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SOME GENERALIZATIONS OF LOGISTIC DISTRIBUTION AND THEIR PROPERTIES

THOMAS MATHEW[1] AND K. JAYAKUMAR[2]

ABSTRACT

The logistic distribution is generalized using the Marshall-Olkin scheme and its generalization. Some properties are studied. First order autoregressive time series model with Marshall-Olkin semi-logistic distribution as marginal is developed and studied.

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1. INTRODUCTION

Logistic distribution has attracted the attention of many researchers due to the application of this distribution in various fields. Glasbey (1979) applied the generalized logistic curve to the weight gain analysis of Ayshire steer calves, which were recorded weekly from birth to slaughter at 880 ponds. Oliver (1982) applied the logistic curve to human population. Wijesinha *et al.* (1983) applied the polychotomous logistic regression model to large data set of patients where there were many distinct diagnostic categories. Johnson (1985) applied logistic regression to estimate the survival time of diagnosed leukemia patients. Morgan (1985) proposed and applied the cubic logistic model to quantal assay data.

By various methods new parameters can be introduced to expand families of distributions. Introduction of a scale parameter leads to accelerate life model and taking powers of a survival function introduces a parameter that leads to proportional hazards model. Marshall and Olkin (1997) introduced a new method of adding a parameter to expand families of distribution. In particular, starting

Received February 2006; accepted October 2006.
[1]Corresponding author. Department of Statistics, M.D. College, Pazhanji-680542, Thrissur district, Kerala, India (e-mail: t_tmathew@rediffmail.com)
[2]Department of Statistics, University of Calicut, Calicut University P.O.-673 635, Kerala, India

with survival function $\bar{F}$ they have derived the one-parameter family of survival functions

$$\bar{H}(x) = \left\{\frac{\alpha \bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right\}, \quad -\infty < x < \infty, \ 0 < \alpha < \infty, \ \bar{\alpha} = 1-\alpha. \tag{1.1}$$

We call $\bar{H}$ the Marshall-Olkin distribution generated from $\bar{F}$. Marshall and Olkin (1997) have applied this to exponential and Weibull case. The density and hazard rate of $\bar{H}$ in terms of F are given by the expression $h(x) = \alpha f(x)/(1 - \bar{\alpha}\bar{F}(x))^2$, $r(x) = f(x)/\{\bar{F}(x)\left(1-\bar{\alpha}\bar{F}(x)\right)\}$ respectively. They established that this family of distributions is geometric extreme stable.

A generalization to the method suggested by Marshall and Olkin (1997) is starting with a survival function $\bar{F}$ and density function f, the two-parameter family of survival function is

$$\bar{G}(x) = \left\{\frac{\alpha \bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right\}^{\gamma}, \quad -\infty < x < \infty, \ 0 < \alpha < \infty, \ 0 < \gamma < \infty. \tag{1.2}$$

When $\alpha = 1$, we get $\bar{G}(x) = \left[\bar{F}(x)\right]^{\gamma}$ and in particular when $\alpha = \gamma = 1$, we get $\bar{G}(x) = \bar{F}(x)$.

$$g(x) = \gamma\left\{\frac{\alpha \bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right\}^{\gamma-1} \frac{\alpha f(x)}{\left\{1-\bar{\alpha}\bar{F}(x)\right\}^{2}},$$

where $\bar{G}$ and g are the survival function and the density function of new family of distribution. The hazard rate function is

$$r(x) = \frac{g(x)}{\bar{G}(x)} = \frac{\gamma f(x)}{\bar{F}(x)\left\{1-\bar{\alpha}\bar{F}(x)\right\}}.$$

The study on minification processes began with the work of Tavares (1980). In his work, the observations are generated by the equation

$$X_n = k \min(X_{n-1}, \varepsilon_n), \quad n \geq 1, \tag{1.3}$$

where $k > 1$ is a constant and $\{\varepsilon_n\}$ is an innovation sequence of *i.i.d.* random variables chosen to ensure that $\{X_n\}$ is a stationary Markov process with a given marginal distribution. Because of the structure of (1.3), the process $\{X_n\}$ is called minification process.

Sim (1986) developed a first order autoregressive Weibull process and studied its properties. Giving slight modifications to (1.3), several other minification models have been constructed so far. Pillai (1991) studied semi-Pareto minification process. Pillai *et al.* (1995) introduced a minification process having the form

$$X_n = \begin{cases} \varepsilon_n, & w.p. \quad p \\ k \min(X_{n-1}, \varepsilon_n), & w.p. \quad 1-p \end{cases}, \quad 0 < p < 1.$$

Lewis and McKenzie (1991) obtained necessary and sufficient conditions on the hazard rate of the marginal distribution for a minification process to exist.

In Section 2, Marshall-Olkin semi-logistic distribution is introduced and its properties are studied. As a special case, Marshall-Olkin logistic distribution is studied in detail and estimation of parameters is done. Generalized Marshall-Olkin semi-logistic and logistic distributions are studied in Section3. In Section 4, first order autoregressive minification process with Marshall-Olkin semi-logistic distribution as marginal is introduced and its properties are studied. Some applications are discussed in Section 5.

