JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **20**, No. 4, December 2007

# ALGEBRAIC MEET CONTINUOUS LATTICE

Seung On Lee* and Yong Ho Yon **

Abstract. This paper is sequel to [3]. In this paper, we discuss some properties of an algebraic meet-continuous lattice and study a complete lattice which can be embedded into an algebraic meet-continuous lattice.

## 1. Strong atomistic complete lattice

We recall that an algebraic lattice is just a sublattice of a power of the two point chain with respect to an arbitrary meets and directed joins. In 1969, Lawson has shown that continuous lattices are precisely sublattices of a power of the unit interval under the same operations as above. In 1972, Scott has named a concept of continuous lattices and has shown the equivalence between continuous lattices and injective $T_0$-spaces([4]).

Every complete lattice $L$ can be embedded into an atomistic Boolean lattice $\mathcal{P}(L)$, where $\mathcal{P}(L)$ is the power set of $L$, but the complete lattice $L$ is not algebraic or continuous. We study a complete lattice which is embedded into an algebraic meet-continuous lattice.

In this paper, every semilattice has the bottom element 0 and every meet-semilattice has the top element $e$. That is, a lattice means a bounded lattice.

Recall that a complete lattice $L$ is said to be *continuous* if $x = \bigvee \downarrow_w x$ for all $x \in L$, and $L$ is said to be *algebraic* if $x = \bigvee(\downarrow x \cap K(L))$ for all $x \in L$, where $\downarrow x = \{y \in L \mid y \leq x\}$ and $K(L)$ is the set of all compact elements in $L$. Clearly, every algebraic lattice is continuous.

For terminology not introduced in this paper, we refer to [2, 3].

Received September 28, 2007.
2000 Mathematics Subject Classification: Primary 01A05, 01A50, 01A70, 01A85.
Key words and phrases: join-irreducible, $J$-lattice, meet-continuous lattice, adjunction, algebraic lattice.

Let $(P, \leq)$ be a poset and $x, y \in P$. We say that $x$ is *covered by* $y$ (or $y$ *covers* $x$), denoted by $x \prec y$ or $y \succ x$, if $x < y$ and $x \leq z < y$ implies $z = x$.

An element $a$ in a lattice $L$ is called an *atom* if $0 \prec a$. We denote the set of all atoms in $L$ by $A(L)$.

A lattice $L$ is said to be *atomic* if the interval $[0, x]$ contains an atom for every $x > 0$, and *atomistic* if for each $x \in L$, $x = \bigvee S$ for a subset $S$ of $A(L)$.

An element $u$ in a lattice $L$ is called *join-irreducible* if for any $x, y \in L$, $u = x \vee y$ implies $u = x$ or $u = y$. We denote the set of all non-zero join-irreducible elements in $L$ by $J(L)$. If a join-irreducible element has a lower cover, then it is unique.

A lattice $L$ is said to be *strong* if for any $x, y \in L$ and any $u \in J(L)$, $b < u \leq x \vee y$ imply $u \leq y$.

A lattice $L$ is called a $J$-*lattice* if for each $x \in L$, $x = \bigvee S$ for a subset $S$ of $J(L)$.

Further discussions of an atomistic lattice and an atomic $J$-lattice can be found in [5].

LEMMA 1.1. ([3]) *If $L$ is an atomistic meet-continuous lattice, then $L$ is a continuous lattice.*

DEFINITION 1.2. A complete lattice $L$ is said to be *irredundant atomistic* if for each $x \in L$, there is a non-empty subset $S$ of $A(L)$ such that

$$x = \bigvee S \quad \text{and} \quad \bigvee S' < x$$

for every non-empty proper subset $S'$ of $S$.

LEMMA 1.3. *If $L$ is an irredundant atomistic complete lattice, then $J(L) = A(L)$.*

*Proof.* It is clear that every atom is join-irreducible, that is, $J(L) \subseteq A(L)$.

