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DILATION OF PROJECTIVE ISOMETRIC REPRESENTATION ASSOCIATED WITH UNITARY MULTIPLIER

Man Kyu Im*, Un Cig Ji**, Young Yi Kim***, and Su Hyung Park****

Abstract. For a unital $*$-subalgebra of the space $\mathfrak{L}^a(X)$ of all adjointable maps on a Hilbert $\mathfrak{B}$-module X with a C^*-algebra $\mathfrak{B}$, we study unitary operator (in such algebra)–valued multiplier σ on a normal, generating subsemigroup S of a group G with its extension to G. A dilation of a projective isometric σ–representation of S is established as a projective unitary ρ–representation of G for a suitable unitary operator (in some algebra)–valued multiplier ρ associated with the multiplier σ which is explicitly constructed.

1. Introduction

In recent years, many authors have been studing the problem of extending (unit circle-valued) multipliers on a subsemigroup S of a group G to multipliers on G, see [1], [2], [3], [8] and the references cited therein. In particular, in [11], Murphy gave an elementary proof of the multiplier extension theorem in [8] and studied a number of other extension theorems. Also, the author proved a more general dilation theorem than that proved by Laca and Raeburn using Kolmogorov decompositions of positive definite kernels. For the study of dilation theory, we refer to [4], [7], [10]. In [6], we obtained extension theorems of unitary operator (in a von Neumann subalgebra) valued multiplier with suitable motivation and dilation theory.

In this paper, we study problems of extending multipliers and dilation theory obtained in [6] to the case of a $*$-subalgebra of the space

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of all adjointable maps on a Hilbert C^*-module. Recently, in [5], the author studied the dilations of projective $\sigma-$representations of a normal generating subsemigroup of a group G with unit circle-valued multiplier. Since Hilbert C^*-modules introduced by Kaplansky [12] is the generalizations of Hilbert space by allowing the inner product to take values in a C^*-algebra such study is natural. However, since several important properties held in the case of Hilbert space are not to be held in the case of Hilbert C^*-modules, it has to be handled carefully. For the more study, we refer to [9].

The paper is organized as follows: In Section 2 we study unitary operator (in a $*$-subalgebra of the space of all adjointable maps on Hilbert C^*-module)–valued multipliers and its extension theorem. In Section 3 we recall the Kolmogorov decompositions of positive definite kernels in [13] with connection to unitary operator–valued multipliers. In Section 4 we study the dilation theory.

2. Unitary multipliers

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be right Hilbert $\mathfrak{B}-$modules with a C^*-algebra $\mathfrak{B}$. We define $\mathcal{L}^a(\mathfrak{X}, \mathfrak{Y})$ to be the set of adjointable bounded linear maps from $\mathfrak{X}$ into $\mathfrak{Y}$. For the notational convenience, we denote $\mathcal{L}^a(\mathfrak{X}, \mathfrak{X})$ by $\mathcal{L}^a(\mathfrak{X})$. Then it is well-known that $\mathcal{L}^a(\mathfrak{X})$ is a $*-$algebra, in fact, C^*-algebra.

Let S denote a semigroup with unity, i.e., S is a set with an associative binary operation usually written multiplicatively and a unit element e. A subsemigroup S of a group G is said to be *normal*, if $xSx^{-1} \in S$ for any $x \in G$. If for all x in G, there exist $s, t \in S$ such that $x = s^{-1}t$, then we say that S *generates* G. A $*$*-semigroup* is a semigroup with an involution $* : S \ni s \mapsto s^* \in S$ such that $(st)^* = t^*s^*$ and $(s^*)^* = s$ for all $s, t \in S$.

Let $\mathfrak{B}$ be a C^*-algebra and $\mathfrak{X}$ a Hilbert $\mathfrak{B}$-module. For a unital $*-$subalgebra $\mathfrak{A}$ of $\mathcal{L}^a(\mathfrak{X})$, let $\mathfrak{A}'$ be the commutant of $\mathfrak{A}$ and $\mathfrak{A}'' = (\mathfrak{A}')'$ the double commutant of $\mathfrak{A}$.

The following definition of unitary operator valued multiplier is motivated by the definition in [6].