2. MARSHALL-OLKIN SEMI-LOGISTIC DISTRIBUTION

We say that a random variable X defined on $R = (-\infty, \infty)$ has semi-logistic distribution (Jayakumar and Mathew, 2004) and write $X \stackrel{d}{=} L_s(\beta, p)$ if its survival function is

$$\bar{F}_X(x) = \frac{1}{1+\eta(x)}, \tag{2.1}$$

where $\eta(x)$ satisfies the functional equation

$$\eta(x) = \frac{1}{p}\eta\left(\frac{1}{\beta}\ln(p) + x\right), \quad \beta > 0, \quad 0 < p < 1. \tag{2.2}$$

It can be shown that $\eta(x) = e^{\beta x}h(x)$, where $h(x)$ is periodic in x with period $\ln(p)/\beta$. For proof see Kagan *et al.* (1973). For example, if $h(x) = e^{\theta\cos(\beta x)}$, it satisfies (2.2) with $p = e^{-2\pi}$ and $\eta(x)$ is monotone increasing with $0 < \theta < 1$.

Substituting (2.1) in (1.1), we get the Marshall-Olkin semi-logistic ($MOSL(\alpha, \beta)$) distribution whose survival function is given by

$$\bar{G}(x) = \frac{\alpha}{\alpha + \eta(x)}, \quad \alpha, \ \beta > 0,$$

where $\eta(x)$ is as defined in (2.2). That is,

$$\bar{G}(x) = \frac{1}{1 + \frac{1}{\alpha}\eta(x)}, \quad \alpha, \ \beta > 0.$$

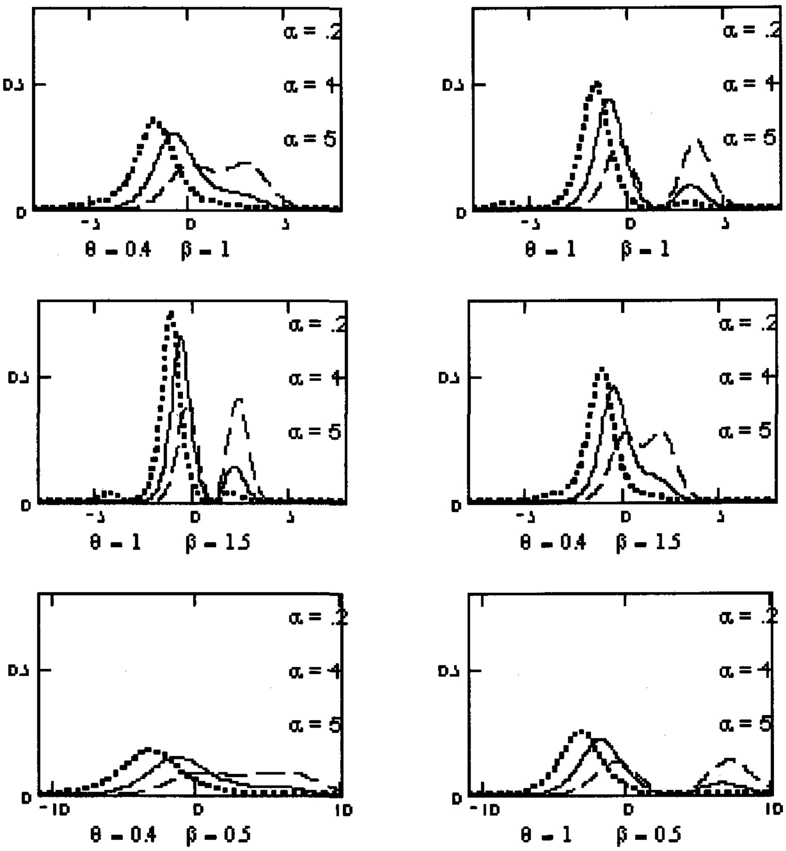

FIGURE 2.1 *Density of Marshall-Olkin semi-logistic distribution.*

The probability density function is

$$g(x) = \frac{\alpha\eta'(x)}{(\alpha+\eta(x))^2}, \quad -\infty < x < \infty, \quad \alpha > 0.$$

Taking $h(x) = e^{\theta\cos(\beta x)}$, the density plots of Marshall-Olkin semi-logistic distribution for $\theta = 0.4$ and $\beta = 1$ is presented in Figure 2.1. The solid line corresponds to $\alpha = 1$, dotted line for $\alpha = 0.2$ and dashed line for $\alpha = 5$. The graph is symmetric when $\alpha = 1$ and $\theta = 0$ and exhibit periodic nature for all other values of the parameters. From the Figure 2.1 it can be observed that for fixed β as θ increases periodicity become more dominant.

