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GENERATING FUNCTION OF TRACES OF SINGULAR MODULI

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Abstract. Let p be a prime and $f(z) = \sum_n a(n)q^n$ be a weakly holomorphic modular function for $\Gamma_0^*(p)$ with $a(0) = 0$. We use Bruinier and Funke's work to find the generating series of modular traces of $f(z)$ as Jacobi forms.

1. Introduction

The classical modular function $j(z)$ for the modular group $\Gamma(1) = PSL_2(\mathbb{Z})$ is defined on the complex upper half plane $\mathbb{H}$ and has a Fourier expansion

$$j(z) = q^{-1} + 744 + 196884q + \cdots$$

where $q = e(z) = e^{2\pi iz}$. For a positive integer d congruent to 0 or 3 modulo 4, we denote by $\mathcal{Q}_d$ the set of positive definite integral binary quadratic forms

$$Q(x, y) = [a, b, c] = ax^2 + bxy + cy^2$$

with discriminant $-d = b^2 - 4ac$. The group $\Gamma(1)$ acts on $\mathcal{Q}_d$ by $Q \circ \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) = Q(\alpha x + \beta y, \gamma x + \delta y)$. For each $Q \in \mathcal{Q}_d$, we let

$$\alpha_Q = \frac{-b + i\sqrt{d}}{2a},$$

the corresponding CM point in $\mathbb{H}$ and we write $\Gamma(1)_Q$ for the stabilizer of Q in $\Gamma(1)$. The Hurwitz-Kronecker class number $H(d)$ and the trace

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$\mathbf{t}_J(d)$ for $J(z) = j(z) - 744$ are defined as

$$H(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma(1)} \frac{1}{|\Gamma(1)_Q|};\ \ \mathbf{t}_J(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma(1)} \frac{1}{|\Gamma(1)_Q|} J(\alpha_Q).$$

In [7, Theorem 1], Zagier proved that the generating series for the traces of singular moduli

$$-q^{-1} + 2 + \sum_{\substack{d>0 \\ d \equiv 0,3 (\mathrm{mod} 4)}} \mathbf{t}_J(d) q^d = -q^{-1} + 2 - 248q^3 + 492q^4 + \cdots$$

is a weakly holomorphic modular form (that is, meromorphic with poles only at the cusps) of weight 3/2 on $\Gamma_0(4)$.

For a generalization of Zagier's result we consider the following setting. Let N be a positive integer and $\Gamma_0^*(N)$ be the group generated by $\Gamma_0(N)$ and all Atkin-Lehner involutions W_e for $e||N$. Here $e||N$ denotes that $e|N$ and $(e, N/e) = 1$, and W_e can be represented by a matrix $\frac{1}{\sqrt{e}} \left(\begin{smallmatrix} ex & y \\ Nz & ew \end{smallmatrix} \right)$ with $\det W_e = 1$ and $x, y, z, w \in \mathbb{Z}$. Let d denote a positive integer such that $-d$ is congruent to a square modulo $4N$. Let $\mathcal{Q}_{d,N} = \{[a,b,c] \in \mathcal{Q}_d \mid a \equiv 0 \pmod N\}$ on which $\Gamma_0^*(N)$ acts. We choose an integer $h \pmod{2N}$ with $h^2 \equiv -d \pmod{4N}$ and consider the set $\mathcal{Q}_{d,N,h} = \{[a,b,c] \in \mathcal{Q}_d \mid a \equiv 0 \pmod N,\ b \equiv h \pmod{2N}\}$ on which $\Gamma_0(N)$ acts. Let f be a weakly holomorphic modular function for $\Gamma_0^*(N)$. The class number $H_N^{(h)}(d)$ and the trace $\mathbf{t}_f^{(h)}(d)$ are defined by

$$H_N^{(h)}(d) = \sum_{Q \in \mathcal{Q}_{d,N,h}/\Gamma_0(N)} \frac{1}{|\Gamma_0(N)_Q|};\ \ \mathbf{t}_f^{(h)}(d) = \sum_{Q \in \mathcal{Q}_{d,N,h}/\Gamma_0(N)} \frac{1}{|\Gamma_0(N)_Q|} f(\alpha_Q).$$

We note that if N is a prime number p, then the definition of $\mathbf{t}_f^{(h)}(d)$ is independent of the choice of h and therefore one can define $\mathbf{t}_f(d) = \mathbf{t}_f^{(h)}(d)$.

