - 2\}$. Then we have

$$[\mathcal{O}_F^* : \widetilde{\mathcal{C}}_F] = \frac{R_\eta}{R(\mathcal{O}_K)},$$

where $R(\mathcal{O}_K)$ is the regulator of $\mathcal{O}_K$. Thus, we may regard Theorem 1.1 as a generalization of [3, Theorem 16.15].

2. Proof of Theorem 1.1

To give the proof of Theorem 1.1, we need some lemmas.

Lemma 2.1. $F = k(\sqrt[\ell]{P}) \subseteq K_P^+$.

Proof. Since G_P is a cyclic group of order $q^d - 1$, there is a unique cyclic extension of k of degree ℓ inside K_P. By Lemma 3.2 in [2], $k(\sqrt[\ell]{P})$ is contained in K_P. Since $d = \deg(P)$ is divisible by ℓ, so is $|G_P^+| = \frac{q^d-1}{q-1}$. Thus k has a cyclic extension of degree ℓ inside K_P^+. By the uniqueness, it must be $k(\sqrt[\ell]{P})$. □

Fix a primitive root $Q \in \mathcal{M}_P$ of P. For any $B \in \mathbb{A}$ with $P \nmid B$, write $\delta_Q(B)$ for the index of B relative to Q.

Lemma 2.2. *Let B be a polynomial in $\mathbb{A}$ such that $P \nmid B$. Then $\sigma_B \in Gal(K_P/F)$ if and only if $(B/P)_\ell = 1$.*

Proof. Note that the restriction of σ_Q to F is a generator of $Gal(F/k)$. Thus $\sigma_B \in Gal(K_P/F)$ if and only if $\delta_Q(B) \equiv 0 \bmod \ell$, which is equivalent to $(B/P)_\ell = 1$ by Proposition 3.1 in [3]. □

Lemma 2.3. (i) *η is independent of the choice of $\mathcal{R}$.*
(ii) $\eta \in \mathcal{O}_F^*$.

Proof. (i) Let $\mathcal{Y} = \{A \in \mathcal{M}_P : 0 \le \delta_Q(A) < (q^d-1)/(q-1)\}$. It is easy to see that $\mathcal{M}_P$ is the disjoint union of $\alpha \cdot \mathcal{Y}$, where α runs over all elements of $\mathbb{F}_q^*$. Thus, $\mathcal{Y}$ is a complete set of representatives of $(\mathbb{A}/P\mathbb{A})^*/\mathbb{F}_q^*$. Let $\mathcal{R}$ be any complete set of representatives of $(\mathbb{A}/P\mathbb{A})^*/\mathbb{F}_q^*$. For any $A \in \mathcal{R}$, there exist unique $\tilde{A} \in \mathcal{Y}$ and $c \in \mathbb{F}_q^*$ such that $A \equiv c\tilde{A} \bmod P$. Thus we have

$$\frac{\lambda_P^A}{sgn_P(A)\lambda_P} = \frac{\lambda_P^{c\tilde{A}}}{sgn_P(c\tilde{A})\lambda_P} = \frac{\lambda_P^{\tilde{A}}}{sgn_P(\tilde{A})\lambda_P}.$$

Also, $(A/P)_\ell = (\tilde{A}/P)_\ell$, since $(c/P)_\ell = 1$ by Proposition 3.2 in [3]. Thus we have

$$\begin{aligned}\eta &= \frac{\left(\prod_{A\in\mathcal{R}_0} \lambda_P^A/sgn_P(A)\lambda_P\right)^\ell}{\prod_{A\in\mathcal{R}} \lambda_P^A/sgn_P(A)\lambda_P} \\ &= \frac{\left(\prod_{A\in\mathcal{Y}_0} \lambda_P^A/sgn_P(A)\lambda_P\right)^\ell}{\prod_{A\in\mathcal{Y}} \lambda_P^A/sgn_P(A)\lambda_P}.\end{aligned}$$

(ii) Recall that $\mathcal{Y}_0 = \{A \in \mathcal{Y} : (A/P)_\ell = 1\}$. Note that λ_P^A/λ_P lies in K_P^+ for any $A \in \mathcal{M}_P$. Thus, it suffices to show that $\eta^{\sigma_B} = \eta$ for all $B \in \mathcal{Y}_0$. For any $B \in \mathcal{Y}_0$, we have

$$\begin{aligned}
\eta^{\sigma_B} &= \frac{\left(\prod_{A\in\mathcal{Y}_0} \lambda_P^{AB}/sgn_P(A)\lambda_P^B\right)^\ell}{\prod_{A\in\mathcal{Y}} \lambda_P^{AB}/sgn_P(A)\lambda_P^B} \\
&= \frac{\left(\prod_{A\in\mathcal{Y}_0} \lambda_P^{AB}/sgn_P(AB)\lambda_P\right)^\ell}{\prod_{A\in\mathcal{Y}} \lambda_P^{AB}/sgn_P(AB)\lambda_P} \cdot \frac{\left(\prod_{A\in\mathcal{Y}_0} sgn_P(AB)/sgn_P(A)\right)^\ell}{\prod_{A\in\mathcal{Y}} sgn_P(AB)/sgn_P(A)}. \qquad (2.1)
\end{aligned}$$

