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A SIGN TEST FOR UNIT ROOTS IN A SEASONAL MTAR MODEL†

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ABSTRACT

This study suggests a new method for testing seasonal unit roots in a momentum threshold autoregressive (MTAR) process. This sign test is robust against heteroscedastic or heavy tailed errors and is invariant to monotone data transformation. The proposed test is a seasonal extension of the sign test of Park and Shin (2006). In the case of partial seasonal unit root in an MTAR model, a Monte-Carlo study shows that the proposed test has better power than the seasonal sign test developed for AR model.

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1. INTRODUCTION

Asymmetry in time series data has attracted considerable attention from many researchers. To accommodate the asymmetry, Enders and Granger (1998) adopted the threshold autoregressive (TAR) model and proposed a modified version of the TAR model, the momentum TAR (MTAR) model, which consists of two regimes of autoregressive processes depending on levels of previous changes of the time series process. For MTAR models, various tests for unit roots hypothesis were developed by Enders and Granger (1998), Caner and Hansen (2001) and Shin and Lee (2003), which are based on the ordinary least squares estimator (OLSE). But such OLS-based procedures are neither invariant to monotone data transformations nor robust against heteroscedastic or heavy-tailed errors. Campbell and Dufour (1995) and So and Shin (2001) suggested invariant and robust sign tests for unit roots for AR processes.

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All the above studies dealt with nonseasonal models. Unit root inference in seasonal models would be also important as indicated by many studies of Dickey *et al.* (1984), Hylleberg *et al.* (1990), and others on seasonal unit root tests. For MTAR models, Shin and Lee (2003, 2007) developed seasonal unit root tests using instrumental variable approaches. However, they addressed neither robustness nor invariance.

The purpose of this study is to develop a robust sign test for seasonal unit roots in an MTAR model following the spirit of the sign test of Park and Shin (2006). Invariance and consistency of the test is established and a Monte-Carlo study shows that the proposed test is robust against heavy-tailed and / or conditional heteroscedastic errors. It also reveals that the power of the test is better than the general sign test for unit roots developed for AR model.

Section 2 introduces a robust sign test for seasonal unit roots for an MTAR process. Section 3 compares the proposed test with the seasonal version of the test of So and Shin (2001) via a Monte-Carlo study. Section 4 concludes this study with summary.

2. AN MTAR PROCESS AND A SIGN TEST

Consider an MTAR model defined as

$$\Delta_d y_t = \rho_1(y_{t-d} - \mu)I_{1t} + \rho_2(y_{t-d} - \mu)I_{2t} + u_t, \tag{2.1}$$

where $d > 0$ is given integer such as $1, 4, 12$ and others, $\{y_t\}_{t=1}^{n}$ is the set of observations, $\Delta_d y_t = y_t - y_{t-d}$, I_{1t} is the indicator function of the event $\{\Delta_d y_{t-1} > \lambda\}$, λ is a given constant, $I_{2t} = 1 - I_{1t}$, $\rho_i \in (-2, 0)$, $i = 1, 2$, μ is an unknown mean parameter and $\{u_t\}$ is an error sequence. If $\rho_1 = \rho_2 = 0$, y_t is nonstationary. If there is partial unit root $(\rho_1 = 0, -2 < \rho_2 < 0)$ or $(-2 < \rho_1 < 0, \rho_2 = 0)$, then y_t has dynamic asymmetric and, according to Lee and Shin (2001), y_t is stationary.

When $d = 1$, model (2.1) is a nonseasonal model and is related with the model of Caner and Hansen (2001). We consider general seasonal case $d \geq 1$. For the error term, we assume the following:

(A1) $\{u_t\}_{t=1}^{n}$ are independent and identically distributed (i.i.d.) having continuous distribution which is symmetric about zero.

We are interested in testing the null hypothesis of unit roots $H_0 : \rho_1 = \rho_2 = 0$ against the alternative hypothesis $H_1 : \rho_1 < 0$ or $\rho_2 < 0$, which state nonstationarity and stationarity of y_t, respectively.