Using Bruinier and Funke's work [2] we will prove:

THEOREM 1.1. *Let $f = \sum_n a(n) q^n$ be a weakly holomorphic modular function for $\Gamma_0^*(p)$ with $a(0) = 0$. We put*

$$\boldsymbol{t}_f(0) = 2 \sum_{n \in \mathbb{Z}_{\geq 0}} a(-n)(\sigma_1(n) + p\sigma_1(n/p)),$$

and for negative d

$$\boldsymbol{t}_f(d) = \begin{cases} -2^{\mu(\kappa)}\kappa \sum_{\kappa \mid m} a(-m), & \text{if } d = -\kappa^2 \text{ for some positive integer } \kappa; \\ 0, & \text{otherwise} \end{cases}$$

where $\mu(\kappa)$ *is defined to be* 1 *or* 0 *according as* $p \mid \kappa$. *Then*

$$\sum_{n,r} \boldsymbol{t}_f(4pn - r^2) q^n \zeta^r$$

is a weak Jacobi form of weight 2 and index p *where* $q = e(\tau)$, $\zeta = e(z)$ *for* $\tau \in \mathbb{H}$, $z \in \mathbb{C}$.

This paper is organized as follows. In the next section, we give a brief review of Bruinier and Funke's work in [2], and Jacobi forms. In Section 3 we prove Theorem 1.1.

2. Preliminaries

2.1. Bruinier and Funke's work on traces of CM values of modular functions

2.1.1. Definition of modular traces. In this subsection we follow the expositions in [2]. We consider a rational quadratic space (V, q) of dimension 3 given by

$$V(\mathbb{Q}) := \left\{ X = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \in M_2(\mathbb{Q}) \right\},$$

with the associated quadratic form $q(X) := p \cdot \det(X)$ and the bilinear form $(X, Y) := -p \cdot \operatorname{tr}(XY)$. The group $G(\mathbb{Q}) = SL_2(\mathbb{Q})$ acts on V by $g.X := gXg^{-1}$ for $X \in V$ and $g \in G(\mathbb{Q})$. Let D be the space of positive lines in $V(\mathbb{R})$, that is,

$$D = \{\operatorname{span}(X) \subset V(\mathbb{R}) \mid (X, X) > 0\}$$

which is identified with $\mathbb{H}$ as follows. We pick as a base point of D the line spanned by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For $z = x + iy \in \mathbb{H}$, we choose $g_z \in SL_2(\mathbb{R})$ such that $g_z i = z$. We now have the isomorphism $\mathbb{H} \to D$ which assigns $z \in \mathbb{H}$ the positive line in D spanned by

$$X(z) := g_z . \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{y} \begin{pmatrix} -\frac{1}{2}(z + \bar{z}) & z\bar{z} \\ -1 & \frac{1}{2}(z + \bar{z}) \end{pmatrix}.$$

Note that $q(X(z)) = 1$ and $g.X(z) = X(gz)$ for $g \in SL_2(\mathbb{R})$.

