The Estimation of the Coverage Probability in a Redundant System with a Control Module

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Abstract The concept of the coverage has been played an important role in the area of reliability evaluation of a system. The widely used measures of reliability include the mean time between failures, the availability and so on. In this paper, we propose an estimator of the coverage probability in a redundant system with a control unit and investigate some moments of the proposed estimator. And, assuming exponential distribution of all units, we conduct a simulation study for calculating the estimates of the coverage probability and its confidence bounds. An example of evaluating the availability of an optical transportation system is illustrated.

Key Words : Coverage Probability, Redundant System, mean time between failures, Availability

I. Introduction

The systems with very high reliability requirements are generally designed to use of redundancy for critical units. For example, most of electronic switching systems contain two central processors - one active and one standby. [1] [2] The reliability model of a redundant system, for calculating the mean time between failures(hereafter, MTBF), the availability calculation, or other reliability measures, incorporates a parameter called coverage to reflect the error recovery ability of the system. The coverage is defined as follows:

(1) The proportion of faults from which a system automatically recovers. [3]

(2) The conditional probability of successful error recovery given that an error has occurred. [4]

Many authors have studied the concept of the coverage since Arnold[3] studied the effect of coverage probability on the MTFF(Mean Time to First Failure) and the expected downtime in a repairable system. They are Dugan and Trivedi[5], Pham[6], and Doyle & Dugan[7] among others.

The importance of the coverage parameter has been demonstrated by Arnold[3], Bouricius et. al.[4] and Dugan and Trivedi[5], among others. Dugan and Trivedi[5] define the MTIF(Mean Time Improvement Factor) for capturing the effect of the coverage on MTTF of a system.

Several methods for calculating the coverage probability have been proposed in Amer and McClusky[8], Constantinescu[9] and Powell et. al.[10].

One of the most widely used methods for evaluating the reliability, the availability and the MTBF of a system is Markov model, a probability model that can account for many simultaneous active event processes in a system. <Figure 1> shows a Markov model for availability calculation of a duplicated repairable

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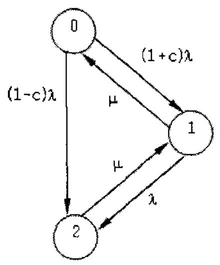
system with a concept of coverage. It is noted that the state 2 includes the uncovered outage .

In <Figure 1>, λ and μ represent the failure rate and the repair rate, respectively, and c denotes the coverage probability. Assuming the independence and exponentiality of the life time and the repair time, we obtain the availability, the steady state probability of $P_0(t)+P_1(t)$, as follows:

$$\frac{\mu^2+2\lambda\mu}{(\lambda+\mu)(2\lambda+\mu)-c\lambda\mu}$$



State	Description
0	System is normally operating
1	One unit fails but system is still operating
2	System fails

<Figure 1> State Transition Diagram for Availability Calculation

<Table 1> shows the effect of increasing coverage on the system availability for various values of c and the ratio of the repair rate to the failure rate. It is clear that the effect of coverage probability is predominant. Since, in evaluating the availability of a system, the value of the coverage probability is usually unknown, the knowledge of the coverage probability is much of interest to the system analyst.

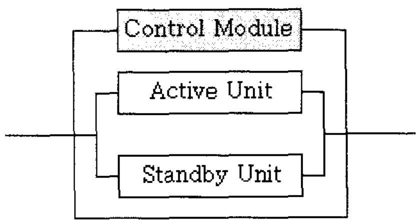
In this paper, we consider a problem of estimating the coverage probability in a redundant system with a control module. In Section 2, we describe a reference model of a redundant system with a control module and propose an estimator of the coverage probability. Section 3 is devoted to investigate the first and the second moments of the proposed estimator. In Section 4, we conduct a simulation study for calculating the coverage probability for the various values of the failure rate and the repair rate. An example of evaluating the system availability is illustrated in Section 5.

<Table 1> Availability of two-components redundant system

	Ratio of repair rate to failure rate		
c	5,000	10,000	20,000
0.90	0.999980	0.999990	0.999995
0.92	0.999984	0.999992	0.999996
0.94	0.999988	0.999994	0.999997
0.96	0.999992	0.999996	0.999998
0.98	0.999996	0.999998	0.999999

2. The Estimation of the Coverage probability

<Figure 2> shows a redundant system with a control module, which is a widely adapted structure in the design of a reliable system. This system consists of three units which are an active unit, a hot standby unit and a control unit. Initially the active unit conducts the major mission of the system.



<Figure 2> Reference Model of a redundant system with a Control Module.

And the standby unit is switched to perform the required mission when the active unit fails. This switchover processing is controlled by the control module, which monitors the state of the active unit and makes the standby unit perform the mission as soon as the active unit fails. (That is called the recovery from the fault.) The system returns to the initial state after the failed active unit is repaired. We assume the following statements.

(i) all units are independently operating,
(ii) the switchover processing is successful as far as the control unit is normally operating, and
(iii) it takes negligible time to conduct switchover process.