We define the sign function and discuss an identity related with the sign function, from which a sign test is constructed. If $x \neq 0, \text{sign}(x) = x/ \mid x \mid$ and $\text{sign}(0) = 0$. By $(A1)$, $E[\text{sign}(u_t)] = 0$. Therefore we obtain

$$E_i = E[\text{sign}(\Delta_d y_t (y_{t-d} - \mu) I_{it})],$$

which is zero if $\rho_i = 0, \ i = 1, 2$. The sample analogues of $E_i's$ are

$$D_i = \sum_{t=d+2}^{n} \text{sign}(\Delta_d y_t (y_{t-d} - m_{t-d}) I_{it}), \quad i = 1, 2 \ , \tag{2.2}$$

where m_t is the sample median of $\{y_k\}_{k=1}^t$ as a $\mathcal{F}_t$-measurable function estimating μ and $\mathcal{F}_t$ is the σ-field generated by $y_t, y_{t-1}, \ldots$. The adjustment $y_{t-d} - m_{t-d}$ is called a recursive median adjustment. It is a median version of the recursive mean adjustment of Shin and So (1999, 2001) which significantly reduces biases of estimators of unit roots and improves powers of unit root tests and seasonal unit root tests. Note that D_i measures a departure of ρ_i from 0. We construct our test, D say, so that it rejects H_0 against H_1 if $D_1 < c$ or $D_2 < c$ for some c. The critical value can be obtained from Theorem 2.1 below, which states the null distributions of $D_1, \ D_2$. We need two preliminary lemmas in order to establish Theorem 2.1. Proofs of all lemmas and theorems are omitted.

LEMMA 2.1. *Under (A1) and* H_0, $E(D_i) = 0, Var(D_i) = (n-2)/2, \ i = 1, 2.$

LEMMA 2.2. *Let* $\{s_t\}_{t=1}^n$ *be i.i.d. random variables with* $P(s_t = 1 \mid \mathcal{F}_{t-1}) = P(s_t = -1 \mid \mathcal{F}_{t-1}) = 1/4, P(s_t = 0 \mid \mathcal{F}_{t-1}) = 1/2$. *If we let* $S_n = \sum_{t=1}^{n} s_t$, *then* $[P(S_n = x) = \binom{2n}{n+x} 4^{-n}, \ x = -n, -n+1, \ldots, 0, \ldots, n-1, n]$.

Note that $\text{sign}(\Delta_d y_t (y_{t-d} - m_{t-d}) I_{it})$ takes one value out of $1, 0, -1$ with probabilities (1/4), (1/2), (1/4) respectively, for $i = 1, 2$. Under H_0 and $(A1)$, applying Lemmas 2.1 and 2.2 with $\text{sign}(\Delta_d y_t (y_{t-d} - m_{t-d}) I_{it})$ in place of s_t, we get the null distribution of $D_i, \ i = 1, 2$, given below.

THEOREM 2.1. (Exact null distribution). *Consider (2.1) with (A1). Under* H_0,

i) The distribution of D_i *is the same as that of* S_{n-d-1}, $i = 1, 2$.

ii) D_1, D_2 *are independent.*

Theorem 2.1 shows that the proposed test statistic has non-standard null distribution, for which Park and Shin (2006)'s method can be used in computing the probability distribution function.

Two test statistics D_1 and D_2 are independent and have the same null distribution. Therefore our level-α test rejects H_0 against H_1 if $D_1 \leq c_\alpha$ or $D_2 \leq c_\alpha$, with an integer c_α satisfying

$$P(S_{n-d-1} \leq c_\alpha) = 1 - \sqrt{1-\alpha}.$$

If sample size n is large, we can use asymptotic normality instead of the exact null distribution for computing critical values.

REMARK 2.1. (Asymptotic distribution). Under $(A1)$ and H_0, as $n \to \infty$, $\sqrt{(n-2)/2}D_1$ and $\sqrt{(n-2)/2}D_2$ have independent standard normal distribution.

Park and Shin (2006) confirmed asymptotic normality by computing numerically quantiles of the exact and the asymptotic distributions for $n = 100$ and also showed that their proposed test satisfies invariance property for a monotone data transformation and consistency property under a weak condition. We show that these properties also hold for our proposed seasonal sign unit root test for seasonal model.

