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GENERALIZED HYERS-ULAM STABILITY OF FUNCTIONAL EQUATIONS

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability of the following linear functional equations

$$f(x+iy)+f(x-iy)+f(y+ix)+f(y-ix)=2f(x)+2f(y)$$

and $f((1+i)x)=(1+i)f(x)$, and of the following quadratic functional equations

$$f(x+iy)+f(x-iy)+f(y+ix)+f(y-ix)=0$$

and $f((1+i)x)=2if(x)$ in complex Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [31] concerning the stability of group homomorphisms: Let $(G_1,*)$ be a group and let $(G_2,\diamond,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta(\epsilon)>0$ such that if a mapping $h:G_1\to G_2$ satisfies the inequality

$$d(h(x*y),h(x)\diamond h(y))<\delta$$

for all $x,y\in G_1$, then there is a homomorphism $H:G_1\to G_2$ with

$$d(h(x),H(x))<\epsilon$$

for all $x\in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x*y)=H(x)\diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how

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do the solutions of the inequality differ from those of the given functional equation?

Hyers [5] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X$ and some $\varepsilon \ge 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon$$

for all $x \in X$.

Th.M. Rassias [21] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

THEOREM 1.1. (Th.M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

The above inequality (1.1) that was introduced for the first time by Th.M. Rassias [21] for the proof of the stability of the linear mapping bewteen Banach spaces has provided a lot of influence in the development of what is now known as a *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [4] extended the Hyers-Ulam stability by proving the following theorem in the spirit of Th.M. Rassias' approach.

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic function*. A generalized Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [30] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [3] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1], [8]–[20], [23]–[29]).

In this paper, we prove the generalized Hyers-Ulam stability of the following linear functional equations

$$f(x+iy) + f(x-iy) + f(y+ix) + f(y-ix) = 2f(x) + 2f(y) \tag{1.2}$$

and $f((1+i)x) = (1+i)f(x)$, whose solution is called an *additive mapping*, and the generalized Hyers-Ulam stability of the following quadratic functional equations

$$f(x+iy) + f(x-iy) + f(y+ix) + f(y-ix) = 0 \tag{1.3}$$

and $f((1+i)x) = 2if(x)$, whose solution is called a *quadratic mapping*.

Throughout this paper, assume that X is a complex normed vector space with norm $||\cdot||$ and that Y is a complex Banach space with norm $\|\cdot\|$.

2. Generalized Hyers-Ulam stability of linear functional equations

For a given mapping $f : X \to Y$, we define

$$Cf(x,y) := f(x+iy) + f(x-iy) + f(y+ix) + f(y-ix) - 2f(x) - 2f(y)$$

for all $x, y \in X$.

If a mapping $f : X \to Y$ satisfies the linear functional equation

$$f(x+y) = f(x) + f(y)$$

and $f(ix) = if(x)$ for all $x, y \in X$, then

$$f(x+iy) + f(x-iy) + f(y+ix) + f(y-ix) = 2f(x) + 2f(y)$$

for all $x, y \in X$. In fact, $f : \mathbb{C} \to \mathbb{C}$ with $f(x) = x$ satisfies (1.2).

We prove the generalized Hyers-Ulam stability of the linear functional equation $Cf(x, y) = 0$.

THEOREM 2.1. *Let $p < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f((1+i)x) = (1+i)f(x)$ and*

$$\|Cf(x, y)\| \le \theta(||x||^p + ||y||^p) \tag{2.1}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{\sqrt{2}\theta}{2 - 2^p}||x||^p \tag{2.2}$$

for all $x \in X$.

Proof. Since $f((1+i)x) = (1+i)f(x)$ for all $x \in X$, $f(0) = 0$ and $f(2x) = (1+i)f((1-i)x)$ for all $x \in X$.

Letting $y = x$ in (2.1), we get

$$\|2f((1+i)x) + 2f((1-i)x) - 4f(x)\| \le 2\theta||x||^p$$

for all $x \in X$. Hence

$$\|(1-i)f(2x) - (2-2i)f(x)\| \le 2\theta||x||^p$$

for all $x \in X$. So

$$\|f(x) - \frac{1}{2}f(2x)\| \le \frac{\theta}{\sqrt{2}}||x||^p \tag{2.3}$$

for all $x \in X$. Hence

$$\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\| \le \sum_{j=l}^{m-1} \frac{2^{pj}\theta}{2^j\sqrt{2}}||x||^p \tag{2.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.4) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x)$$

for all $x \in X$.

By (2.1),

$$\|CA(x,y)\| = \lim_{n\to\infty} \frac{1}{2^n}\|Cf(2^n x, 2^n y)\| \le \lim_{n\to\infty} \frac{2^{pn}\theta}{2^n}(||x||^p + ||y||^p) = 0$$

for all $x, y \in X$. So $CA(x,y) = 0$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.4), we get (2.2).