To show that $A(L) \subseteq J(L)$, suppose that $u \in J(L)$ and $u$ is not an atom. Since $L$ is irreducible atomistic, there is a non-empty subset $S$ of $A(L)$ such that $u = \bigvee S$ and $\bigvee S' < u$ for every non-empty proper subset $S'$ of $S$. So

$$u = a \vee (\bigvee(S - \{a\})) \quad \text{and} \quad \bigvee(S - \{a\}) \neq u$$

for any $a \in S$. Since $u$ is irreducible, $u = a$, and it is contradiction. Hence $u$ is an atom. $\square$

PROPOSITION 1.4. *Any irredundant atomistic lattice is strong.*

*Proof.* Suppose that $x < u \leq x \vee y$ for $x, y \in L$ and $u \in J(L)$. Then $x = 0$ since $u$ is an atom, hence $u \leq x \vee y = 0 \vee y = y$. □

Let $L$ be a lattice and $x, y \in L$ with $x \leq y$. Then we denote

$$r(x; y) = \{z \in L \mid y \leq x \vee z \text{ and } x \wedge z = 0\}.$$

For any $x, y \in L$ with $x \leq y$, if $x = y$, then

$$r(x; y) = \{z \in L \mid x \wedge z = 0\},$$

and $r(x; y) \neq \emptyset$ since $0 \in r(x; y)$; otherwise, $0 \notin r(x; y)$. And it is clear that $y_1 \leq y_2$ implies $r(x; y_1) \supseteq r(x; y_2)$, hence if $x$ has the complement $x'$, then $x' \in r(x; 1) \subseteq r(x; y)$ for all $y \in\uparrow x$ since $x \wedge x' = 0$ and $y \leq 1 = x \vee x'$.

LEMMA 1.5. *Let $L$ be an atomic and a strong lattice. If $r(a; u) \neq \emptyset$ for each $u \in J(L)$ and each $a \in\downarrow u \cap A(L)$, then $J(L) = A(L)$.*

*Proof.* It is clear that $A(L) \subseteq J(L)$.

Let $u \in J(L)$. Since $L$ is atomic, the interval $[0, u]$ has an atom $a_u$. If $a_u < u$, then $r(a_u; u) \neq \emptyset$, hence there is $z \in L$ such that $u \leq a_u \vee z$ and $a_u \wedge z = 0$. Since $L$ is strong, $u \leq z$, and $a_u \leq z$. It contradicts to $a_u \wedge z = 0$. Hence $u = a_u \in A(L)$. □

PROPOSITION 1.6. *Let $L$ be an atomic strong complete $J$-lattice. If $r(a; u) \neq \emptyset$ for each $u \in J(L)$ and each $a \in\downarrow u \cap A(L)$, then $L$ is atomistic.*

*Proof.* If $x \in L$ and $x = 0$, then $\bigvee(\downarrow x \cap A(L)) = \vee\emptyset = 0$.

Let $x \in L$ with $x \neq 0$. Since $L$ is $J$-lattice, there is a non-empty subset $S$ of $J(L)$ such that $x = \bigvee S$. $S \subseteq A(L)$ and $u \leq x$ for all $u \in S$. Hence we have

$$x = \bigvee S \leq \bigvee(\downarrow x \cap A(L)) \leq x,$$

that is, $x = \bigvee(\downarrow x \cap A(L))$. □

COROLLARY 1.7. *If $L$ is an atomic strong meet-continuous $J$-lattice with $r(a; u) \neq \emptyset$ for each $u \in J(L)$ and each $a \in\downarrow u \cap A(L)$, then $L$ is continuous.*

*Proof.* If $L$ is an atomic strong meet-continuous $J$-lattice with $r(a; u) \neq \emptyset$ for each $u \in J(L)$ and each $a \in\downarrow u \cap A(L)$, then $L$ is an atomic meet-continuous lattice, and hence $L$ is continuous. □

PROPOSITION 1.8. *Let $L$ be a meet-continuous lattice. If every join-irreducible element of $L$ has the lower cover, then $J(L) \subseteq K(L)$.*

*Proof.* Let $u \in J(L)$ and $u_0$ the lower cover of $u$ and $D$ a directed subset of $\downarrow u$ with $u = \bigvee D$. Then $d \leq u$ for all $d \in D$.