THEOREM 2.2. (Consistency). *Consider model (2.1) with H_1. Let $u_t = \nu_t \epsilon_t$ where ν_t is $\mathcal{F}_t$-measurable positive sequence, ϵ_t is i.i.d. random process with a distribution function F_t. Let m be the median of the distribution of y_t and let*

$$\psi_i = E[(1 - 2F(\frac{h_{it}}{\nu_t}))sign(y_{t-d} - m)I_{it}], \quad h_{it} = -\rho_i(y_{t-d} - \mu), \quad i = 1, 2.$$

If y_t has no atoms at m and $(\psi_1 < 0, \rho_1 < 0)$ or $(\psi_2 < 0, \rho_2 < 0)$, then,

$$P(D_1 \leq c_\alpha \ \ or \ \ D_2 \leq c_\alpha) \to 1 \quad as \quad n \to \infty, \quad for \ \alpha \in (0,1). \qquad (2.3)$$

THEOREM 2.3. (Invariance).

i) If f is a strictly monotone function, the values of D_1, D_2 are invariant to a data transformation $y_t \to f(y_t)$ in that (D_1, D_2) constructed using $f(y_t)$ in place of y_t has the identical value with (D_1, D_2) constructed using y_t.

ii) Let $g_{t-1}(\cdot)$ be a monotone $\mathcal{F}_{t-1}$-measurable function satisfying

$$E[sign(g_{t-1}(u_t)) \mid \mathcal{F}_{t-1}] = 0.$$

Then the exact null distribution of D_1, D_2, is invariant to the change of the error distribution $u_t \to g_{t-1}(u_t)$ in that the null distribution remains the same if u_t in (2.1) is replaced by $g_{t-1}(u_t)$.

The proposed method is valid even if the observed series is an unknown monotone transformation of an MTAR process under general heteroscedastic or heavy-tailed errors. Theorem 2.3 states that the values of D_1 and D_2 do not change by monotone data transformation of the observation y_t and test statistics D_1 and D_2 have the exact null distribution given in Theorem 2.1 under general conditional heteroscedastic or heavy-tailed errors.

3. A MONTE-CARLO STUDY

For the MTAR model, we compare size and power of the proposed sign test with those of the seasonal version of the test of So and Shin (2001). We consider model

$$\Delta_d y_t = \rho_1(y_{t-d} - \mu)I_{1t} + \rho_2(y_{t-d} - \mu)I_{2t} + u_t$$

along with the quarterly case $d = 4$ and the monthly case $d = 12$. The threshold parameter λ is set to zero. For the error term, we consider homoscedastic error $u_t = \varepsilon_t$ and autoregressive conditional heteroscedastic (ARCH) error $u_t = \varepsilon\sqrt{1 + 0.6u_{t-d}^2}$. The *i.i.d.* error terms ε_t have one of the following distributions: the standard normal distribution $N(0,1)$; the variance mixture VM(1,10), say, of two normals $0.9N(0,1) + 0.1N(0,10)$; the $t-$distribution with 3 degrees of freedom, $t(3)$; or the standard Cauchy distribution. Compared to the normal distribution $N(0,1)$, distributions VM(1,10), $t(3)$, and the Cauchy distributions have heavier tails.

For each parameter combination, we simulate 10,000 independent series with $n = 100$, $y_0 = 0$, and $\mu = 0$. We compare the proposed sign test D for the MTAR model with the seasonal version S of the sign test of So and Shin (2001) developed for AR model, given by

$$S = \sum_{t=d+1}^{n} \text{sign}(\Delta_d y_t(y_{t-d} - m_{t-d})).$$

Nominal size is set to 5%.