The hazard rate is

$$r(x) = \frac{\eta'(x)}{\alpha+\eta(x)}.$$

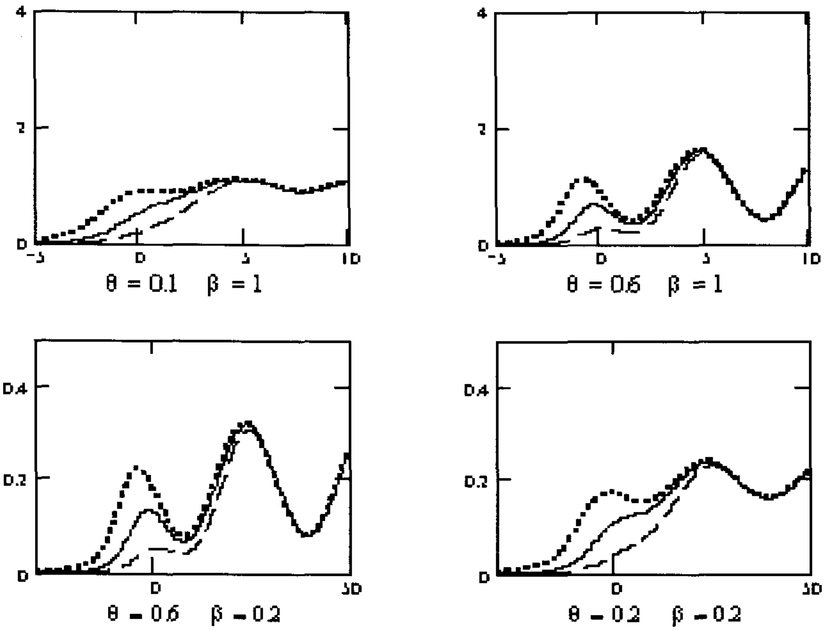

FIGURE 2.2 *Hazard rate of Marshall-Olkin semi-logistic distribution for α = 0.2, 1 and 5.*

When $h(x) = e^{\theta\cos(\beta x)}$, the hazard rate function of the Marshall-Olkin semi-logistic distribution is

$$r(x) = \frac{\frac{1}{\alpha}\beta e^{\beta x+\theta\cos(\beta x)}\left(1-\theta\sin(\beta x)\right)}{\left(1+\frac{1}{\alpha}e^{\beta x+\theta\cos(\beta x)}\right)}.$$

The solid line corresponds to $\alpha = 1$, dotted line for $\alpha = 0.2$ and dashed line for $\alpha = 5$. The hazard rate of Marshall-Olkin semi-logistic distribution is periodic in nature. This characteristic of the hazard rate function is useful in many situations. In the case of automobiles the failure rate is increasing and as older parts are replaced by new ones, failure rate decreases and so on. The periodic nature of the hazard rate may be useful in modeling such a situation.

From the Figure 2.2 it can be observed that for fixed β as θ increases periodicity become more dominant.

DEFINITION 2.1. *Let $X_1, X_2, ..., X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables with distribution function F. Suppose N is independent of X_i's with geometric(p) distribution. That is,*

$$P(N=n) = p(1-p)^{n-1}, \quad n = 1,2,..., \; 0<p<1, \; q = 1-p$$

and let $U_N = \min(X_1, X_2, ..., X_N)$ *and* $V_N = \max(X_1, X_2, ..., X_N)$. *If* $F \in \Im$ *implies that distribution of* $U_N (V_N)$ *is in* $\Im$, *then* $\Im$ *is said to be geometric minimum stable (geometric maximum stable). If* $\Im$ *is both geometric minimum stable and geometric maximum stable, then* $\Im$ *is said to be geometric extreme stable.*

THEOREM 2.1. *The Marshall-Olkin family of semi-logistic distribution is geometric extreme stable.*

PROOF. We have,

$$\begin{aligned} P(U_N \geq x) &= P\left(\min(X_1, X_2, ..., X_N) \geq x\right) \\ &= \sum_{n=1}^{\infty} P\left(\min\left(X_1, X_2, ..., X_N\right) > x | N = n\right) P\left(N = n\right) \\ &= \sum_{n=1}^{\infty} \left(\bar{F}(x)\right)^n P\left(N = n\right) \\ &= \sum_{n=1}^{\infty} \left(\bar{F}(x)\right)^n q^{n-1} p \\ &= \frac{p\bar{F}(x)}{1 - q\bar{F}(x)}. \end{aligned}$$

Also,

$$\begin{aligned} P(V_N \leq x) &= P\left(\max(X_1, X_2, ..., X_N) \leq x\right) \\ &= \frac{p\eta(x)}{1 + p\eta(x)}. \end{aligned}$$

□

THEOREM 2.2. *Let* $X_1, X_2, ...$ *be a sequence of i.i.d. random variables with common survival function* $\bar{F}(x)$ *and* N *is geometric random variable with parameter* p, *which is independent of* X_i *for all* $i \geq 1$. *Let* $U_N = \min(X_1, X_2, ..., X_N)$. *Then* U_N *is distributed as Marshall-Olkin semi-logistic if and only if* $\{X_i\}$ *is distributed as semi-logistic.*

PROOF. Let

$$\begin{aligned} \bar{H}(x) &= P\left(U_N > x\right) \\ &= \sum_{n=1}^{\infty} \left(\bar{F}(x)\right)^n pq^{n-1} \end{aligned}$$

$$= \frac{p\bar{F}(x)}{1-(1-p)\bar{F}(x)}.$$

Suppose

$$\bar{F}(x) = \frac{1}{1+\eta(x)}.$$

Then

$$\bar{H}(x) = \frac{1}{1+\frac{1}{p}\eta(x)}$$

which is the survival function of Marshall-Olkin semi-logistic distribution. This proves the sufficiency part of the theorem. Converse easily follows. □

DEFINITION 2.2. *We say that a random variable X on $(-\infty, \infty)$ is said to follow semi-extreme value distribution if its survival function is*

$$\bar{F}(x) = e^{-\eta(x)},$$

where $\eta(x)$ satisfies the functional equation (2.2).

