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COEFFICIENTS OF UNIVALENT HARMONIC MAPPINGS

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Abstract. In this paper, we obtain some coefficient bounds of harmonic univalent mappings by using properties of the analytic univalent function on $\Delta = \{z : |z| > 1\}$.

1. Introduction

Let Σ be the class of all complex-valued, harmonic, orientation-preserving, univalent mappings, which are normalized at infinity by $f(\infty) = \infty$,

$$f(z) = h(z) + \overline{g(z)} + Alog|z| \tag{1.1}$$

of $\Delta = \{z : |z| > 1\}$, where

$$h(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in Δ and $A \in \mathbb{C}$. The mapping f can be viewed as a solution of the partial differential equation $\overline{f_{\bar{z}}} = af_z$ where the function a is analytic in Δ and satisfies $|a(z)| < 1$.[3]

The coefficient problem for this class appears to be difficult. In the full class Σ, a few estimates are known only for lower order coefficients: $|A| \leq 2$ and $|b_1| \leq 1$ hold for the full class Σ, and $|b_2| \leq \frac{1}{2}(1 - |b_1|^2) \leq \frac{1}{2}$ holds if $A = 0$. These coefficient bounds [3] are all sharp and a consequence of Schwarz's lemma. If we restrict our attention to some subclass of Σ, we can obtain good results; for $f \in \Sigma$ with $f(\Delta) = \Delta$, $|1 + b_1| \leq 1$, $|b_n| \leq \frac{1}{n}$ for $n \geq 2$, and $|a_n| \leq \frac{1}{n}$ for all n. These sharp coefficient bounds are

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obtained by Jun[4]. Therefore, in this paper, we shall consider the subclass $\Sigma_R = \{f \in \Sigma : f \text{ is convex in the real axis}\}$ of Σ and get some coefficient estimates by using properties of the analytic univalent function defined in Δ.

2. Mappings which are convex in the real axis

DEFINITION 2.1. A set D is called convex in the real axis if every line parallel to the real axis has a connected intersection with D.

DEFINITION 2.2. A mapping f is convex in the real axis if $f(\Delta)$ is convex in the real axis.

Before getting into the main subject, we need to mention the area theorem which is fundamental to the theory of univalent functions. The name of this theorem comes from the proof.

THEOREM 2.3(AREA THEOREM). ([2]) *If $H(z) = z + \sum_{n=0}^{\infty} c_n z^{-n}$ is analytic and univalent in Δ, then $\sum_{n=1}^{\infty} n|c_n|^2 \leq 1$.*

Let Σ_R be the class of all mappings $f \in \Sigma$ which is convex in the real axis.

THEOREM 2.4. *If $f \in \Sigma_R$ with real A, then $h - g$ is conformal univalent in Δ.*

Proof. Since f is univalent, there exists a mapping $z = z(w)$ such that $f(z(w)) = w$ and $z(f(z)) = z$. Thus we have $h - g = f - Alog|z| - 2Re\{g\}$ and $h(z(w)) - g(z(w)) = w + \phi(w)$ where $\phi(w) = -Alog|z(w)| - 2Re\{g(z(w))\}$ is a continuous real valued function. Since $a(z) = \frac{2zg'(z)+A}{2zh'(z)+A}$ satisfies $|a(z)| < 1$, we have $h'(z) - g'(z) \neq 0$ in Δ. Thus the mapping $h(z(w)) - g(z(w)) = w + \phi(w)$ is locally univalent since $z(w)$ is 1-1. If $w_1 + \phi(w_1) = w_2 + \phi(w_2)$ with $w_1 \neq w_2 (w_1 = u_1 + iv_1, w_2 = u_2 + iv_2)$, then $v_1 = v_2 = v$ and $u_1 + \phi(u_1 + iv) = u_2 + \phi(u_2 + iv)$. The real valued function $\psi(u) = u + \phi(u + iv)$, which is defined on some interval I since f is convex in the real axis, is not strictly monotonic and therefore not locally 1-1. Thus $w + \phi(w) = h - g$ is 1-1 and so conformal univalent. $\square$

Sharp coefficient bounds of the analytic univalent function $H(z) = z + \sum_{n=0}^{\infty} c_n z^{-n}$ in Δ are known only for $1 \leq n \leq 3$: $|c_1| \leq 1$ [2], $|c_2| \leq \frac{2}{3}$ [5], $|c_3| \leq \frac{1}{2} + e^{-6}$ [1]. From these, we can easily get the lower order coefficient bounds for the harmonic univalent mapping $f \in \Sigma_R$ with real A as follows;

$$|a_1 - b_1| \leq 1,\ |a_2 - b_2| \leq \frac{2}{3},\ |a_3 - b_3| \leq \frac{1}{2} + e^{-6}.$$

In the following Corollary 2.5, we obtain the coefficient bounds for all orders.