TABLE 3.1. *Empirical sizes(%) and powers(%) of level 5% tests for quarterly case of d=4*

Dist. of ϵ_t		Homoscedastic error							
		N(0,1)		VM(1,10)		t(3)		Cauchy	
ρ_1	ρ_2	D	S	D	S	D	S	D	S
0	0	6.4	4.6	6.8	4.6	6.5	4.2	6.5	4.7
0	−0.01	6.9	4.9	7.6	5.4	7.8	5.6	22.7	15.7
0	−0.10	20.1	13.9	27.9	18.5	33.1	21.8	83.2	64.6
0	−0.50	83.0	59.5	88.5	67.4	91.2	71.1	98.7	87.1
0	−0.90	98.2	85.1	98.5	86.8	98.7	86.5	99.4	90.4
−0.01	−0.01	8.0	6.4	8.1	5.8	9.0	6.5	34.6	35.7
−0.01	−0.1	20.2	15.5	27.0	19.6	32.7	24.9	83.6	78.8
−0.01	−0.5	82.9	61.8	87.9	69.1	91.3	73.3	98.5	92.3
−0.01	−0.9	98.2	86.2	98.4	87.1	98.6	88.3	99.4	93.0
−0.1	−0.1	25.6	27.6	32.8	38.4	39.6	48.0	91.9	97.1
−0.1	−0.5	79.5	73.7	85.6	81.1	88.8	85.7	98.4	99.1
−0.1	−0.9	97.8	92.4	98.2	93.9	98.4	95.0	99.5	98.9
Dist. of ϵ_t		Heteroscedastic error							
		N(0,1)		VM(1,10)		t(3)		Cauchy	
ρ_1	ρ_2	D	S	D	S	D	S	D	S
0	0	6.5	4.4	6.8	4.6	6.3	4.4	6.4	4.8
0	−0.01	7.2	5.3	8.9	6.3	11.8	8.5	59.2	47.4
0	−0.10	24.7	16.2	43.0	29.3	54.7	37.5	87.0	75.5
0	−0.50	86.4	64.0	92.1	72.3	94.5	76.8	93.7	84.2
0	−0.90	98.0	84.3	98.6	85.8	98.8	86.5	97.8	86.6
−0.01	−0.01	8.1	5.7	11.0	9.1	15.4	13.8	77.6	79.8
−0.01	−0.1	25.0	18.5	41.7	33.2	53.3	44.9	89.2	92.4
−0.01	−0.5	85.8	66.0	91.5	75.8	93.9	80.1	93.7	94.9
−0.01	−0.9	98.5	86.4	98.7	87.6	98.7	88.7	97.5	94.9
−0.1	−0.1	31.0	36.0	50.9	59.5	64.8	73.1	93.9	96.9
−0.1	−0.5	82.9	77.7	90.4	88.6	92.5	92.6	95.8	97.9
−0.1	−0.9	97.6	92.6	98.1	94.5	98.4	96.3	97.9	98.9

Empirical sizes and powers are reported in Table 3.1 and Table 3.2. Empirical sizes of the two tests are reasonably close to the nominal level 5% in both cases of homoscedasticity and heteroscedasticity. In any case, these results enable us to compare empirical powers of the two tests D and S without any size adjustments.

We now investigate empirical powers of the tests. First consider the quarterly case of $d = 4$. Consider the homoscedastic cases. For error distributions $N(0,1)$, VM(1,10), and $t(3)$, D is more powerful than S, except for symmetric cases $\rho_1 = \rho_2$. For example, if $\rho_1 = 0, \rho_2 = -0.1$ under the error of $t(3)$, empirical power value of D is 33.1% while that of S is 21.8%. On the other hands, under the symmetric case $\rho_1 = \rho_2 = -0.1$ and the same distribution $t(3)$ for ε_t, D is less powerful than S. The power value 39.6% of D is smaller than that 48.0% of S. It is clear that power performances of D depend on the differences of the value of $\rho_i's$. That is, under the symmetric case, powers of D and S are relatively close to each other. However, power advantage of D over S tends to increase as ρ_2

TABLE 3.2. *Empirical sizes(%) and powers(%) of level 5% tests for monthly case of d=12*