Now, let $L : X \to Y$ be another additive mapping satisfying (1.2) and (2.2). Then we have

$$\begin{aligned}\|A(x) - L(x)\| &= \frac{1}{2^n}\|A(2^n x) - L(2^n x)\| \\ &\le \frac{1}{2^n}(\|A(2^n x) - f(2^n x)\| + \|L(2^n x) - f(2^n x)\|) \\ &\le \frac{2\sqrt{2}\theta}{2 - 2^p} \cdot \frac{2^{pn}}{2^n}||x||^p,\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $A(x) = L(x)$ for all $x \in X$. This proves the uniqueness of A. So there exists a unique quadratic mapping $A : X \to Y$ satisfying (1.2) and (2.2). □

THEOREM 2.2. *Let $p > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.1) *and $f((i+i)x) = (1+i)f(x)$ for all $x \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{\sqrt{2}\theta}{2^p - 2}||x||^p \tag{2.5}$$

for all $x \in X$.

Proof. It follows from (2.3) that

$$\|f(x) - 2f(\frac{x}{2})\| \le \frac{\sqrt{2}\theta}{2^p}||x||^p$$

for all $x \in X$. Hence

$$\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})\| \le \sum_{j=l}^{m-1} \frac{2^{j+\frac{1}{2}}\theta}{2^{pj+p}}||x||^p \tag{2.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$.

By (2.1),

$$\|CA(x,y)\| = \lim_{n\to\infty} 2^n \|Cf(\frac{x}{2^n}, \frac{y}{2^n})\| \le \lim_{n\to\infty} \frac{2^n\theta}{2^{pn}}(||x||^p + ||y||^p) = 0$$

for all $x, y \in X$. So $CA(x,y) = 0$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.5).

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

THEOREM 2.3. *Let $p < \frac{1}{2}$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f((1+i)x) = (1+i)f(x)$ and*

$$\|Cf(x,y)\| \le \theta \cdot ||x||^p \cdot ||y||^p \tag{2.7}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{\theta}{(2-4^p)\sqrt{2}}||x||^{2p} \tag{2.8}$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.7), we get

$$\|2f((1+i)x) + 2f((1-i)x) - 4f(x)\| \le \theta||x||^{2p}$$

for all $x \in X$. Hence

$$\|(1-i)f(2x) - (2-2i)f(x)\| \le \theta||x||^{2p}$$

for all $x \in X$. So

$$\|f(x) - \frac{1}{2}f(2x)\| \le \frac{\theta}{2\sqrt{2}}||x||^{2p} \tag{2.9}$$

for all $x \in X$. Hence

$$\|\frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x)\| \le \sum_{j=l}^{m-1} \frac{4^{pj}\theta}{2^{j+1}\sqrt{2}}||x||^{2p} \tag{2.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.10) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x)$$

for all $x \in X$.

By (2.7),

$$\|CA(x,y)\| = \lim_{n\to\infty} \frac{1}{2^n}\|Cf(2^n x, 2^n y)\| \le \lim_{n\to\infty} \frac{4^{pn}\theta}{2^n} \cdot ||x||^p \cdot ||y||^p = 0$$

for all $x, y \in X$. So $CA(x,y) = 0$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.10), we get (2.8).

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

THEOREM 2.4. *Let $p > \frac{1}{2}$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (2.7) *and $f((1+i)x) = (1+i)f(x)$ for all $x \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x) - A(x)\| \le \frac{\theta}{(4^p - 2)\sqrt{2}}||x||^{2p} \tag{2.11}$$

for all $x \in X$.

Proof. It follows from (2.9) that

$$\|f(x) - 2f(\frac{x}{2})\| \le \frac{\theta}{4^p\sqrt{2}}||x||^{2p}$$

for all $x \in X$. Hence

$$\|2^l f(\frac{x}{2^l}) - 2^m f(\frac{x}{2^m})\| \le \sum_{j=l}^{m-1} \frac{2^j\theta}{4^{pj+p}\sqrt{2}}||x||^{2p} \tag{2.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.12) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$.

By (2.7),

$$\|CA(x,y)\| = \lim_{n\to\infty} 2^n\|Cf(\frac{x}{2^n}, \frac{y}{2^n})\| \le \lim_{n\to\infty} \frac{2^n\theta}{4^{pn}} \cdot ||x||^p \cdot ||y||^p = 0$$

for all $x, y \in X$. So $CA(x,y) = 0$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.12), we get (2.11).

The rest of the proof is similar to the proof of Theorem 2.1. $\square$

3. Generalized Hyers-Ulam stability of quadratic functional equations

For a given mapping $f : X \to Y$, we define

$$Cf(x, y) := f(x + iy) + f(x - iy) + f(y + ix) + f(y - ix)$$

for all $x, y \in X$.

If a mapping $f : X \to Y$ satisfies the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

and $f(ix) = -f(x)$ for all $x, y \in X$, then

$$f(x + iy) + f(x - iy) + f(x + y) + f(x - y) = 0$$

for all $x, y \in X$. In fact, $f : \mathbb{C} \to \mathbb{C}$ with $f(x) = x^2$ satisfies (1.3).

We prove the generalized Hyers-Ulam stability of the quadratic functional equation $Cf(x, y) = 0$.