Let $L \subset V(\mathbb{Q})$ be an even lattice of full rank and write $L^{\#}$ for the dual lattice of L. Let Γ be a congruence subgroup of $\mathrm{Spin}(L)$ which preserves L. We assume that Γ acts trivially on the discriminant group $L^{\#}/L$ and set the modular curve $M := \Gamma\backslash D$. The set $\mathrm{Iso}(V)$ of all isotropic lines in V corresponds to $P^1(\mathbb{Q}) = \mathbb{Q}\cup\{\infty\}$ via the bijective map $\psi : P^1(\mathbb{Q}) \to \mathrm{Iso}(V)$, which is defined by $\psi((\alpha : \beta)) = \mathrm{span}\begin{pmatrix} -\alpha\beta & \alpha^2 \\ -\beta^2 & \alpha\beta \end{pmatrix} \in \mathrm{Iso}(V)$. Since $\psi(g(\alpha : \beta)) = g.\psi((\alpha : \beta))$ for $g \in G(\mathbb{Q})$, the cusps of M, i.e., the Γ-classes of $P^1(\mathbb{Q})$, can be identified with the Γ-classes of $\mathrm{Iso}(V)$. In particular, the cusp $\infty \in P^1(\mathbb{Q})$ is mapped to the isotropic line ℓ_0 which is spanned by $X_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We orient all lines $\ell \in \mathrm{Iso}(V)$ by regarding $\sigma_\ell.X_0$ as a positively oriented basis vector of ℓ, where $\sigma_\ell \in SL_2(\mathbb{Z})$ such that $\sigma_\ell.\ell_0 = \ell$. For each isotropic line $\ell \in \mathrm{Iso}(V)$, there exist positive rational numbers α_ℓ and β_ℓ such that $\sigma_\ell^{-1}\Gamma_\ell\sigma_\ell = \left\{\pm\begin{pmatrix} 1 & k\alpha_\ell \\ 0 & 1 \end{pmatrix} \;\middle|\; k \in \mathbb{Z}\right\}$, where Γ_ℓ denotes the stabilizer of the line ℓ, and $\begin{pmatrix} 0 & \beta_\ell \\ 0 & 0 \end{pmatrix}$ is a primitive element of $\ell_0 \cap \sigma_\ell^{-1}L$, respectively. Finally, we write $\epsilon_\ell = \alpha_\ell/\beta_\ell$. Note that α_ℓ is the width of the cusp ℓ with respect to Γ and the quantities α_ℓ, β_ℓ, and ϵ_ℓ only depend on the Γ-class of ℓ.

We now define the modular trace function of a weakly holomorphic modular function f for Γ. We recall that f has a Fourier expansion at the cusp ℓ of the form

$$f(\sigma_\ell z) = \sum_{n \in \frac{1}{\alpha_\ell}\mathbb{Z}} a_\ell(n)e(nz),$$

with $a_\ell(n) = 0$ for $n \ll 0$. Let us first define CM points as, for $X \in V(\mathbb{Q})$ of positive norm, $D_X = \mathrm{span}(X) \in D$. Note that the corresponding point in $\mathbb{H}$ satisfies a quadratic equation over $\mathbb{Q}$. For $m \in \mathbb{Q}_{>0}$ and $h \in L^{\#}$, the group Γ acts on

$$L_{h,m} = \{X \in L + h \mid q(X) = m\}$$

with finitely many orbits. We define the modular trace of f for positive index by

$$\mathbf{t}_f(h, m) = \sum_{X \in \Gamma\backslash L_{h,m}} \frac{1}{|\overline{\Gamma}_X|} f(D_X).$$

On the other hand, for a vector $X \in V(\mathbb{Q})$ of negative norm, we define a geodesic c_X in D by

$$c_X = \{z \in D \mid z \perp X\}.$$

If $q(X) \in -p \cdot (\mathbb{Q}^\times)^2$, then the quotient $c(X) := \Gamma_X \backslash c_X$ in M is an infinite geodesic, and X is orthogonal to the two isotropic lines $\ell_X = \mathrm{span}(Y)$ and $\tilde{\ell}_X = \mathrm{span}(\tilde{Y})$, with Y and $\tilde{Y}$ positively oriented. We say ℓ_X is the line associated to X if the triple $(X, Y, \tilde{Y})$ is a positively oriented basis for V, and we write $X \sim \ell_X$. Note that $\tilde{\ell}_X = \ell_{-X}$. Let $\sigma_{\ell_X}^{-1}.X = \begin{pmatrix} m & r \\ 0 & -m \end{pmatrix}$ for some $m \in \mathbb{Q}_{>0}$ and $r \in \mathbb{Q}$. In this case the geodesic c_X is given in $D \simeq \mathbb{H}$ by