Dist. of ε_t		Homoscedastic error							
		N(0,1)		VM(1,10)		t(3)		Cauchy	
ρ_1	ρ_2	D	S	D	S	D	S	D	S
0	0	5.1	3.3	5.8	3.3	5.4	3.6	5.1	3.5
0	−0.01	5.8	4.0	6.0	3.9	6.0	3.5	8.2	5.7
0	−0.10	11.1	7.8	12.9	9.0	15.1	10.4	39.9	26.8
0	−0.50	62.0	43.1	69.3	47.8	72.5	51.8	88.4	69.7
0	−0.90	90.9	72.2	91.6	74.0	92.5	74.6	95.3	78.6
−0.01	−0.01	6.2	4.0	6.1	4.2	6.2	4.4	11.4	10.1
−0.01	−0.1	11.9	8.8	13.6	10.4	15.2	11.7	41.9	36.3
−0.01	−0.5	62.5	44.4	68.9	48.8	72.9	53.4	88.9	75.9
−0.01	−0.9	90.8	74.1	91.5	74.9	92.5	76.1	94.9	82.9
−0.1	−0.1	14.6	14.9	18.7	19.6	21.8	24.2	58.5	70.7
−0.1	−0.5	61.0	56.9	67.6	65.5	72.7	70.5	90.1	93.7
−0.1	−0.9	89.4	82.5	90.5	83.9	92.1	86.7	95.3	95.3
Dist. of ε_t		Heteroscedastic error							
		N(0,1)		VM(1,10)		t(3)		Cauchy	
ρ_1	ρ_2	D	S	D	S	D	S	D	S
0	0	5.1	3.4	5.4	3.4	5.6	3.3	5.2	3.5
0	−0.01	5.9	4.0	6.4	4.1	7.2	5.1	41.7	31.9
0	−0.10	12.2	8.7	21.0	14.6	28.8	19.9	71.3	57.7
0	−0.50	67.5	46.8	76.9	57.2	80.5	61.6	88.1	75.1
0	−0.90	91.0	73.1	92.2	74.6	93.0	74.7	91.8	77.5
−0.01	−0.01	5.8	3.9	7.5	5.2	8.5	6.8	62.6	59.6
−0.01	−0.1	13.1	9.3	20.6	16.6	28.5	23.5	80.4	78.2
−0.01	−0.5	67.7	49.2	76.3	58.8	81.0	64.6	91.0	89.0
−0.01	−0.9	91.4	74.5	91.9	75.8	92.8	78.4	92.5	89.7
−0.1	−0.1	18.2	19.3	30.9	34.8	42.5	47.6	89.0	90.5
−0.1	−0.5	66.2	62.6	76.1	76.1	81.6	82.6	94.7	96.0
−0.1	−0.9	89.6	83.5	91.5	87.8	92.7	90.4	95.0	96.2

goes away from ρ_1. For example, if $\rho_1 = 0$ and $\rho_2 = -0.5$, the power value 83.0% of D is substantially greater than that 59.5% of S. For all errors, D dominates S in power performance for all the cases of partial unit root, *i.e.*, $\rho_1 = 0$. This implies that the proposed test performs particularly well under a partial unit root situation. For heteroscedastic errors, similar situations happen. Power advantage of D over S gets larger as $\mid \rho_1 - \rho_2 \mid$ increases. For example, when $\rho_1 = 0$ and the distribution of ε_t is VM(1,10), as ρ_2 varies from -0.01 to -0.90, power value of D increases considerably from 8.9% to 98.6% and D performs better than S.

Next consider the monthly case of $d = 12$. Similar situations happen although relative magnitudes of powers are smaller than those of case $d = 4$. First, for homoscedastic errors, D is more powerful than S, except for symmetric cases $\rho_1 = \rho_2 = -0.1$. If $\rho_1 = 0, \rho_2 = -0.1$ under the error of $t(3)$, for example, empirical power value of D is 15.1% while that of S is 10.4%. On the other hands, under the symmetric case $\rho_1 = \rho_2 = -0.1$ and $t(3)$ error, D is less powerful than

S. The power value 21.8% of D is smaller than that 24.2% of S. Power advantage of D over S tends to increase as ρ_2 goes away from ρ_1 as in the case of $d = 4$. For heteroscedastic errors, when $\rho_1 = 0$ and the distribution VM(1,10) for ε_t, as ρ_2 varies from -0.01 to -0.90, power value of D increases considerably from 6.4% to 92.2% and D performs better than S.

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