If $d \leq u_0$ for all $d \in D$, then $\bigvee D \leq u_0 < u$,and it is a contradiction for $u = \bigvee D$. Hence there is $d \in D$ with $d \nleq u_0$, and $u_0 < d \vee u_0 \leq u$. Since $u_0 \prec u = d \vee u_0$ and $u \neq u_0$, $u = d$ and $u \in K(L)$. □

COROLLARY 1.9. *If $L$ is a meet-continuous $J$-lattice and every join-irreducible element has the lower cover, then $L$ is algebraic.*

*Proof.* If $L$ is a meet-continuous $J$-lattice of which every join-irreducible element has the lower cover and $x \in L$, then $x = \bigvee S$ for some $S \subseteq J(L)$. Since $J(L) \subseteq K(L)$ by Proposition 1.8, $x = \bigvee S$ for some $S \subseteq K(L)$. □

## 2. $(\bigwedge, f)$-structure in an algebraic meet-continuous lattice

DEFINITION 2.1. Let $P$ and $Q$ be posets. A map $f : P \to Q$ is said to be

(1) *monotone* if $x \leq y$ in $P$ implies $f(x) \leq f(y)$ in $Q$;
(2) *order-embedding* if $x \leq y$ in $P$ if and only if $f(x) \leq f(y)$ in $Q$.

If a map $f : L \to K$ between complete lattices is 1-1 and and preserves arbitrary joins (or meets), then $f$ is an order-embedding map. Conversely, If $f : L \to K$ is an order-embedding map, then $f$ is 1-1 and monotone, but $f$ preserves neither arbitrary joins (including $e$) nor arbitrary meets (including 0) in general.

For an adjunction $(g, f)$ between posets $P$ and $Q$, we denote $g \dashv f : P \to Q$ or $g \dashv f$ briefly, and $g$ is called the *left adjoint* of $f$ and $f$ is the *right adjoint* of $g$([1]).

Let $f : P \to Q$ be a map between posets and $P$ is a complete lattice, then $f$ preserves arbitrary meets if and only if $f$ is monotone and $f$ has a left adjoint.

Let $g : Q \to P$ be a map between posets and $Q$ is a complete lattice, then $g$ preserves arbitrary joins if and only if $g$ is monotone and $g$ has a right adjoint.

We remark that the complete lattice $SubV$ of all subspaces of a vector space $V$ is not a distributive lattice, so $SubV$ is not a frame, but $SubV$ is a meet-continuous lattice. So we can conclude that a meet-continuous lattice need not be distributive.

LEMMA 2.2. *Let $P$ and $Q$ be posets and $f : P \to Q$ an order-embedding map.*

(1) *If $g \dashv f$, then $g$ is a left inverse of $f$, i.e., $gf = 1_P$.*

(2) *If* $f \dashv g$, *then* $g$ *is a left inverse of* $f$.

LEMMA 2.3. *Let* $P$ *and* $Q$ *be posets and* $f : P \to Q$ *a monotone map. If* $D$ *is a directed subset of* $P$, *then* $f(D)$ *is a directed subset of* $Q$.

PROPOSITION 2.4. *Let* $L$ *and* $K$ *be complete lattices and* $g \dashv f : L \to K$. *If* $f$ *preserves directed joins, then* $x \ll y$ *in* $K$ *implies* $g(x) \ll g(y)$ *in* $L$.

*Proof.* See [3]. □

The converse of the above proposition is not true in general.

COROLLARY 2.5. *Let* $L$ *and* $K$ *be complete lattices and* $g \dashv f : L \to K$. *If* $f$ *preserves directed joins, then* $y \ll f(x)$ *in* $K$ *implies* $g(y) \ll x$ *in* $L$. *Hence* $g(\downarrow_w f(x)) \subseteq \downarrow_w x$ *for all* $x \in L$.