The following theorem establishes the relationship between the semi-extreme value distribution and Marshall-Olkin semi-logistic distribution.

THEOREM 2.3. *If $X_1, X_2, ..., X_n$ are i.i.d. $MOSL(\alpha, \beta)$ then*

$$Z_n = \min\Big(X_1 - \frac{1}{\beta}\ln\left(\frac{\alpha}{n}\right), X_2 - \frac{1}{\beta}\ln\left(\frac{\alpha}{n}\right), ..., X_n - \frac{1}{\beta}\ln\left(\frac{\alpha}{n}\right)\Big),\ \alpha,\ \beta > 0,\ n > 1,\ n > \alpha$$

is asymptotically distributed as semi-extreme value.

PROOF. If X is distributed as Marshall-Olkin semi-logistic ($MOSL(\alpha, \beta, p)$) then

$$\bar{G}(x) = \frac{1}{1+\frac{1}{\alpha}\eta(x)}, \quad \alpha,\ \beta > 0\ ,\ 0 < p < 1,$$

where $\eta(x)$ satisfies (2.2).

$$\begin{aligned}
\bar{F}_{Z_n}(x) &= P\left(\min\left(X_1 - \frac{1}{\beta}\ln\left(\frac{\alpha}{n}\right), X_2 - \frac{1}{\beta}\ln\left(\frac{\alpha}{n}\right), ..., X_n - \frac{1}{\beta}\ln\left(\frac{\alpha}{n}\right)\right) > x\right) \\
&= \left\{\bar{G}_X\left(x + \frac{1}{\beta}\ln\left(\frac{\alpha}{n}\right)\right)\right\}^n \\
&= \left(\frac{1}{1+\frac{\eta(x)}{n}}\right)^n.
\end{aligned}$$

Taking limit when $n \to \infty$, we get

$$\bar{F}_{Z_n}(x) = e^{-\eta(x)}.$$

This establishes the theorem. □

As a special case of Marshall-Olkin semi-logistic distribution, we now study some properties of Marshall-Olkin logistic distribution. Consider the logistic distribution with survival function

$$\bar{F}(x) = \frac{1}{1+e^{\beta x}}, \quad -\infty < x < \infty, \quad \beta > 0.$$

Substituting this in (1.1) we get Marshall-Olkin logistic distribution. The survival function of Marshall-Olkin logistic distribution is

$$\begin{aligned} \bar{G}(x) &= \frac{\alpha}{\alpha + e^{\beta x}}, \\ &= \frac{1}{1+\frac{1}{\alpha}e^{\beta x}}, \quad -\infty < x < \infty, \quad \alpha > 0, \quad \beta > 0. \end{aligned}$$

The density function is,

$$g(x) = \frac{\alpha\beta e^{\beta x}}{(\alpha + e^{\beta x})^2}, \quad -\infty < x < \infty, \quad \alpha > 0, \quad \beta > 0.$$

Moment generating function corresponding to Marshall-Olkin logistic distribution is

$$M_X(t) = E\left(e^{tX}\right) = \alpha^{\frac{t}{\beta}}\Gamma\left(1-\frac{t}{\beta}\right)\Gamma\left(1+\frac{t}{\beta}\right)$$

and the cumulant generating function is

$$K_X(t) = \frac{t}{\beta}\ln(\alpha) + \ln\left(\Gamma\left(1-\frac{t}{\beta}\right)\right) + \ln\left(\Gamma\left(1+\frac{t}{\beta}\right)\right).$$

From this, we get

$$E(X) = \frac{\ln(\alpha)}{\beta}.$$

It can be easily seen that

$$\text{Mode}(X) = \frac{\ln(\alpha)}{\beta},$$

and

$$\text{Median}(X) = \frac{\ln(\alpha)}{\beta}.$$

That is, mean = median = mode = $\ln(\alpha)/\beta$.

The distribution is symmetric about $\ln(\alpha)/\beta$. All odd order moments are zero and even order moments are given by

$$\mu_{2r} = \int_{-\infty}^{\infty} \left(x - \frac{\ln(\alpha)}{\beta}\right)^{2r} \frac{\beta\frac{1}{\alpha}e^{\beta x}}{\left(1+\frac{1}{\alpha}e^{\beta x}\right)} dx.$$

When $y = x - \ln(\alpha)/\beta$,

$$\begin{aligned}\mu_{2r} &= \int_{-\infty}^{\infty} y^{2r} \frac{\beta e^{\beta y}}{(1+e^{\beta y})} dy \\ &= \frac{2\Gamma(2r+1)}{\beta^{2r}} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^{2r}} \\ &= \frac{2\Gamma(2r+1)}{\beta^{2r}} \left(1 - \frac{1}{2^{2r-1}}\right) \zeta(2r), \quad \text{for } r = 1, 2, \ldots,\end{aligned}$$

where $\zeta(s) = \sum_{j=1}^{\infty} j^{-s}$ is Riemann zeta function.
In particular, we have $V(X) = \pi^2/(3\beta^2)$, $\mu_4 = 7\pi^4/(15\beta^4)$, $\beta_1 = 0$ and $\beta_2 = 21/5 = 4.2$.