Under the assumptions mentioned, the system recovers from a fault if the control unit is normally operating at the moment of the failure of the active unit. In other words, the system crashes if the active unit fails while the control unit is in the failure state.

On the basis of the definition of the coverage probability in Section 1, we propose a naive estimator of the coverage probability as follows:

$$\hat{c} = \frac{1}{n}\sum_{i=1}^{n} D_i$$

where D_i's are dichotomous random variables having the following values.

$$D_i = \begin{cases} 1 & \text{if a successful switchover occurs at the i the failure of the active unit} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots, n$.

3. Some Moments of the Estimator

In this section, we use the following notations.

Notation	Desription
$\{X_i\}_{i=1}^{\infty}$	The sequence of i.i.d. random variables from a distribution F, where X_i represents the i-th life length of the active unit.
$\{Y_i\}_{i=1}^{\infty}$	The sequence of i.i.d. random variables from a distribution G, where Y_i represents the i-th life length of the control unit.
$\{Z_{ij}\}_{j=1}^{\infty}$ $i = 1,2$	The sequence of i.i.d. random variables from a distribution H, where Z_{ij} and Z_{2j} represent the repair times for the j-th failures of the active unit and the control unit, respectively.
$F^{(n)}, G^{(n)}, H^{(n)}$	The n-th convolution of F, G, and H, respectively.
$F \otimes G$	The convolution of F and G.

Let $S_i = S_{i-1} + (X_i + Z_{1,i-1})$ and $T_i = t_{i-1} + (Y_i + Z_{2,i-1})$ for i=1,2,... where $S_0 = T_0 = Z_{1,0} = Z_{2,0} = 0$. Then we have the following Lemma.

Lemma 1. Let $S_i = S_{i-1} + (X_i + Z_{1,i-1})$ and $T_i = t_{i-1} + (Y_i + Z_{2,i-1})$ for i=1,2,.... Then $P[S_i < t] = F^{(i)} \otimes H^{(i-1)}(t)$ and $P[T_i < t] = G^{(i)} \otimes H^{(i-1)}(t)$.

Proof. It is straightforward from the fact that Si is the sum of the independent random variables.

Lemma 2. For any j, the probability that the i-th failure of the active unit occurs in the j-th repair period of the control unit is given by

$$\iint \{F^{(i)} \otimes H^{(i-1)}(y+z) - F^{(i)} \otimes H^{(i-1)}(y)\} \, dG^{(j)} \otimes H^{(j-1)}(y) dH(z)$$

Proof.

It is easy to prove from Lemma 1 and the convolution theory that

$$\begin{aligned} P[S_i \in (T_j, T_j + Z_{2,j})] &= P[T_j < S_i < T_j + Z_{2,j}] \\ &= P[0 < S_i - T_j < Z_{2,j}] = \iint P[0 < S_i - T_j < z] dH(z) \\ &= \iint P[y < S_i < z + y] dG^{(j)} \otimes H^{(j-1)}(y) dH(z) \\ &= \iint \{F^{(i)} \otimes H^{(i-1)}(y+z) - F^{(i)} \otimes H^{(i-1)}(y)\} \\ &\quad dG^{(j)} \otimes H^{(j-1)}(y) dH(z). \end{aligned}$$

We note that, for all j, S_j belongs to at most one interval $(T_j, T_j + Z_{2,j})$ since the i-th failure of the active unit occurs only once either in the failure state of the control unit or in the normal state. Therefore, for $i \neq k$, two events $\{w | S_i(w) \in (T_j, T_j + Z_{2,j})\}$ and $\{w | S_i(w) \in (T_k, T_k + Z_{2,j})\}$ are mutually exclusive.

From Lemma 1 and Lemma 2, we have the following theorem for the expectation of D_i.

$$q_i = \sum_{j=1}^{\infty} \iint \{F^{(i)} \otimes H^{(i-1)}(y+z) - F^{(i)} \otimes H^{(i-1)}(y)\} \, dG^{(j)} \otimes H^{(j-1)}(y) dH(z).$$

Theorem 1. Let $q_i = P[D_i = 0]$. Then $E[D_i] = 1 - q_i$ and $Var[D_i] = q_i(1 - q_i)$, where

Proof. Since Di is a dichotomous random variable, it is clear that $E[D_i] = P[D_i = 1] = 1 - q_i$, $E[D_i^2] = E[D_i]$ and $Var[D_i] = E[D_i]\{1 - E[D_i]\}$. And

$$\begin{aligned} &P[D_i = 0] \\ &= P[S_i \in (T_j, T_j + Z_{2,j}) \text{ for any } j] \\ &= P[\bigcup_{j=1}^{\infty} \{S_i \in (T_j, T_j + Z_{2,j})\}] \\ &= \sum_{j=1}^{\infty} \iint \{F^{(i)} \otimes H^{(i-1)}(y+z) - F^{(i)} \otimes H^{(i-1)}(y)\} \\ &\quad dG^{(j)} \otimes H^{(j-1)}(y) dH(z). \end{aligned}$$

The last equality holds by mutually exclusiveness and Lemma 2.