Every atom in an atomistic meet-continuous lattice $H$ is compact, that is, $A(H) \subseteq K(H)$. Hence every atomistic meet-continuous lattice is algebraic since $x = \bigvee(\downarrow x \cap A(H)) \leq \bigvee(\downarrow x \cap K(H)) \leq x$.

The power set lattice $\mathcal{P}(L)$ of a complete lattice $L$ is a frame; hence a meet-continuous lattice, which is atomistic with $A(\mathcal{P}(L)) = \{\{x\} \mid x \in L\}$, that is, $\mathcal{P}(L)$ is algebraic meet-continuous. The map $\downarrow: L \to \mathcal{P}(L)$ $(x \mapsto \downarrow x)$ is 1-1 and has the left adjoint $\bigvee : \mathcal{P}(L) \to L$ $(S \mapsto \bigvee S)$, i.e., $\downarrow$ is 1-1 meet-preserving map. Hence we can consider an 1-1 meet-preserving map from a complete lattice $L$ to an algebraic meet-continuous lattice $H$.

Let $L$ be a complete lattice and $H$ an algebraic complete lattice. If $f : L \to H$ is a map with $g \dashv f$, then we denote

$$K_f(L) = \{g(k) \in L \mid k \in K(H)\}.$$

LEMMA 2.6. *Let* $L$ *be a complete lattice,* $H$ *an algebraic complete lattice and* $g \dashv f : L \to H$. *Then we have the following : for any* $x \in L$,

(1) $\downarrow x \cap K_f(L) = g(\downarrow f(x) \cap K(H))$,

(2) $f(x) = \bigvee(\downarrow f(x) \cap K(H)) \leq \bigvee f(\downarrow x \cap K_f(L))$.

*Proof.* (1) Let $u \in \downarrow x \cap K_f(L)$. Then there is $k \in K(H)$ with $g(k) = u \leq x$. Since $g \dashv f$, $k \leq f(x)$. Hence $k \in \downarrow f(x) \cap K(H)$ and $u = g(k) \in g(\downarrow f(x) \cap K(H))$.

Conversely, let $u \in g(\downarrow f(x) \cap K(H))$. Then there is $k \in K(H)$ with $k \leq f(x)$ and $u = g(k)$. Since $g \dashv f$, $u = g(k) \leq x$. Hence $u \in \downarrow x \cap K_f(L)$.

(2) It is clear that $f(x) = \bigvee(\downarrow f(x) \cap K(H))$ since $H$ is algebraic. We need to show that $\bigvee(\downarrow f(x) \cap K(H)) \leq \bigvee f(\downarrow x \cap K_f(L))$.

Since $1_H \leq fg$ and $\downarrow x \cap K_f(L) = g(\downarrow f(x) \cap K(H))$ by (1), we have

$$\bigvee(\downarrow f(x) \cap K(H)) \leq \bigvee fg(\downarrow f(x) \cap K(H)) = \bigvee f(\downarrow x \cap K_f(L)).$$

□

Proposition 2.7. *Let $L$ be a complete lattice, $H$ an algebraic meet-continuous lattice and $g \dashv f : L \to H$. Then $f$ preserves directed joins if and only if every element of $K_f(L)$ is compact, that is, $K_f(L) \subseteq K(L)$.*

*Proof.* Suppose that $f$ preserves directed joins and $u \in K_f(L)$. Let $D$ be a directed subset of $L$ with $u \leq \bigvee D$. Then there is $k \in K(H)$ with $g(k) = u$. Since $g \dashv f$ and $f$ preserves directed joins,

$$k \leq f(u) \leq f(\bigvee_L D) = \bigvee_H f(D).$$

Since $k$ is compact and $f(D)$ is directed, there is $d \in D$ with $k \leq f(d)$. Hence $u = g(k) \leq g(f(d)) \leq d$, so $u \ll u$.