Definition 2.1. Let $\mathfrak{A}$ be a unital $*-$subalgebra of $\mathcal{L}^a(\mathfrak{X})$. The $\mathcal{U}(\mathfrak{A})-$*multiplier* on a semigroup S is a $\mathcal{U}(Z(\mathfrak{A}))$-valued map defined on $S \times S$ satisfying that

(i) $\sigma(s, e) = \sigma(e, s) = 1$ for any $s \in S$;
(ii) $\sigma(s, t)\sigma(st, u) = \sigma(s, tu)\sigma(t, u)$ for any $s, t, u \in S$,

where $Z(\mathfrak{A}) \equiv \mathfrak{A} \cap \mathfrak{A}'$ is the center of $\mathfrak{A}$ and $\mathcal{U}(\mathfrak{C})$ the group of all unitary operators in a $*-$subalgebra $\mathfrak{C}$.

Definition 2.2. Let σ be a $\mathcal{U}(\mathfrak{A})$-multiplier on a semigroup S. A map $V : S \ni s \mapsto V_s \in \mathfrak{A}$ is a *projective isometric σ-representation* of S if for any $s, t \in S$

(i) V_s is an isometry and $V_e = 1$;
(ii) $V_{st} = \sigma(s,t)V_sV_t$.

If V_s is unitary for all $s \in S$, we say that V is a *projective unitary σ-representation of* S. It is well–known that V is a projective isometric σ-representation of a group S, then V is automatically a projective unitary σ-representation, in fact, $V_s^* = \sigma(s^{-1}, s)V_{s^{-1}}$ for all $s \in S$.

Remark 2.3. Let S be a semigroup and let σ a $\mathcal{U}(\mathfrak{A}')$-valued map defined on $S \times S$. If a map $V : S \to \mathfrak{A}$ satisfies conditions (i) and (ii) in Definition 2.2, then σ becomes a $\mathcal{U}(Z(\mathfrak{A}))$-valued map. In fact, for any $s, t \in S$, $\sigma(s,t) = V_t^*V_s^*V_{st} \in \mathfrak{A}$.

Theorem 2.4. *Let S be a normal, generating subsemigroup of a group G and let σ a $\mathcal{U}(\mathfrak{A})$-multiplier on S. Then σ can be extended to G as a $\mathcal{U}(\mathfrak{A})$-multiplier.*

Proof. The proof is a simple modification of the proof of Theorem 2.8 in [6]. □

3. Kolmogorov decomposition and unitary multiplier

Let G be a (non-empty) set. A map $K : G \times G \longrightarrow \mathcal{L}^a(\mathfrak{X})$ is called a *kernel* and the set of all such kernels is denoted by $\mathcal{K}(G; \mathcal{L}^a(\mathfrak{X}))$. A kernel $K \in \mathcal{K}(G; \mathcal{L}^a(\mathfrak{X}))$ is said to be *positive definite* if for any positive integer $n \in \mathbb{N}$ and $g_1, \cdots, g_n$ in G, the matrix $(K(g_i, g_j))_{1 \leq i,j \leq n}$ is positive. Let $\mathfrak{X}'$ be another Hilbert $\mathfrak{B}$-module and let V a map from G into $\mathcal{L}^a(\mathfrak{X}, \mathfrak{X}')$. Put

$$K(g,h) = V(g)^*V(h), \qquad g, h \in \mathfrak{X}. \tag{3.1}$$

Then K is positive definite.

Let $K \in \mathcal{K}(G; \mathcal{L}^a(\mathfrak{X}))$. If there exists a Hilbert $\mathfrak{B}$-module $\mathfrak{X}'$, denoted by $\mathfrak{X}_V$, and a map $V : G \to \mathcal{L}^a(\mathfrak{X}, \mathfrak{X}_V)$ such that (3.1) holds, then V is called a *Kolmogorov decomposition* of K. If

$$\mathfrak{X}_V = \overline{LS\{V(g)x \mid g \in G, x \in \mathfrak{X}\}}, \tag{3.2}$$

then V is said to be *minimal*. The right hand side of (3.2) is the closure of the linear span of $\{V(g)x \mid g \in G, x \in \mathfrak{X}\}$. Two Kolmogorv decomposition V and V' are said to be *equivalent* if there is a unitary mapping $U : \mathfrak{X}_V \to \mathfrak{X}_{V'}$ such that $V'(g) = UV(g)$ for any $g \in G$.

THEOREM 3.1 ([13]). *Let $K \in \mathcal{K}(G; \mathcal{L}^a(\mathfrak{X}))$. The kernel has a Kolmogorov decomposition if and only if it is a positive definite kernel.*

THEOREM 3.2. *Let $K : G \times G \to \mathfrak{A}$ be a positive definite kernel and V a minimal Kolmogorov decomposition of K. Then there exists a *-homomorphism $\Phi : \mathcal{U}(\mathfrak{A}') \to \mathcal{L}^a(\mathfrak{X}_V)$ such that for any $g \in G$,*

$$V(g)T = \Phi(T)V(g), \qquad T \in \mathcal{U}(\mathfrak{A}').$$

Moreover, for each $T \in \mathcal{U}(\mathfrak{A}')$, $\Phi(T)$ is unitary on $\mathfrak{X}_V$.