In Figure 2.3 the density plots of Marshall-Olkin logistic distribution $\alpha = 0.1$ (dotted line), $\alpha = 1$ (solid line) and $\alpha = 12$ (dashed line) is presented with $\beta = 1$ and $\beta = 2$.

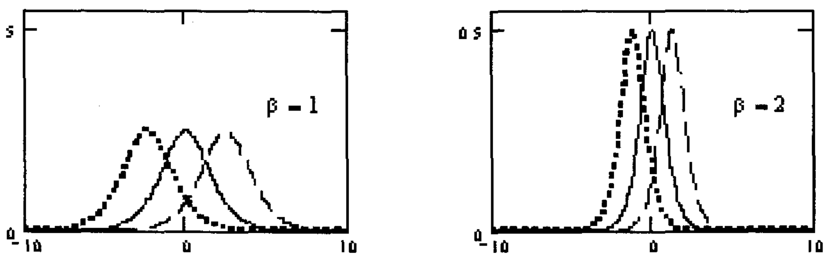

FIGURE 2.3 *density of Marshall-Olkin logistic distribution.*

Figure 2.3 describes how the scale parameter β and the location parameter α affects the distribution. The hazard rate function of Marshall-Olkin logistic distribution is

$$r(x) = \frac{\beta e^{\beta x}}{\alpha + e^{\beta x}}.$$

The parameters of Marshall-Olkin logistic distribution can be estimated as in Johnson *et al.* (1995). The moment estimates of α and β are obtained by solving

$$\bar{x} = \frac{\ln(\alpha)}{\beta} \text{ and } S^2 = \frac{\pi^2}{3\beta^2}.$$

The maximum likelihood estimates of α and β are obtained by solving the equations

$$\alpha = \frac{n}{2\sum_{i=1}^{n} \left(\alpha + e^{\beta x_i}\right)^{-1}}$$

and

$$\frac{n}{\beta} + \sum_{i-1}^{n} x_i = 2\sum_{i=1}^{n} \frac{x_i e^{\beta x_i}}{\alpha + e^{\beta x_i}}.$$

To find an estimate for α first fix the value of β . Then by Newton-Ralphson method, find the value of α. This gives the estimate for α. Then again apply Newton-Ralphson method to find the value of β.

The method of quantiles estimation in this case is as follows. Select two numbers between 0 and 1 and obtain estimators of respective quintiles $\hat{X}_{p_1}$ and $\hat{X}_{p_2}$. Estimates of β and α are obtained by solving the two simultaneous equations

$$P_j = 1 - \frac{1}{1 + \frac{1}{\alpha} e^{\beta \hat{X}_{P_j}}}, \quad j = 1, 2.$$

The estimate of β is

$$\hat{\beta} = \frac{1}{\hat{X}_{P_1} - \hat{X}_{P_2}} \ln\left(\frac{P_1(1-P_2)}{P_2(1-P_1)}\right).$$

Corresponding estimators of α can be obtained from the other equation.

3. Generalized Marshall-Olkin Semi-Logistic Distribution

Substituting (2.1) in (1.2) we get a generalization to Marshall-Olkin semi-logistic distribution ($GMOSL(\alpha, \beta, \gamma)$) whose survival function is given by

$$\bar{G}(x) = \left(\frac{\alpha}{\alpha + \eta(x)}\right)^{\gamma}, \quad \alpha,\ \beta,\ \gamma > 0,$$

where $\eta(x)$ defined in (2.2). That is,

$$\bar{G}(x) = \left(\frac{1}{1 + \frac{1}{\alpha}\eta(x)}\right)^{\gamma}, \quad \alpha,\ \beta,\ \gamma > 0.$$