Conversely, suppose that $u$ is a compact element for every $u \in K_f(L)$ and $D$ is a directed subset of $L$. Since $f(\bigvee_L D) \geq \bigvee_H f(D)$, it remain to show that $f(\bigvee_L D) \leq \bigvee_H f(D)$.

Let $\alpha = \bigvee_L D$ and $k \in \downarrow f(\alpha) \cap K(H)$. Then $k \leq f(\alpha)$, and

$$g(k) \leq \alpha = \bigvee_L D.$$

Since $g(k)$ is a compact element in $L$ and $D$ is directed, there is $d \in D$ with $g(k) \leq d$, hence $k \leq f(d) \leq \bigvee_H f(D)$. That is, $k \leq \bigvee_H f(D)$ for every $k \in \downarrow f(\alpha) \cap K(H)$, so

$$f(\bigvee_L D) = f(\alpha) = \bigvee_H(\downarrow f(\alpha) \cap K(H)) \leq \bigvee_H f(D).$$

□

Definition 2.8. Let $H$ be a complete lattice. Then $L$ is said to have $(\bigwedge, f)$-*structure* in $H$ if $L$ is a complete lattice with a map $f : L \to H$ which 1-1 and preserves arbitrary meets. In particular, if $L$ is a complete lattice with $L \subseteq H$ and the inclusion map $i : L \to H$ preserves arbitrary meets, then $L$ is said to have $\bigwedge$-*structure* in $H$.

If $L$ has $(\bigwedge, f)$-structure in $H$, then $f$ is an order-embedding map and has a unique left adjoint. We denoted the left adjoint of $f$ by $f^l$.

We note that $f$ need not preserve arbitrary joins (meets, resp.).

If $L$ has $\bigwedge$-structure in $H$, then $L$ has $(\bigwedge, i)$-structure in $H$, where $\bigwedge_L S = \bigwedge_H S$ for every $S \subseteq L$, because $\bigwedge_L S = i(\bigwedge_L S) = \bigwedge_H i(S) = \bigwedge_H S$.

Lemma 2.9. *If $L$ has $\bigwedge$-structure in a complete lattice $H$, then $\bigvee_L S \geq \bigvee_H S$ in $H$ for any $S \subseteq L$.*

*Proof.* Let $i : L \to H$ be the inclusion map. Then $i$ is monotone, and

$$\bigvee_L S = i(\bigvee_L S) \geq \bigvee_H i(S) = \bigvee_H S$$

for any $S \subseteq L$. $\square$

Let $L$ have $\bigwedge$-structure in $H$. Then the inclusion map $i : L \to H$ preserves arbitrary meet and $i^l \dashv i : L \to H$. We denote $\hat{k} = i^l(k)$ for each $k \in K(H)$ and $K_i(L) = \{\hat{k} \in L \mid k \in K(H)\}$.

PROPOSITION 2.10. *Let $L$ have $\bigwedge$-structure in an algebraic meet-continuous lattice $H$. Then $\bigvee_H D \in L$ for each directed subset $D$ of $L$ if and only if $\hat{k}$ is compact for every $k \in K(H)$.*

*Proof.* Let $D$ be a directed subset of $L$. Since the inclusion map $i : L \to H$ is order-embedding and $D \subseteq L$, $i^l(d) = i^l(i(d)) = d$ for all $d \in D$, hence $i^l(D) = D$. Since $\bigvee_H D \in L$ and $i^l$ preserves arbitrary joins,

$$\bigvee_H D = i^l(\bigvee_H D) = \bigvee_L i^l(D) = \bigvee_L D.$$

Hence we have

$$i(\bigvee_L D) = \bigvee_L D = \bigvee_H D = \bigvee_L i(D),$$

that is, $i$ preserves directed joins, and we have the equivalence of this proposition by Proposition 2.7. $\square$

EXAMPLE 2.11. Let $IdR$ be the complete lattice of all ideals of a ring $R$. Since $\bigwedge_{IdR} \mathcal{S} = \bigcap \mathcal{S} = \bigwedge_{\mathcal{P}(R)} \mathcal{S}$ for every $\mathcal{S} \subseteq IdR$, the inclusion map $i : IdR \to \mathcal{P}(R)$ preserves arbitrary meets. Hence $IdR$ has $\bigwedge$-structure in $\mathcal{P}(R)$, and for every directed subset $\mathcal{D}$ of $IdR$,

$$\bigvee_{\mathcal{P}(R)} \mathcal{D} = \bigcup \mathcal{D} \in IdR.$$

By Proposition 2.10, every principal ideal is compact in $IdR$ since $\{a\} \in K(\mathcal{P}(R))$ for all $a \in R$.