$$c_X = \sigma_{\ell_X}\{z \in \mathbb{H} \mid \mathrm{Re}(z) = -r/2m\}$$

We call the quantity $-r/2m$ the *real part of* $c(X)$, denoted $\mathrm{Re}(c(X))$. For $m \in \mathbb{Q}_{>0}$ and a fixed cusp ℓ, we find that the group Γ_ℓ acts on

$$L_{h,-pm^2,\ell} = \{X \in L_{h,-pm^2} \mid X \sim \ell\}$$

and

$$L_{h,-pm^2} = \coprod_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \coprod_{\gamma \in \Gamma_\ell \backslash \Gamma} \gamma^{-1} L_{h,-pm^2,\ell}. \tag{1}$$

Hence

$$\#\Gamma \backslash L_{h,-pm^2} = \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \#\Gamma_\ell \backslash L_{h,-pm^2,\ell} \tag{2}$$

where

$$\#\Gamma_\ell \backslash L_{h,-pm^2,\ell} = \begin{cases} 2m\epsilon_\ell, & \text{if } L_{h,-pm^2,\ell} \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

If $n \in \mathbb{Q}_{<0}$ is not of the form $n = -pm^2$ with $m \in \mathbb{Q}_{>0}$, we put the modular trace of f for negative index $\mathbf{t}_f(h, n) = 0$. If $n = -pm^2$ with $m \in \mathbb{Q}_{>0}$, we define $\mathbf{t}_f(h, -pm^2)$ in terms of geodesic cycles connecting two cusps of which the precise definition is given in [2, Definition 4.4].

Lastly, the modular trace of f for zero index is defined by

$$\mathbf{t}_f(h, 0) = -\frac{\delta_{h,0}}{2\pi} \int_M^{reg} f(z) \frac{dx\, dy}{y^2},$$

where $\delta_{h,0}$ is the Kronecker delta and the regularized integral is defined as

$$\int_M^{reg} f(z) \frac{dx\, dy}{y^2} = \lim_{\epsilon \to 0} \int_{M(\epsilon)} f(z) \frac{dx\, dy}{y^2}.$$

Here $M(\epsilon)$ denotes the manifold with boundary obtained by removing an ϵ-disc around each cusp from M.

2.1.2. Main Theorem of Bruinier and Funke.

THEOREM 2.1. *([2, Theorem 4.5 and Proposition 4.7], [5], [1]) (a) There exists a (non-holomorphic) modular form $I_h(\tau, f)$ with Fourier expansion*

$$\begin{aligned} I_h(\tau, f) = & \sum_{\substack{m\in\mathbb{Z}+q(h)\\ m\gg-\infty}} \mathbf{t}_f(h,m)q^m \\ & + \frac{1}{2\pi\sqrt{vp}} \sum_{\substack{\ell\in\Gamma\backslash\ Iso(V)\\ \ell\cap L+h\neq\emptyset}} a_\ell(0)\epsilon_\ell \\ & + \sum_{m>0}\sum_{X\in\Gamma\backslash L_{h,-m^2}} \frac{a_{\ell_X}(0)+a_{\ell_{-X}}(0)}{8\pi\sqrt{vp}m}\beta(4\pi vpm^2)q^{-dm^2} \end{aligned}$$

where $q = e(\tau)$ for $\tau = u + iv \in \mathbb{H}$ and $\beta(s) = \int_1^\infty t^{-3/2}e^{-st}dt$.
(b) For $k \in \mathbb{Q}_{>0}$,