Theorem 3.1. *Let $p < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f((1 + i)x) = 2if(x)$ and*

$$\|Cf(x, y)\| \le \theta(||x||^p + ||y||^p) \tag{3.1}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{2\theta}{4 - 2^p}||x||^p \tag{3.2}$$

for all $x \in X$.

Proof. Since $f((1+i)x) = 2if(x)$ for all $x \in X$, $f(0) = 0$ and $f(2x) = 2if((1 - i)x)$ for all $x \in X$.

Letting $y = x$ in (3.1), we get

$$\|2f((1 + i)x) + 2f((1 - i)x)\| \le 2\theta||x||^p$$

for all $x \in X$. Hence

$$\| - if(2x) + 4if(x)\| \le 2\theta||x||^p$$

for all $x \in X$. So

$$\|f(x) - \frac{1}{4}f(2x)\| \le \frac{\theta}{2}||x||^p \tag{3.3}$$

for all $x \in X$. Hence

$$\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\| \le \sum_{j=l}^{m-1} \frac{2^{pj}\theta}{2 \cdot 4^j}||x||^p \tag{3.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.4) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$.

By (3.1),

$$\|CQ(x, y)\| = \lim_{n\to\infty} \frac{1}{4^n} \|Cf(2^n x, 2^n y)\| \le \lim_{n\to\infty} \frac{2^{pn}\theta}{4^n} (||x||^p + ||y||^p) = 0$$

for all $x, y \in X$. So $CQ(x, y) = 0$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.4), we get (3.2).

The rest of the proof is similar to the proof of Theorem 2.1. □

THEOREM 3.2. *Let $p > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.1) *and $f((i + i)x) = 2if(x)$ for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{2^p - 4} ||x||^p \tag{3.5}$$

for all $x \in X$.

Proof. It follows from (3.3) that

$$\|f(x) - 4f(\frac{x}{2})\| \le \frac{2\theta}{2^p} ||x||^p$$

for all $x \in X$. Hence

$$\|4^l f(\frac{x}{2^l}) - 4^m f(\frac{x}{2^m})\| \le \sum_{j=l}^{m-1} \frac{2 \cdot 4^j \theta}{2^{pj+p}} ||x||^p \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$.

By (3.1),

$$\|CQ(x, y)\| = \lim_{n\to\infty} 4^n \|Cf(\frac{x}{2^n}, \frac{y}{2^n})\| \le \lim_{n\to\infty} \frac{4^n\theta}{2^{pn}} (||x||^p + ||y||^p) = 0$$

for all $x, y \in X$. So $CQ(x, y) = 0$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.1. □

THEOREM 3.3. *Let $p < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f((1+i)x) = 2if(x)$ and*

$$\|Cf(x, y)\| \leq \theta \cdot ||x||^p \cdot ||y||^p \tag{3.7}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta}{4 - 4^p}||x||^{2p} \tag{3.8}$$

for all $x \in X$.

Proof. Letting $y = x$ in (3.7), we get

$$\|2f((1+i)x) + 2f((1-i)x)\| \leq \theta||x||^{2p}$$

for all $x \in X$. Hence

$$\| - if(2x) + 4if(x)\| \leq \theta||x||^{2p}$$

for all $x \in X$. So

$$\|f(x) - \frac{1}{4}f(2x)\| \leq \frac{\theta}{4}||x||^{2p} \tag{3.9}$$

for all $x \in X$. Hence

$$\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\| \leq \sum_{j=l}^{m-1} \frac{4^{pj}\theta}{4^{j+1}}||x||^{2p} \tag{3.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.10) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n}f(2^n x)$$

for all $x \in X$.

By (3.7),

$$\|CQ(x, y)\| = \lim_{n\to\infty} \frac{1}{4^n}\|Cf(2^n x, 2^n y)\| \leq \lim_{n\to\infty} \frac{4^{pn}\theta}{4^n} \cdot ||x||^p \cdot ||y||^p = 0$$

for all $x, y \in X$. So $CQ(x, y) = 0$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.10), we get (3.8).

The rest of the proof is similar to the proof of Theorem 2.1. □

THEOREM 3.4. *Let $p > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.7) *and $f((1+i)x) = 2if(x)$ for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{\theta}{4^p - 4}||x||^{2p} \tag{3.11}$$

for all $x \in X$.

Proof. It follows from (3.9) that

$$\|f(x) - 4f(\frac{x}{2})\| \le \frac{\theta}{4^p}||x||^{2p}$$

for all $x \in X$. Hence

$$\|4^l f(\frac{x}{2^l}) - 4^m f(\frac{x}{2^m})\| \le \sum_{j=l}^{m-1} \frac{4^j \theta}{4^{pj+p}}||x||^{2p} \tag{3.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.12) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$.

By (3.7),

$$\|CQ(x,y)\| = \lim_{n\to\infty} 4^n \|Cf(\frac{x}{2^n}, \frac{y}{2^n})\| \le \lim_{n\to\infty} \frac{4^n\theta}{4^{pn}} \cdot ||x||^p \cdot ||y||^p = 0$$

for all $x, y \in X$. So $CQ(x,y) = 0$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.12), we get (3.11).

The rest of the proof is similar to the proof of Theorem 2.1. □

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