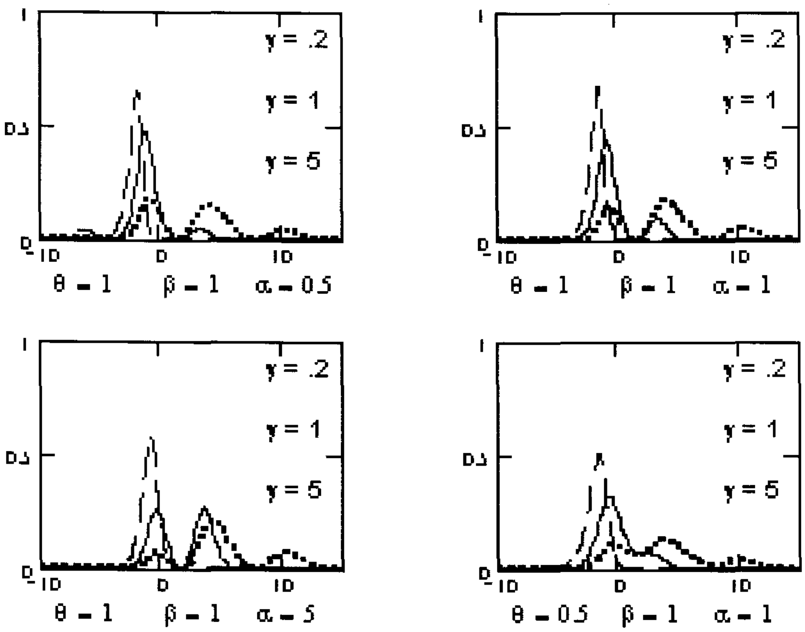

FIGURE 3.1 *Density of the* $GMOSL(\alpha, \beta, \gamma)$ *distribution.*

The probability density function of $GMOSL(\alpha, \beta, \gamma)$ is given by

$$g(x) = \frac{\gamma}{\alpha}\left(\frac{\alpha}{\alpha + \eta(x)}\right)^{\gamma+1} \eta'(x), \quad \alpha,\ \beta,\ \gamma > 0.$$

The hazard rate function is

$$r(x) = \frac{\gamma}{\alpha}\left(\frac{\alpha}{\alpha + \eta(x)}\right) \eta'(x), \quad \alpha,\ \beta,\ \gamma > 0.$$

Here we study the special case when $h(x) = e^{\theta \cos(\beta x)}$, $0 < \theta < 1$.

$$\bar{G}(x) = \left(\frac{\alpha}{\alpha + e^{\beta x + \theta \cos(\beta x)}}\right)^{\gamma}, \quad \alpha,\ \beta,\ \gamma > 0.$$

In Figure 3.1 the density plots of generalized Marshall-Olkin semi-logistic distribution $\gamma = 0.2$ (dotted line), $\gamma = 1$ (solid line) and $\gamma = 5$ (dashed line) is presented. From Figure 3.1 it can be observed how the periodicity parameter θ, skewness parameter γ and the location parameter α affects the distribution.

$$g(x) = \frac{\gamma\beta}{\alpha}\left(\frac{\alpha}{\alpha + e^{\beta x + \theta \cos(\beta x)}}\right)^{\gamma+1} e^{\beta x + \theta \cos(\beta x)}(1 - \theta \sin(\beta x)), \quad \alpha,\ \beta\ ,\gamma > 0.$$

The hazard rate is

$$r(x) = \frac{\gamma\beta}{\alpha}\left(\frac{\alpha}{\alpha + e^{\beta x+\theta\cos(\beta x)}}\right)e^{\beta x+\theta\cos(\beta x)}(1-\theta\sin(\beta x)), \quad \alpha\ ,\beta\ ,\gamma > 0.$$

For $\theta = 0$, the Generalized Marshall-Olkin semi-logistic distribution reduces to Generalized Marshall-Olkin logistic ($GMOL(\alpha,\beta,\gamma)$) distribution whose survival function is

$$\bar{G}(x) = \left(\frac{\alpha}{\alpha + e^{\beta x}}\right)^{\gamma}, \quad \alpha,\ \beta,\ \gamma > 0,$$

where $\eta(x)$ defined in (2.2).
That is,

$$\bar{K}(x) = \left(\frac{1}{1 + \frac{1}{\alpha}e^{\beta x}}\right)^{\gamma}, \quad \alpha,\ \beta,\ \gamma > 0.$$

The probability density function of $GMOL(\alpha,\beta,\gamma)$ is given by

$$g(x) = \frac{\gamma\beta}{\alpha}\left(\frac{\alpha}{\alpha + e^{\beta x}}\right)^{\gamma+1} e^{\beta x}, \quad \alpha,\ \beta,\ \gamma > 0.$$

The hazard rate function is

$$r(x) = \frac{\gamma\beta}{\alpha}\left(\frac{\alpha}{\alpha + e^{\beta x}}\right) e^{\beta x}, \quad \alpha,\ \beta,\ \gamma > 0.$$

In Figure 3.2 the density plots of $GMOL(\alpha,\beta,\gamma)$ distribution for $\gamma = 0.2$ (dotted line), $\gamma = 1$ (solid line) and $\gamma = 5$ (dashed line) is presented.

It is interesting to observe how the skewness parameter γ affects the probability distribution.

The moment generating function and the expectation are given by

$$M_X(t) = \frac{\gamma\alpha^{\frac{t}{\beta}}\Gamma\left(\gamma - \frac{t}{\beta}\right)\Gamma\left(1 + \frac{t}{\beta}\right)}{\Gamma(1+\gamma)} \quad \text{and}$$

$$E(X) = \frac{\ln(\alpha)}{\beta} + \frac{\gamma}{\beta}\int_0^1 y^{\gamma-1}\ln\left(\frac{1-y}{y}\right)dy.$$