Proof. For the complete proof we refer to the proof of Theorem 3.2 in [6]. Now we sketch the proof. For each $T \in \mathcal{U}(\mathfrak{A}')$, we define a unitary $\tilde{T}$ by

$$\tilde{T}(V(g)x) = V(g)Tx, \qquad g \in G, \quad x \in \mathfrak{X}.$$

Then by direct computation we see that for any $T, S \in \mathcal{U}(\mathfrak{A}')$, $g, h \in G$ and $x, y \in \mathfrak{X}$

$$\widetilde{TS}(V(g)x) = \tilde{T}\tilde{S}(V(g)x)$$

and

$$\langle \widetilde{T^*}(V(g)x), V(h)y \rangle_{\mathfrak{X}_V} = \langle \tilde{T}^* V(g)x, V(h)y \rangle_{\mathfrak{X}_V}.$$

Hence $\widetilde{TS} = \tilde{T}\tilde{S}$ and $\widetilde{T^*} = \tilde{T}^*$ for any $T, S \in \mathcal{U}(\mathfrak{A}')$. Now, define a *-homomorphism Φ from $\mathcal{U}(\mathfrak{A}')$ into $\mathcal{L}^a(\mathfrak{X}_V)$ by $\Phi(T) = \tilde{T}$ for $T \in \mathcal{U}(\mathfrak{A}')$. □

THEOREM 3.3. *Let S be a semigroup and Φ the *-homomorphism given as in Theorem 3.2. For each $\mathcal{U}(\mathfrak{A})-$multiplier σ on S, $\Phi(\sigma)$ is a $\mathcal{U}(\mathfrak{N})$-multiplier on S, where $\mathfrak{N}$ is the unital $*$-algebra generated by $\Phi(\mathcal{U}(Z(\mathfrak{A})))$ and $\Phi(\sigma)(s,t) = \Phi(\sigma(s,t))$ for any $s, t \in S$.*

Proof. The proof is immediate since Φ is a $*$-homomorphism and σ is a $\mathcal{U}(\mathfrak{A})$–multiplier. □

REMARK 3.4. Since $ST = TS$ for any $S, T \in \Phi(\mathcal{U}(Z(\mathfrak{A})))$, the unital $*$-algebra $\mathfrak{N}$ generated by $\Phi(\mathcal{U}(Z(\mathfrak{A})))$ is commutative.

4. Dilation theory

Let S be a normal, generating subsemigroup of a group G. Let $\mathfrak{A}$ be a $*-$subalgebra of $\mathcal{L}^a(\mathfrak{X})$ and σ a $\mathcal{U}(\mathfrak{A})-$multiplier on G. A map $W : G \to \mathfrak{A}$ is *σ-positive definite* if the map K on $G \times G$ defined by setting $K(g,h) = \sigma(g^{-1}, g)\sigma(g^{-1}, h)^* W_{g^{-1}h}$ is positive definite. We define a *(minimal) Kolmogorov decomposition* for W to be a (minimal) Kolmogorov decomposition for K.

Theorem 4.1. *Let σ be a $\mathcal{U}(\mathfrak{A})-$multiplier and $W : G \to \mathfrak{A}$ a σ-positive definite map. There exist a Hilbert $\mathfrak{B}$-module $\mathfrak{X}'$, an operator $T \in \mathcal{L}^a(\mathfrak{X}, \mathfrak{X}')$ and a unitary $\Phi(\sigma)$-representation U of G on $\mathfrak{X}'$ such that $W_g = T^* U_g T$ for any $g \in G$, where Φ is the $*$-homomorphism given as in Theorem 3.2. Moreover, $\mathfrak{X}'$ is the closed linear span of the set $\bigcup_{g\in G} U_g T(\mathfrak{X})$.*

Proof. The proof is same as the proof of Theorem 4.1 in [6]. □

The projective unitary $\Phi(\sigma)$-representation U is called a *dilation* of W. If W is unital, i.e., $W_e = 1$, then $T^*T = T^*U_eT = W_e = 1$. Thus T is an isometry.

The following corollary is immediate by applying Theorem 4.1 with the unital $*-$algebra $\mathfrak{A} = \mathcal{L}^a(\mathfrak{X})$.