COROLLARY 2.5. *If $f \in \Sigma_R$ with real A, then*

$$\sum_{n=1}^{\infty} n|a_n - b_n|^2 \leq 1 \text{ and } |a_n - b_n| \leq \frac{1}{\sqrt{n}}.$$

Proof. $f \in \Sigma_R$ with real A implies that $h - g = z + \sum_{k=1}^{\infty}(a_k - b_k)z^{-k}$ is a univalent analytic function in Δ. Thus we get $|a_1 - b_1| \leq 1$ from [2] and $\sum_{n=1}^{\infty} n|a_n - b_n|^2 \leq 1$ from the area theorem. $n|a_n - b_n|^2 \leq 1 - |a_1 - b_1|^2 \leq 1$ for $n \geq 2$ and so $|a_n - b_n| \leq \frac{1}{\sqrt{n}}$. □

COROLLARY 2.6. *If $f \in \Sigma_R$ with real A and $Re\{a_1 - b_1\} \leq \frac{nt^2-1}{nt^2+1}$ for $t > 0$, then*

$$Re\{t(a_1 - b_1) - (a_n - b_n)\} \leq t \text{ for } n \geq 2.$$

Proof. In the proof of Corollary 2.5, we know that $n|a_n - b_n|^2 \leq 1 - |a_1 - b_1|^2 \leq 1$ for $n \geq 2$ and so $|a_n - b_n| \leq \frac{\sqrt{1-|a_1-b_1|^2}}{\sqrt{n}}$. Hence

$$\begin{aligned} Re\{t(a_1 - b_1) - (a_n - b_n)\} &\leq tRe\{a_1 - b_1\} + \frac{1}{\sqrt{n}}\sqrt{1 - |a_1 - b_1|^2} \\ &\leq tRe\{a_1 - b_1\} + \frac{1}{\sqrt{n}}\sqrt{1 - (Re\{a_1 - b_1\})^2}. \end{aligned}$$

Let $x = Re\{a_1 - b_1\}$, then $Re\{t(a_1 - b_1) - (a_n - b_n)\} \leq tx + \frac{1}{\sqrt{n}}\sqrt{1 - x^2}$. The function $G(x) = tx + \frac{1}{\sqrt{n}}\sqrt{1 - x^2}$ is increasing for $-1 \leq x \leq \frac{nt^2-1}{nt^2+1}$ and therefore $Re\{t(a_1 - b_1) - (a_n - b_n)\} \leq t$ for $n \geq 2$. □

Corollary 2.7. *If $f \in \Sigma_R$ with real A and $Re\{a_1 - b_1\} \leq \frac{n^3-1}{n^3+1}$, then*

$$Re\{n(a_1 - b_1) - (a_n - b_n)\} \leq n.$$

Proof. Set $t = n$ in Corollary 2.6. □

Corollary 2.8. *If $f \in \Sigma_R$ with real A, then $Re\{n(a_1 - b_1) - (a_n - b_n)\} \leq n$ for all n sufficiently large depending on f.*

Proof. Fix f. If $Re\{a_1 - b_1\} = 1$, then $a_n - b_n = 0$ for all $n \geq 2$ by the area theorem and the result holds for all $n \geq 2$. If $Re\{a_1 - b_1\} < 1$, then $Re\{a_1 - b_1\} \leq \frac{nn^2-1}{nn^2+1}$ for all n sufficiently large since $(n^3 - 1)/(n^3 + 1) \to 1$ as $n \to \infty$. In this case the result follows from Corollary 2.6. □

A function H is said to be typically real if $H(z)$ is real if, and only if, z is real.

Theorem 2.9. *Let $f \in \Sigma_R$. If A and coefficients are real, then $h - g$ is typically real in Δ.*

Proof. For $\overline{f(z)} = f(\bar{z})$ and so $\overline{f(z)} = f(z)$ if, and only if, $z = \bar{z}$ because of the univalence. $\overline{(h-g)(z)} = (h-g)(z)$ if, and only if, $\overline{f(z)} = f(z)$. These imply that $\overline{(h-g)(z)} = (h-g)(z)$ if, and only if, $z = \bar{z}$. Thus $h - g$ is typically real in Δ. □

Corollary 2.10. *Let $f \in \Sigma_R$. If A and coefficients are real, and $h - g \neq 0$ for all $z \in \Delta$, then*

$$||c_1|^2 - |c_3|| \leq 5, \ ||2c_1c_2| - |c_4|| \leq 6,$$

$$||2c_1c_3 + c_2^2 - c_1^3| - |c_5|| \leq 7,$$

$$||2c_2c_3 + 2c_1c_4 - 3c_1^2c_2| - |c_6|| \leq 8,$$

where $c_n = a_n - b_n$.

Proof. Let $(h - g)(z) = z + \sum_{n=1}^{\infty} c_n z^{-n}$, where $c_n = a_n - b_n$. The function G defined by $G(\zeta) = \{(h - g)(1/\zeta)\}^{-1}$ is analytic in the unit disk

$\mathbb{D} = \{\zeta : |\zeta| < 1\}$ since $h - g \neq 0$ for all $z \in \Delta$. $\overline{G(\zeta)} = G(\zeta)$ if, and only if, $\overline{(h-g)(1/\zeta)} = (h-g)(1/\zeta)$. $\overline{(h-g)(1/\zeta)} = (h-g)(1/\zeta)$ if, and only if, $\zeta = \bar{\zeta}$ since $h - g$ is typically real in Δ. Thus G is analytic and typically real in the unit disk $\mathbb{D}$. Let $G(\zeta) = \zeta + \sum_{n=2}^{\infty} s_n \zeta^n$, then $|s_n| \leq n$ by the known coefficient bound for analytic typically real functions in the unit disk $\mathbb{D}$. Therefore the results are obtained from

$$\begin{aligned} G(\zeta) &= \zeta - c_1\zeta^3 - c_2\zeta^4 + (c_1^2 - c_3)\zeta^5 + (2c_1c_2 - c_4)\zeta^6 \\ &\quad + (2c_1c_3 - c_5 + c_2^2 - c_1^3)\zeta^7 + (2c_2c_3 + 2c_1c_4 - c_6 - 3c_1^2c_2)\zeta^8 + \cdots . \end{aligned}$$

□

THEOREM 2.11. *Let $f \in \Sigma_R$ with real A. $h - g$ is typically real if and only if f is typically real.*

Proof. The result holds because of $Im\{f\} = Im\{h - g\}$. □

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