In the same way, a subspace generated by a singleton set is compact in the complete lattice $SubV$ of all subspace of a vector space $V$

LEMMA 2.12. *Let $L$ have $(\bigwedge, f)$-structure in an algebraic complete lattice $H$. Then $x = \bigvee(\downarrow x \cap K_f(L))$ for every $x \in L$.*

*Proof.* From the definition of an algebraic lattice, $f(x) = \bigvee(\downarrow f(x) \cap K(H))$ for each $x \in L$. Since $f^l \dashv f$ and $f$ is order-embedding, $f^l$ preserves arbitrary joins and $f^l f = 1_L$. Hence

$$x = f^l f(x) = f^l(\bigvee(\downarrow f(x) \cap K(H))) = \bigvee f^l(\downarrow f(x) \cap K(H)) = \bigvee(\downarrow x \cap K_f(L))$$

for every $x \in L$. □

Proposition 2.13. *Let $L$ have $(\bigwedge, f)$-structure in an algebraic meet-continuous lattice $H$. If $f$ preserves directed joins, then $L$ is algebraic.*

*Proof.* Suppose that $f$ preserves directed joins and let $x \in L$. Then $K_f(L) \subseteq K(L)$ by Proposition 2.7, hence we have

$$x = \bigvee(\downarrow x \cap K_f(L)) \leq \bigvee(\downarrow x \cap K(L)) \leq x.$$

That is, $x = \bigvee(\downarrow x \cap K(L))$, and $L$ is algebraic. □

Corollary 2.14. *Let $L$ have $(\bigwedge, f)$-structure in an algebraic meet-continuous lattice $H$. Then we have the following :*

(1) *If $f$ preserves directed joins, then $L$ is continuous.*
(2) *If every element of $K_f(L)$ is compact, then $L$ is algebraic.*
(3) *If every element of $K_f(L)$ is compact, then $L$ is continuous.*

Corollary 2.15. *Let $L$ have $\bigwedge$-structure in an algebraic meet-continuous lattice $H$.*

(1) *If $\hat{k}$ is compact for every $k \in K(H)$, then $L$ is algebraic.*
(2) *If $\hat{k}$ is compact for every $k \in K(H)$, then $L$ is continuous.*
(3) *If $\bigvee_H D \in L$ for every directed subset $D$ of $L$, then $L$ is algebraic.*
(4) *If $\bigvee_H D \in L$ for every directed subset $D$ of $L$, then $L$ is continuous.*

## References

[1] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, Continuous Lattices and Domains, Cambridge University Press, 2003.
[2] P. T. Johnstone, Stone Spaces, Cambridge Studies in Advanced Mathematics No. 3, Cambridge University Press, 1982.
[3] S. O. Lee, Y. H. Yon, Some Structures on a Complete Lattice, J. of the Chungcheong Mathematical Society, **20**, no.3 (2007), 211-222.
[4] D. S. Scott, Continuous Lattices, Lect. Notes in Math., **274**, Springer Verlag, 1972, 97-136.
[5] A. Walendziak, On Atomistic Lattices, Portugaliae Mathematica, **51**, Fasc. 4, 1994, 583-585.

*

Department of Mathematics
Chungbuk National University,
Chungbuk 361-763, Republic of Korea
*E-mail*: solee@chungbuk.ac.kr

**

Engineering Education Innovation Center
Mokwon University
Daejeon, 302-729, Republic of Korea
*E-mail*: yhyon@mokwon.ac.kr