$$\begin{aligned} \boldsymbol{t}_f(h,-pk^2) = & -\sum_\ell \sharp(\Gamma_\ell\backslash L_{h,-pk^2,\ell}) \sum_{n\in\frac{2k}{\beta_\ell}\mathbb{Z}<0} a_\ell(n)e^{2\pi irn} \\ & -\sum_\ell \sharp(\Gamma_\ell\backslash L_{-h,-pk^2,\ell}) \sum_{n\in\frac{2k}{\beta_\ell}\mathbb{Z}<0} a_\ell(n)e^{2\pi ir'n} \end{aligned}$$

with $r = Re(c(X))$ for any $X \in L_{h,-pk^2,\ell}$ and $r' = Re(c(X))$ for any $X \in L_{-h,-pk^2,\ell}$. In particular, $\boldsymbol{t}_f(h,-pk^2) = 0$ for $k \gg 0$.
(c) $I_h(\tau, f)$ satisfies

$$\begin{aligned} I_h(\tau+1, f) &= e((h,h)/2)I_h(\tau, f) \\ I_h(-1/\tau, f) &= \sqrt{\tau}^3\frac{\sqrt{i}}{\sqrt{|L^\sharp/L|}}\sum_{h'\in L^\sharp/L} e(-(h,h'))I_{h'}(\tau, f). \end{aligned}$$

2.2. Jacobi form

A *(holomorphic) Jacobi form of weight k and index N* is defined to be a holomorphic function $\phi : \mathfrak{H}\times\mathbb{C} \to \mathbb{C}$ satisfying the two transformation laws

$$\phi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^k e^{2\pi i N\frac{cz^2}{c\tau+d}}\phi(\tau,z)\quad (\forall \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})),$$

$$\phi(\tau,z+\lambda\tau+\mu)=e^{-2\pi i N(\lambda^2\tau+2\lambda z)}\phi(\tau,z)\quad (\forall(\lambda,\mu)\in\mathbb{Z}^2)$$

and having a Fourier expansion of the form

$$\phi(\tau,z)=\sum_{\substack{n,r\in\mathbb{Z}\\ 4Nn-r^2\geq 0}} c(n,r)q^n\zeta^r\qquad (q=e^{2\pi i\tau},\ \zeta=e^{2\pi i z}),\tag{4}$$

where the coefficient $c(n,r)$ depends only on $4Nn-r^2$ and on the residue class of $r \pmod{2N}$ ([3] Theorem 2.2). In (4), if we relax the condition $4Nn-r^2\geq 0$ to merely requiring that $c(n,r)=0$ if $n<0$, we obtain the space of *weak Jacobi forms*, denoted $\tilde{J}_{k,N}$ in [3]. The structure of the ring $\tilde{J}_{ev,*}$ of all weak Jacobi forms of even weight was determined in [3, Theorem 9.3]: $\tilde{J}_{ev,*}$ is the free polynomial algebra over $M_*(\Gamma(1))=\mathbb{C}[E_4(\tau),E_6(\tau)]$ on two generators $a=\tilde{\phi}_{-2,1}(\tau,z)\in\tilde{J}_{-2,1}$ and $b=\tilde{\phi}_{0,1}(\tau,z)\in\tilde{J}_{0,1}$.

The periodicity property of the Fourier coefficients mentioned above implies that

$$c(n,r)=c_r(4nN-r^2),\ \ c_{r'}(D)=c_r(D)\quad\text{for } r'\equiv r \pmod{2N}.\tag{5}$$

Equation (5) gives us coefficients $c_\mu(D)$ for all $\mu\in\mathbb{Z}/2N\mathbb{Z}$ and all integers $D\geq 0$ satisfying $D\equiv-\mu^2 \pmod{4N}$, namely

$$c_\mu(D):=c\left(\frac{D+r^2}{4N},r\right)\quad (\text{any } r\in\mathbb{Z}, r\equiv\mu \pmod{2N}).$$

We extend the definition to all N by setting $c_\mu(D)=0$ if $D\not\equiv-\mu^2 \pmod{4N}$, and set

$$h_\mu(\tau):=\sum_{D=0}^{\infty}c_\mu(D)q^{D/4N}\qquad(\mu\in\mathbb{Z}/2N\mathbb{Z}).$$