This expression is convergent and numerical evaluation is possible

$$\text{Median}(X) = \frac{\ln(\alpha)}{\beta} + \frac{1}{\beta}\ln\left(2^{\frac{1}{\gamma}} - 1\right),$$

$$\text{Mode}(X) = \frac{\ln\left(\frac{\alpha}{\gamma}\right)}{\beta}.$$

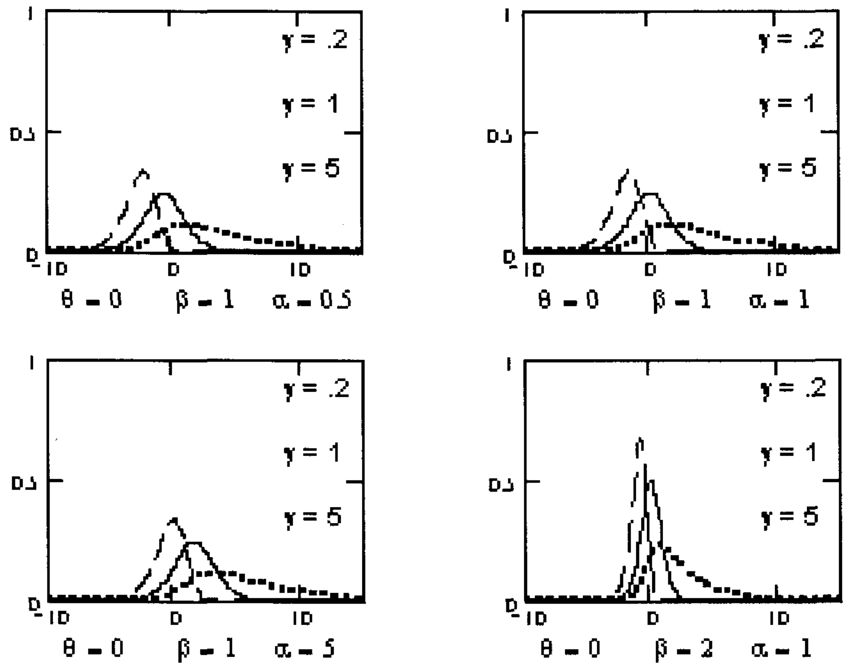

FIGURE 3.2 *Density of the $GMOL(\alpha, \beta, \gamma)$ distribution.*

4. FIRST ORDER AUTOREGRESSIVE MARSHALL-OLKIN SEMI-LOGISTIC PROCESS

Since the logistic distribution has many applications in real life situations, autoregressive processes with logistic marginal are relevant. Sim (1993) introduced and studied first order additive autoregressive model with logistic marginal distribution. Here we develop Marshall-Olkin semi-logistic process as a minification analogue of the TEAR(1) structure in Lawrance and Lewis (1981).

THEOREM 4.1. *Consider the first order autoregressive process $\{X_n\}$ defined by*

$$X_n = \begin{cases} \varepsilon_n, & w.p. \quad \alpha \\ \min(X_{n-1}, \varepsilon_n), & w.p. \quad 1-\alpha \end{cases}, \tag{4.1}$$

where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables independent of X_n. Then $\{X_n\}$ is stationary first order autoregressive process with Marshall-Olkin semi-logistic marginal if and only if $\{\varepsilon_n\}$ is distributed as semi-logistic.

PROOF. Equation (4.1) in terms of survival functions is

$$\bar{F}_{X_n}(x) = \alpha\bar{F}_{\varepsilon_n}(x) + (1-\alpha)\bar{F}_{X_{n-1}}(x)\bar{F}_{\varepsilon_n}(x). \tag{4.2}$$

When the process is stationary, we get

$$\bar{F}_X(x) = \frac{\alpha\bar{F}_{\varepsilon_n}(x)}{1-(1-\alpha)\bar{F}_{\varepsilon_n}(x)}.$$

Let

$$\bar{F}_{\varepsilon_n}(x) = \frac{1}{1+\eta(x)}.$$

Then

$$\bar{F}_X(x) = \frac{1}{1+\frac{1}{\alpha}\eta(x)}$$

which is the survival function of Marshall-Olkin semi-logistic distribution.
For $n = 1$, (4.2) becomes

$$\bar{F}_{X_1}(x) = \alpha\bar{F}_{\varepsilon_1}(x) + (1-\alpha)\bar{F}_{X_0}(x)\bar{F}_{\varepsilon_0}(x). \tag{4.3}$$

If we take $\bar{F}_{X_0}(x)$ as Marshall-Olkin semi-logistic and $\bar{F}_{\varepsilon_1}(x)$ as semi-logistic in (4.3), we get

$$\bar{F}_{X_1}(x) = \frac{1}{1+\frac{1}{\alpha}\eta(x)}.$$

Assuming that X_{n-1} is Marshall-Olkin semi-logistic and ε_n is semi-logistic, we get

$$\bar{F}_{X_n}(x) = \frac{1}{1+\frac{1}{\alpha}\eta(x)}.$$

Hence $\{X_n\}$ is stationary. This completes the proof. □

REMARK 4.1. Even if X_0 is arbitrary, it can be proved that $\{X_n\}$ is asymptotically stationary Marshall-Olkin semi-logistic process.

