Journal of the Korean Statistical Society (2007), **36**: *1, pp 77–91*

DEVELOPING NONINFORMATIVE PRIORS FOR THE FAMILIAL DATA

JUNGEUN HEO[1] AND YEONG-HWA KIM[2]

ABSTRACT

This paper considers development of noninformative priors for the familial data when the families have equal number of offspring. Several noninformative priors including the widely used Jeffreys' prior as well as the different reference priors are derived. Also, a simultaneously-marginally-probability-matching prior is considered and probability matching priors are derived when the parameter of interest is inter- or intra-class correlation coefficient. The simulation study implemented by Gibbs sampler shows that two-group reference prior is slightly edge over the others in terms of coverage probability.

AMS 2000 subject classifications. Primary 62F12; Secondary 62F15.
Keywords. Familial data, Gibbs sampler, interclass correlation coefficient, intraclass correlation coefficient, Jeffreys' prior, matching priors, reference priors.

1. INTRODUCTION

In the analysis of familial data the primary aim is to estimate the degree of resemblance between family members which is measured by a parent-child correlation, the interclass correlation coefficient, and correlation between siblings, the intraclass correlation coefficient. Several estimates of these correlations have been proposed in the literature. In particular, Rosner *et al.* (1977) gave the maximum likelihood estimates when the sib sizes are equal. However, when the sib sizes are not equal, Rosner (1979) proposed an algorithm for finding the maximum likelihood estimates which involves iterative implementation and may not even converge for some sets of data. Mak and Ng (1981) used the linear model approach of Kempthorne and Tandon (1953) to obtain the maximum likelihood

Received October 2005; accepted October 2006.
[1]Office of Directorate, KOSEF(Korea Science and Engineering Foundation), Daejeon 305-350, Korea (e-mail: prettyheo@kosef.re.kr)
[2]Corresponding author. Department of Statistics, Chung-Ang University, Seoul 156-756, Korea (e-mail: gogators@cau.ac.kr)

estimates. However, nothing is known about the convergence of the procedure. Because families have varying number of offspring, the maximum likelihood estimators of correlations are difficult to compute. To avoid this difficulty, several estimators have been proposed. For example, Srivastava (1984) transformed the data and then proposed two alternative estimators. Srivastava and Keen (1988) developed a noniterative method for estimating the interclass correlation coefficient which is derived from the technique of weighted sum of squares. Also, Gleser (1992) provided the formulas for the maximum likelihood estimators of the parameters based on samples from each family, and then combining them in some arbitrary way over the different families.

In this paper, we attempt the Bayesian analysis of familial data in the case when the families considered have equal number of offspring. To this end, we have used certain noninformative priors including the widely used Jeffreys' prior as well as the different reference priors of Berger and Bernardo (1989, 1992a, 1992b). Also, probability matching priors are considered. In Section 2, we find information matrix and the Jeffreys' prior, and also the different reference priors. Also, a simultaneously-marginally-probability-matching prior is derived which is the same as the five group reference prior. In Section 3, we establish the propriety of posteriors under a general class of priors which includes two-group and five-group reference priors and Jeffreys' prior under certain conditions. In Section 4, some simulations are undertaken for comparing reference priors with Jeffreys' prior. The Bayesian procedure is implemented by Gibbs sampler.

2. Development of Noninformative Priors

2.1. Fisher information matrix and Jeffreys' prior

Suppose that there is a family with k offspring, let Y denote the measurement on the mother and $\boldsymbol{X} = (x_1, x_2, \ldots, x_k)$ be the vector of measurement on the k offspring. Further $\mathbf{1}_k$ denotes a $k \times 1$ vector of ones, $\boldsymbol{I}_k$ a $k \times k$ identity matrix and $\boldsymbol{J}_k$ a $k \times k$ matrix containing only ones. It is assumed that

$$\begin{pmatrix} Y \\ \boldsymbol{X} \end{pmatrix} \sim N\left[\begin{pmatrix} \mu_m \\ \mu_s \mathbf{1}_k \end{pmatrix}, \begin{pmatrix} \sigma_m^2 & \rho_{ms}\sigma_m\sigma_s\mathbf{1}_k^T \\ \rho_{ms}\sigma_m\sigma_s\mathbf{1}_k & \sigma_s^2\left\{(1-\rho_{ss})\boldsymbol{I}_k + \rho_{ss}\boldsymbol{J}_k\right\} \end{pmatrix}\right],$$

where ρ_{ms} is an interclass correlation coefficient and ρ_{ss} is an intraclass correlation coefficient. Letting

$$\boldsymbol{\Sigma} \equiv \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} = \begin{bmatrix} \sigma_m^2 & \rho_{ms}\sigma_m\sigma_s\mathbf{1}_k^T \\ \rho_{ms}\sigma_m\sigma_s\mathbf{1}_k & \sigma_s^2\left\{(1-\rho_{ss})\boldsymbol{I}_k + \rho_{ss}\boldsymbol{J}_k\right\} \end{bmatrix}, \tag{2.1}$$

then the joint *pdf* of Y and $\boldsymbol{X}$ for the assumed familial model is given by

$$f(y, \boldsymbol{x}; \mu_m, \mu_s, \sigma_m, \sigma_s, \rho_{ms}, \rho_{ss})$$

$$= (2\pi)^{-\frac{k+1}{2}}|\boldsymbol{\Sigma}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(y-\mu_m, \boldsymbol{x}-\mu_s\mathbf{1}_k)^T\boldsymbol{\Sigma}^{-1}(y-\mu_m, \boldsymbol{x}-\mu_s\mathbf{1}_k)\right\}$$

$$= (2\pi)^{-\frac{k+1}{2}}|\boldsymbol{\Sigma}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\Big[(y-\mu_m)^2\boldsymbol{\Sigma}_{11\cdot2}^{-1} + (\boldsymbol{x}-\mu_s\mathbf{1}_k)^T\boldsymbol{\Sigma}_{22\cdot1}^{-1}(\boldsymbol{x}-\mu_s\mathbf{1}_k)\right.$$

$$\left. -2(y-\mu_m)\boldsymbol{\Sigma}_{11\cdot2}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}-\mu_s\mathbf{1}_k)\Big]\right\},$$

where $\boldsymbol{\Sigma}_{11\cdot2} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$ and $\boldsymbol{\Sigma}_{22\cdot1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$.