According to [3, §5] $h_\mu(\tau)$ satisfies

$$h_\mu(\tau+1)=e\left(-\frac{\mu^2}{4N}\right)h_\mu(\tau)\tag{6}$$

and

$$h_\mu(-1/\tau)=\frac{\tau^k}{\sqrt{2N\tau/i}}\sum_{\nu \pmod{2N}}e\left(\frac{\mu\nu}{2N}\right)h_\nu(\tau).\tag{7}$$

Furthermore we have

THEOREM 2.2. [3, Theorem 5.1] *There is an isomorphism between the space of weak Jacobi forms of weight k and index N and the space of vector valued modular forms $(h_\mu)_{\mu \pmod{2N}}$ satisfying the transformation laws (6) and (7) and some meromorphic condition at $i\infty$.*

3. Proof of Theorem 1.1

We consider

$$L = \left\{ X = \begin{pmatrix} b & c/p \\ a & -b \end{pmatrix} ; a, b, c \in \mathbb{Z} \right\} \tag{8}$$

with $q(X) = p \cdot \det(X)$ and $(X, Y) = -p \cdot \operatorname{tr}(XY)$. The dual lattice $L^\sharp$ is given by

$$L^\sharp = \left\{ \begin{pmatrix} b & c/p \\ a & -b \end{pmatrix} ; a, c \in \mathbb{Z}, b \in \frac{1}{2p}\mathbb{Z} \right\}.$$

Note that $L^\sharp/L \cong \mathbb{Z}/2p\mathbb{Z}$ is cyclic. From now on we identify $h \in \mathbb{Z}/2p\mathbb{Z}$ with the corresponding element $L + \begin{pmatrix} h/2p & 0 \\ 0 & -h/2p \end{pmatrix} \in L^\sharp/L$. Each coset in $L^\sharp/L$ is then of the form

$$\left\{ \begin{pmatrix} b + h/2p & c/p \\ a & -b - h/2p \end{pmatrix} ; a, b, c \in \mathbb{Z} \right\}$$

for $h \in \{0, 1, \cdots, 2p-1\}$. Moreover it is easy to check that $\Gamma = \Gamma_0(p)$ acts on L and acts trivially on $L^\sharp/L$. There are two $\Gamma_0(p)$-inequivalent cusps: $\infty, 0$ which correspond to elements of $\Gamma_0(p)\backslash \operatorname{Iso}(V)$. Let $\ell_0 = \operatorname{span}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\ell_1 = \operatorname{span}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ which correspond to the cusps ∞ and 0, respectively. Then it is easy to compute that

$$\begin{aligned} &\alpha_{\ell_0} = 1, \beta_{\ell_0} = 1/p, \epsilon_{\ell_0} = p, \\ &\alpha_{\ell_1} = p, \beta_{\ell_1} = 1, \epsilon_{\ell_1} = p. \end{aligned}$$

By Theorem 2.1-(c), $I_h(\tau, f)$ satisfies

$$\begin{aligned} I_h(\tau+1, f) &= e((h,h)/2) I_h(\tau, f) \\ I_h(-1/\tau, f) &= \sqrt{\tau}^3 \frac{\sqrt{i}}{\sqrt{|L^\sharp/L|}} \sum_{h' \in L^\sharp/L} e(-(h,h')) I_{h'}(\tau, f) \end{aligned}$$

where

$$(h,h)/2 = q(h) = p \cdot \det \begin{pmatrix} h/2p & 0 \\ 0 & -h/2p \end{pmatrix} = -\frac{h^2}{4p}$$

$$-(h,h') = p \cdot \mathrm{tr}\left(\begin{pmatrix} h/2p & 0 \\ 0 & -h/2p \end{pmatrix} \begin{pmatrix} h'/2p & 0 \\ 0 & -h'/2p \end{pmatrix} \right) = \frac{hh'}{2p}.$$

Thus in the sense of Theorem 2.2 we are led to

LEMMA 3.1. *Let L be the cyclic lattice given in (8). Then $(I_h(\tau, f))_{h \in L^\sharp/L}$ can be interpreted as a Jacobi form.*

Now we need a lemma.