Since $B \in \mathcal{Y}_0$, $\delta_Q(B) = n\ell$ for some $0 \le n < \frac{q^d-1}{\ell(q-1)}$. Then we have

$$\begin{aligned}
\prod_{A\in\mathcal{Y}} \frac{sgn_P(AB)}{sgn_P(A)} &= \prod_{j=0}^{\frac{q^d-1}{\ell(q-1)}-1} \frac{sgn_P(Q^{j\ell+n\ell})}{sgn_P(Q^{j\ell})} \\
&= \frac{\prod_{j=0}^{n-1} sgn_P(Q^{\frac{q^d-1}{q-1}+j\ell})}{\prod_{j=0}^{n-1} sgn_P(Q^{j\ell})} \\
&= (c_0)^n, \qquad (2.2)
\end{aligned}$$

where c_0 is the unique element of $\mathbb{F}_q^*$ such that $c_0 \equiv Q^{\frac{q^d-1}{q-1}} \bmod P$. Similarly,we have

$$\begin{aligned}
\prod_{A\in\mathcal{Y}} \frac{sgn_P(AB)}{sgn_P(A)} &= \prod_{j=0}^{\frac{q^d-1}{q-1}-1} \frac{sgn_P(Q^{j+n\ell})}{sgn_P(Q^{j})} \\
&= \frac{\prod_{j=0}^{n\ell-1} sgn_P(Q^{\frac{q^d-1}{q-1}+j})}{\prod_{j=0}^{n\ell-1} sgn_P(Q^{j})} \\
&= (c_0)^{n\ell}. \qquad (2.3)
\end{aligned}$$

Since $\mathcal{W} = \{AB : A \in \mathcal{Y}\}$ is also a complete set of representatives of $(\mathbb{A}/P\mathbb{A})^*/\mathbb{F}_q^*$ and $\mathcal{W}_0 = \{AB : A \in \mathcal{Y}_0\}$, by (2.1), (2.2) and (2.3), we have $\eta^{\sigma_B} = \eta$, which completes the proof. □

We set

$$\varepsilon := \frac{1}{\sqrt[\ell]{P}} N_{K_P/F}(\lambda_P) \in \mathcal{O}_F^*.$$

Proposition 2.4. $\eta^{\frac{q-1}{\ell}} = \varepsilon$.

Proof. By taking $\mathcal{R} = \mathcal{M}_P^+$, we have

$$\eta = \frac{\left(\prod_{A\in\mathcal{M}_P^+,(A/P)_\ell=1} \lambda_P^A\right)^\ell}{\prod_{A\in\mathcal{M}_P^+} \lambda_P^A}.$$

Since

$$\begin{aligned} N_{K_P/F}(\lambda_P) &= \prod_{A\in\mathcal{M}_P^+,(A/P)_\ell=1} \prod_{c\in\mathbb{F}_q^*} \lambda_{P_i}^{cA} \\ &= (-1)^{\frac{q^d-1}{\ell(q-1)}} \Big(\prod_{A\in\mathcal{M}_P^+,(A/P)_\ell=1} \lambda_P^A \Big)^{q-1}, \end{aligned}$$

we have

$$\eta^{\frac{q-1}{\ell}} = \frac{(-1)^{\frac{q^d-1}{\ell(q-1)}} N_{K_P/F}(\lambda_P)}{(-1)^{\frac{q^d-1}{\ell(q-1)}} \sqrt[\ell]{P}} = \varepsilon,$$

which completes the proof. □

Now, we give the proof of Theorem 1.1. Let $\mathcal{C}_F$ be the subgroup of $\mathcal{O}_F^*$ generated by $\mathbb{F}_q^*$ and $\{\varepsilon^{\tau^i} : 0 \le i \le \ell-1\}$. It is known ([1, Theorem 3.4]) that $\{\varepsilon^{\tau^i} : 0 \le i \le \ell-2\}$ forms a basis for the non-torsion part of $\mathcal{C}_F$ and

$$[\mathcal{O}_F^* : \mathcal{C}_F] = \frac{(q-1)^{\ell-1}}{\ell} \cdot h(\mathcal{O}_F). \tag{2.4}$$

By Proposition 2.4, $\mathcal{C}_F$ is contained in $\widetilde{\mathcal{C}}_F$ and

$$[\widetilde{\mathcal{C}}_F : \mathcal{C}_F] = \left(\frac{q-1}{\ell}\right)^{\ell-1}. \tag{2.5}$$

Thus $\{\eta^{\tau^i} : 0 \le i \le \ell-2\}$ forms a basis for the non-torsion part of $\widetilde{\mathcal{C}}_F$ and, by (2.4) and (2.5), we have

$$[\mathcal{O}_F^* : \widetilde{\mathcal{C}}_F] = \ell^{\ell-2} \cdot h(\mathcal{O}_F),$$

which completes the proof of Theorem 1.1.

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