Corollary 4.2 ([5]). *Let G be a group and let σ be a unit circle valued multiplier of G. If a map $W : G \to \mathcal{L}^a(\mathfrak{X})$ is $\sigma-$positive definite, then there exist a Hilbert $\mathfrak{B}-$module $\mathfrak{X}'$, $T \in \mathcal{L}^a(\mathfrak{X}, \mathfrak{X}')$ and unitary $\sigma-$representation U of G on $\mathfrak{X}'$ such that $W_g = T^*U_gT$ for all $g \in G$. Moreover, $\mathfrak{X}'$ is the closed linear span of $\bigcup_{g\in G} U_g T\mathfrak{X}$.*

Theorem 4.3. *Let σ be a $\mathcal{U}(\mathfrak{A})$-multiplier on G and let $W : S \to \mathfrak{A}$ a projective isometric σ-representation with $\mathcal{U}(\mathfrak{A})$-multiplier σ on S. Then there exists a unique extension $\overline{W}$ of W to G such that*

(i) $\overline{W}_{gs} = \sigma(g,s)\overline{W}_g W_s$ for any $g \in G$ and $s \in S$;

(ii) $\overline{W}^*_g = \sigma(g^{-1}, g)\overline{W}_{g^{-1}}$ for any $g \in G$.

Moreover, $\overline{W}$ is $\sigma-$positive definite.

Proof. For the proof, we refer to the proof of Theorem 4.4 in [6]. □

Theorem 4.4. *Let σ be a $\mathcal{U}(\mathfrak{A})$-multiplier on G and let $W : S \to \mathfrak{A}$ a projective isometric σ-representation with $\mathcal{U}(\mathfrak{A})$-multiplier σ on S. Then there exist a Hilbert $\mathfrak{B}$-module $\mathfrak{X}'$, an isometry $T \in \mathcal{L}^a(\mathfrak{X}, \mathfrak{X}')$ and* *a unitary $\Phi(\sigma)-$representation U such that $W_s = T^* U_s T$ for all $s \in S$. Moreover, $\mathfrak{X}'$ is the closed linear span of the set $\bigcup_{g \in G} U_g T(\mathfrak{X})$.*

Proof. By Theorem 4.3, the extension $\overline{W}$ of W to G is σ-positive definite. Therefore, by applying Theorem 4.1, the proof is immediate. □

The following corollary is immediate by applying Theorem 4.1 with the unital $*-$algebra $\mathfrak{A} = \mathcal{L}^a(\mathfrak{X})$.

COROLLARY 4.5 ([5]). *Let S be a normal generating subsemigroup of a group G and σ be a unit circle valued multiplier of G. If $W : S \to \mathcal{L}^a(\mathfrak{X})$ is a projective isometric representation with the restriction of σ to S as the associated multiplier, then there exist a Hilbert $\mathfrak{B}-$module $\mathfrak{X}'$, an isometry $T \in \mathcal{L}^a(\mathfrak{X}, \mathfrak{X}')$ and unitary $\sigma-$representation U of G on $\mathfrak{X}'$ such that $W_s = T^* U_s T$ for all $s \in S$. Moreover, $\mathfrak{X}'$ is the closed linear span of $\cup_{g \in G} U_g T\mathfrak{X}$.*

Proof. By applying Theorem 4.4 with the unital $*-$algebra $\mathfrak{A} = \mathcal{L}^a(\mathfrak{X})$, the proof is immediate. □

REMARK 4.6. *If C^*-algebra $\mathfrak{B}$ is given by $\mathbf{C}$, then a Hilbert $\mathfrak{B}-$module $\mathfrak{X}$ becomes a Hilbert space and it can easily checked that $\mathcal{L}^a(\mathfrak{X})$ is $\mathfrak{B}(\mathfrak{X})$. Therefore, extension theorem of unitary operator valued multiplier and dilation theorem of projective isometric representation in* [6] *can be considered as a special case of those in this paper.*

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*
Department of Mathematics
Hannam University
Taejon 306-791, Republic of Korea
E-mail: mki@hnu.kr

**
Department of Mathematics
Research Institute of Mathematical Finance
Chungbuk National University
Cheongju 361-763, Republic of Korea
E-mail: uncigji@cbucc.chungbuk.ac.kr

***
Department of Mathematics
Chungbuk National University
Cheongju 361-763, Republic of Korea
E-mail: kimyy@chungbuk.ac.kr

****
Department of Mathematics
Chungbuk National University
Cheongju 361-763, Republic of Korea
E-mail: shpark05@chungbuk.ac.kr