LEMMA 3.2. *For positive d,*

$$\boldsymbol{t}_f(h, d/4p) = 2\boldsymbol{t}_f^{(h)}(d).$$

Proof. If

$$X = \begin{pmatrix} b + h/2p & c/p \\ -a & -b - h/2p \end{pmatrix} \in L + h$$

is a vector of positive norm $d/4p$, then the matrix

$$Q = \begin{pmatrix} pa & pb + h/2 \\ pb + h/2 & c \end{pmatrix} = p \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X$$

defines a definite integral binary quadratic form of discriminant $-d = (2pb+h)^2 - 4pac = -4pq(X)$. Here the $\Gamma_0(p)$-action on $L+h$ corresponds to the natural right action on quadratic forms and the cycle D_X coincides with the CM point α_Q (resp. α_{-Q}) corresponding to Q (resp. $-Q$) $\in \mathcal{Q}_{d,p,h}$ if Q is positive (resp. negative) definite. We then easily see that

$$\mathbf{t}_f(h, d/4p) = 2 \sum_{Q \in \mathcal{Q}_{d,p,h}/\Gamma_0(p)} \frac{1}{|\Gamma_0(p)_Q|} f(\alpha_Q) = 2\mathbf{t}_f^{(h)}(d).$$

□

Let $X = \begin{pmatrix} b + h/2p & c/p \\ a & -b - h/2p \end{pmatrix} \in L_{h,-pk^2}$. It follows from $q(X) = -pk^2 = -p(b + h/2p)^2 - ac$ that $k \in \frac{1}{2p}\mathbb{Z}$. Thus we have $L_{h,-pk^2} = \emptyset$ unless $k \in \frac{1}{2p}\mathbb{Z}$ and so

$$\mathbf{t}_f(h, -pk^2) = 0 \text{ unless } k \in \frac{1}{2p}\mathbb{Z}.$$

We note that $a_{\ell_0}(-m) = a_{\ell_1}(-m/p) = 1$. By Theorem 2.1-(b) we then have

$$\mathbf{t}_f(h, -pk^2) = -\sum_{\ell\in\{\ell_0,\ell_1\}} \sharp(\Gamma_\ell \backslash L_{h,-pk^2,\ell}) \sum_{n\in \frac{2k}{\beta_\ell}\mathbb{Z}<0} a_\ell(n)e^{2\pi i r n}$$
$$(9) \qquad - \sum_{\ell\in\{\ell_0,\ell_1\}} \sharp(\Gamma_\ell \backslash L_{-h,-pk^2,\ell}) \sum_{n\in \frac{2k}{\beta_\ell}\mathbb{Z}<0} a_\ell(n)e^{2\pi i r' n}$$

Now we need two lemmas:

LEMMA 3.3. *For $k \in \frac{1}{2p}\mathbb{Z}_{>0}$, $L_{h,-pk^2,\ell_0}$ or $L_{h,-pk^2,\ell_1}$ is nonempty $\iff$ $h \equiv \pm 2pk \pmod{2p}$.*

Proof. ($\Rightarrow$) For $X = \begin{pmatrix} b+h/2p & c/p \\ a & -b-h/2p \end{pmatrix}$ to lie in $L_{h,-pk^2,\ell_0}$ (resp. $L_{h,-pk^2,\ell_1}$) we need $a = 0$ (resp. $c = 0$). It follows from $q(X) = -pk^2$ that $b + h/2p = \pm k$ which yields $h \equiv \pm 2pk \pmod{2p}$.
($\Leftarrow$) We write $h = \pm 2pk + 2p(-b)$ for some integer b. We then have $h/2p + b = \pm k$. If we set $X = \begin{pmatrix} b+h/2p & 0 \\ 0 & -b-h/2p \end{pmatrix}$, then X belongs to $L_{h,-pk^2,\ell_0}$ (resp. $L_{h,-pk^2,\ell_1}$) if $b + h/2p > 0$ (resp. $b + h/2p < 0$). □