Now we look in to some properties of the stationary Marshall-Olkin semi-logistic process.

$$\begin{aligned} P(X_{n+1} > X_n) &= \alpha P(\varepsilon_{n+1} > X_n) + (1-\alpha)P(\min(X_n, \varepsilon_{n+1}) > X_n) \\ &= \alpha P(\varepsilon_n > X_n) \\ &= \frac{\alpha}{2}. \end{aligned}$$

It can be proved that,

$$\operatorname{Corr}(X_n, X_{n+1}) = 1 - \alpha.$$

The joint survival function of (X_n, X_{n+1}) is

$$\begin{aligned}
\bar{F}_{X_n,X_{n+1}}(x,y) &= P\left(X_n > x, X_{n+1} > y\right) \\
&= \left(\alpha \bar{F}_{X_n}(x) + (1-\alpha)\bar{F}_{X_n}(\max(x,y))\right)\bar{F}_{\varepsilon_n}(y) \\
&= \left(\alpha \frac{1}{1+\frac{1}{\alpha}\eta(x)} + (1-\alpha)\frac{1}{1+\frac{1}{\alpha}\max\left(\eta(x),\eta(y)\right)}\right)\frac{1}{1+\eta(x)}.
\end{aligned}$$

When $h(x) = e^{\theta\cos(\beta x)}$, the joint survival function of (X_n, X_{n+1}) for the Marshall-Olkin semi-logistic distribution for $\beta = 0.6$, $\alpha = 0.6$ and $\theta = 0.9$ is presented in Figure 4.1. The periodicity of the distribution can be observed in the figure.

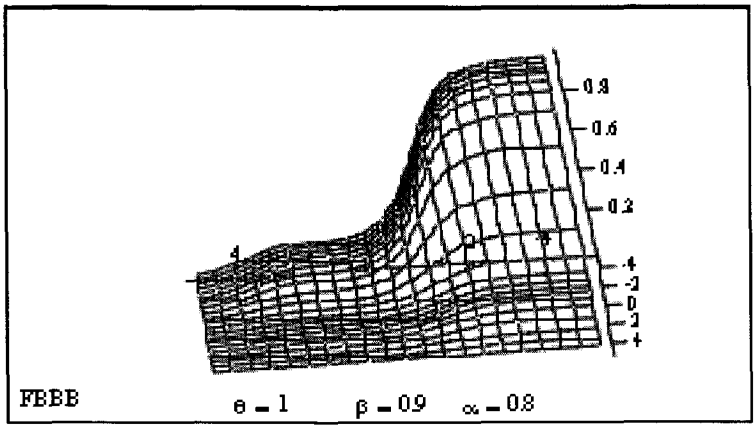

FIGURE 4.1 *Joint survival function of (X_n, X_{n+1}) for the Marshall-Olkin semi-logistic process.*

Sample path behavior of the Marshall-Olkin semi-logistic process for various values of α, β and θ are given Figure 4.2.

Since Marshall-Olkin logistic distribution is a special case of the Marshall-Olkin semi-logistic distribution, all properties of Marshall-Olkin semi-logistic process is also satisfied by Marshall-Olkin logistic process.

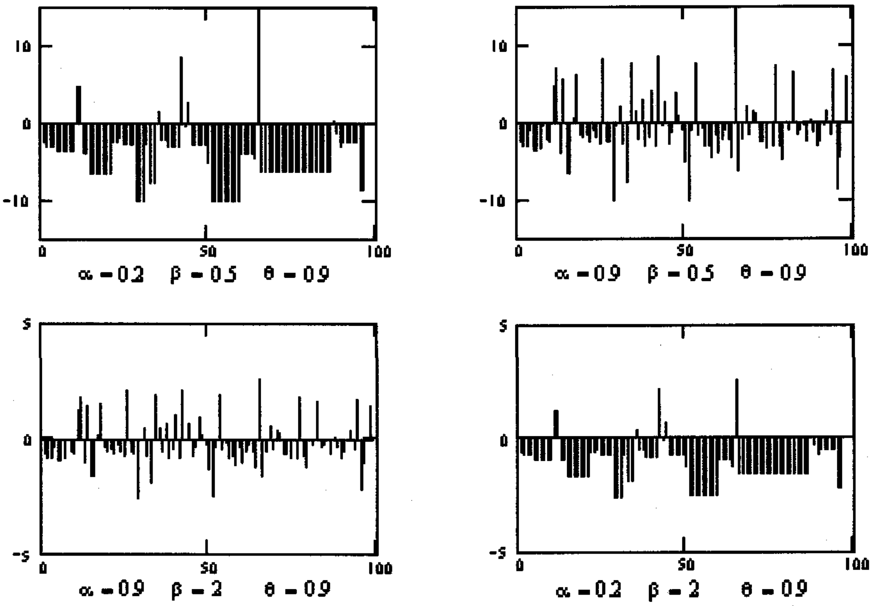

FIGURE 4.2 *Sample path behavior of the first order autoregressive semi-logistic process.*

5. APPLICATIONS

Logistic distribution has applications in agricultural field, medical diagnosis, public health etc. Most of the time series data that we come across in practice are seasonal and periodic in nature. A one-parameter logistic model may not be sufficient enough to model these characteristics. In these situations the two parameter Marshall-Olkin logistic model may be useful. In the case of data exhibiting periodic nature, the Marshall-Olkin semi-logistic distribution may be an appropriate model. Note that if a given time series has periodic nature with the governing process has stationary marginals as MOSL, then it accommodates the seasonality inherent in the series.

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