Corollary. The expectation and the variance of the estimator of the coverage probability is given by

$$E[\hat{c}] = \frac{1}{n} \sum_{i=1}^{n} (1 - q_i) \text{ and } Var[\hat{c}] = \frac{1}{n^2} \sum_{i=1}^{n} q_i (1 - q_i).$$

4. The Simulation Study

Since it is analytically difficult to calculate the moments of the estimator for a given distribution, we conduct a simulation study for obtaining the estimates and confidence bounds of the coverage probability. We assume that the life times of all units are exponentially distributed. The failure rates(λ) of the active units considered vary from 10,000 Fits to 50,000 Fits increasing by 10,000 Fits. And so do the failure rates(ξ) of the control unit. The repair times of both units are assumed to be exponentially distributed with the mean of 2 hours. The unit of the failure rate is given by Fit, which is widely used to represent the failure intensity in the area of telecommunication system. 1 Fits is defined to be one failure per hours.

We generate two sets of random numbers of size 1,000 from an exponential distribution with the mean of $1/\lambda$ and with the mean of 2 hours,

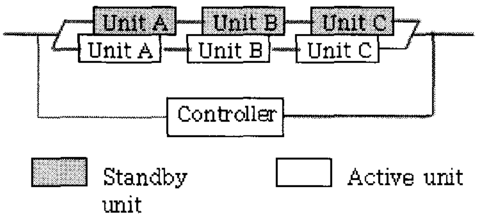
<Table 2> The Simulated Values of the Coverage Probability and Its 95% Confidence Interval

The Failure Rate of the Active Unit	The Failure Rate of the Control Unit (Fits)	The Coverage Probability	Lower Bound	Upper Bound
10000 Fits	10000	0.999942	0.9999272	0.9999568
	20000	0.999896	0.9998763	0.9999157
	30000	0.999852	0.9998286	0.9998754
	40000	0.999788	0.9997605	0.9998155
	50000	0.997040	0.9963670	0.9977130
20000 Fits	10000	0.999965	0.9999533	0.9999767
	20000	0.999936	0.9999208	0.9999512
	30000	0.999886	0.9998652	0.9999068
	40000	0.999869	0.9998461	0.9998919
	50000	0.999825	0.9997996	0.9998504
30000 Fits	10000	0.999974	0.9999641	0.9999839
	20000	0.999956	0.9999430	0.9999690
	30000	0.999920	0.9999029	0.9999371
	40000	0.999907	0.9998874	0.9999266
	50000	0.999849	0.9998246	0.9998734
40000 Fits	10000	0.999973	0.9999626	0.9999834
	20000	0.999952	0.9999376	0.9999664
	30000	0.999910	0.9998914	0.9999286
	40000	0.999904	0.9998853	0.9999227
	50000	0.999846	0.9998223	0.9998697
50000 Fits	10000	0.999974	0.9999641	0.9999839
	20000	0.999957	0.9999441	0.9999699
	30000	0.999920	0.9999025	0.9999375
	40000	0.999896	0.9998750	0.9999170
	50000	0.999866	0.9998425	0.9998895

respectively. (That is, 1,000 failures and repairs of the active unit occur.) And we count the number of the failures at which the switchover processing successfully occurs by constructing the failure-repair process of the control unit. Then a simulated value of the coverage probability can be obtained by computing the ratio of the number of the failures from which the system is recovered to the total number of the failures. We repeats these steps 1000 times and obtain the average and the lower bound and the upper bound with 95% confidence level. <Table 2> shows the simulated value of the coverage probability and its 95% confidence bounds. It is shown from <Table 2> that, for a given failure rate of the active unit, the coverage probability tends to decrease as the failure rate of the control unit increases.

5. An Example

For the purpose of illustration of our results, we consider a part of an optical transportation switch system. Figure 3 shows partially the architecture of the system in which three units are arranged in series and they are forming two rows in parallel. The controller monitors the active units and switches to the standby units as soon as any of active units is detected to fail.



<Figure 3> The Structure of an Optical Transportation System

<Table 3> shows the failure rate of each unit. (Note that the numbers in Table 2 are artificial values.) We assume that all PBAs are independently operating and have exponential life distributions with failure rates in <Table 2>

<Table 3> The Failure Rate of PBAs

(Unit : FIT)

PBA	Unit A	Unit B	Unit C	Controller
Failure Rate	13,000	11,500	15,500	10,000

Since Unit A, B, and C are connected in series, it can be easily shown from elementary statistics that both active units and standby units are exponentially distributed with the failure rate equal to the sum of failure rates of three units, 40,000 Fits. Then we obtain the value of the coverage probability, which is 0.999973, from <Table 2> and calculate availability of the system by using the equation in Section 2. That is 0.99999854.

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