LEMMA 3.4. *Let $k \in \frac{1}{2p}\mathbb{Z}_{>0}$ and $\ell \in \{\ell_0, \ell_1\}$. If both $L_{h,-pk^2,\ell}$ and $L_{-h,-pk^2,\ell}$ are nonempty, then $p \mid h$.*

Proof. First we consider the case $\ell = \ell_0$. Since both $L_{h,-pk^2,\ell_0}$ and $L_{-h,-pk^2,\ell_0}$ are nonempty, we can choose $X \in L_{h,-pk^2,\ell_0}$ and $X' \in L_{-h,-pk^2,\ell_0}$, which are of the form

$$X = \begin{pmatrix} k & c/p \\ 0 & -k \end{pmatrix} = \begin{pmatrix} b+h/2p & c/p \\ 0 & -(b+h/2p) \end{pmatrix}$$

and

$$X' = \begin{pmatrix} k & c'/p \\ 0 & -k \end{pmatrix} = \begin{pmatrix} b'-h/2p & c'/p \\ 0 & -(b'-h/2p) \end{pmatrix}$$

for some integers c, c', b, b'. Since $b + h/2p = b' - h/2p$, we obtain $h \equiv -h \pmod{2p}$ so that $p \mid h$. The proof for the case $\ell = \ell_1$ is similar. □

For $k \in \frac{1}{2p}\mathbb{Z}_{>0}$ we find that

$$(10) \qquad -m \in \frac{2k}{\beta_{\ell_0}}\mathbb{Z} = 2pk\mathbb{Z} \iff 2pk \mid m \quad \text{and} \quad -m/p \in \frac{2k}{\beta_{\ell_1}}\mathbb{Z} = 2k\mathbb{Z} \iff 2pk \mid m.$$

We define $\mu(h) = \begin{cases} 1, & \text{if } p \mid h; \\ 0, & \text{otherwise.} \end{cases}$ By Lemma 3.3, Lemma 3.4, (10), (3) and (9) we obtain that for $k \in \frac{1}{2p}\mathbb{Z}_{>0}$,

$$\mathbf{t}_f(h, -pk^2) = \begin{cases} -2^{\mu(h)}(2k\epsilon_{\ell_0} + 2k\epsilon_{\ell_1}) \sum_{2pk|m} a(-m) & \\ \qquad = -2^{\mu(h)} 4pk \sum_{2pk|m} a(-m), & \text{if } h \equiv \pm 2pk (2p); \\ 0, & \text{otherwise.} \end{cases}$$

We write $\kappa = 2pk \in \mathbb{Z}$. Then we have

$$(11) \quad \mathbf{t}_f(h, -\kappa^2/4p) = \begin{cases} -2^{\mu(h)} 2\kappa \sum_{\kappa|m} a(-m) & \\ \qquad = 2\mathbf{t}_f(-\kappa^2), & \text{if } h \equiv \pm\kappa \pmod{2p}; \\ 0, & \text{otherwise.} \end{cases}$$

As for the trace of zero index, it follows from [2, Remark 4.9 and (6.5)] that

$$(12) \qquad \mathbf{t}_f(h, 0) = 4\delta_{h,0} \sum_{n \in \mathbb{Z}_{\geq 0}} a(-n)(\sigma_1(n) + p\sigma_1(n/p)) = 2\delta_{h,0}\mathbf{t}_f(0).$$

Now Lemma 3.1, Lemma 3.2 together with (11), (12) prove Theorem 1.1 since

$$\sum_{n,r} \mathbf{t}_f(\bar{r}, \frac{4pn - r^2}{4p}) q^n \zeta^r$$

is a weak Jacobi form corresponding to $(I_h(\tau, f))_{h \in L^\sharp/L}$.

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