In this case,

$$|\boldsymbol{\Sigma}| = |\boldsymbol{\Sigma}_{11\cdot2}|\,|\boldsymbol{\Sigma}_{22}| = \sigma_m^2\sigma_s^{-2k}(1-\rho_{ss})^{k-1}[1+(k-1)\rho_{ss} - k\rho_{ms}^2],$$

$$\boldsymbol{\Sigma}_{11\cdot2}^{-1} = \sigma_m^{-2}\left[\frac{1+(k-1)\rho_{ss}}{1+(k-1)\rho_{ss}-k\rho_{ms}^2}\right],$$

$$\boldsymbol{\Sigma}_{22\cdot1}^{-1} = \frac{1}{\sigma_s^2(1-\rho_{ss})}\left[\boldsymbol{I}_k - \frac{\rho_{ss}-\rho_{ms}^2}{1+(k-1)\rho_{ss}-k\rho_{ms}^2}\right],$$

$$\boldsymbol{\Sigma}_{11\cdot2}^{-1}\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1} = \frac{\rho_{ms}}{\sigma_m\sigma_s[1+(k-1)\rho_{ss}-k\rho_{ms}^2]}\mathbf{1}_k^T.$$

Note that in order that all covariance matrices in (2.1) are positive definite, it must be assumed that

$$\sigma_m^2, \sigma_s^2 > 0, \quad -\frac{1}{k-1} < \rho_{ss} < 1, \quad \frac{\rho_{ms}^2 - \rho_{ss}}{1-\rho_{ss}} \leq \frac{1}{k}.$$

The stronger conditions $0 \leq \rho_{ss} < 1$, $\rho_{ms}^2 \leq \rho_{ss}$ are imposed by Srivastava and Keen (1988).

On simplification, the above reduces to

$$f(y, \boldsymbol{x}; \mu_m, \mu_s, \sigma_m, \sigma_s, \rho_{ms}, \rho_{ss})$$

$$= \left\{(2\pi\sigma_m^2)^{-\frac{1}{2}}\exp\left[-\frac{(y-\mu_m)^2}{2\sigma_m^2}\right]\right\}$$

$$\times\Big\{(2\pi\sigma_s^2(1-\rho_{ss}))^{-\frac{k-1}{2}}\exp\Big[-\frac{\sum_{i=1}^k(x_i-\bar{x})^2}{2\sigma_s^2(1-\rho_{ss})}\Big]\Big\}$$
$$\times\Big[2\pi\sigma_s^2(1+(k-1)\rho_{ss}-k\rho_{ms}^2)\Big]^{-\frac{1}{2}}$$
$$\times\Big\{\exp\Big[-\frac{k}{2\sigma_s^2[1+(k-1)\rho_{ss}-k\rho_{ms}^2]}\Big((\bar{x}-\mu_s)-\frac{\rho_{ms}\sigma_s}{\sigma_m}(y-\mu_m)\Big)^2\Big]\Big\}.$$

Consider the following reparameterization of $(\mu_m,\mu_s,\sigma_m,\sigma_s,\rho_{ms},\rho_{ss})$.

$$\beta=\frac{\rho_{ms}\sigma_s}{\sigma_m},$$
$$\sigma_1^2=\sigma_s^2(1-\rho_{ss}),$$
$$\sigma_2^2=\sigma_s^2[1+(k-1)\rho_{ss}-k\rho_{ms}^2].$$

With this transformation, the joint *pdf* of Y and $\boldsymbol{X}$ reduces to

$$f(y,\boldsymbol{x};\mu_m,\mu_s,\beta,\sigma_m,\sigma_1,\sigma_2)$$
$$=\Big\{(2\pi\sigma_m^2)^{-\frac{1}{2}}\exp\Big[-\frac{(y-\mu_m)^2}{2\sigma_m^2}\Big]\Big\}\Big\{(2\pi\sigma_1^2)^{-\frac{k-1}{2}}\exp\Big[-\frac{\sum_{i=1}^k(x_i-\bar{x})^2}{2\sigma_1^2}\Big]\Big\}$$
$$\times\Big\{(2\pi\sigma_2^2)^{-\frac{1}{2}}\exp\Big[-\frac{k}{2\sigma_2^2}\Big(\bar{x}-\mu_s-\beta(y-\mu_m)\Big)^2\Big]\Big\}.$$

As one knows, Jeffreys' prior is proportional to the positive square root of the determinant of the Fisher information matrix. The derivation of the reference priors also stems from the same matrix. Thus, we find the per unit Fisher information matrix under the new parametrization $\boldsymbol{\theta}=(\mu_m,\mu_s,\beta,\sigma_m,\sigma_1,\sigma_2)$ as

$$\boldsymbol{I}(\boldsymbol{\theta})=\begin{bmatrix}\boldsymbol{I}_2^* & \mathbf{0}^T\\ \mathbf{0} & \boldsymbol{I}_4^*\end{bmatrix}, \tag{2.2}$$

where

$$\boldsymbol{I}_2^*=\begin{bmatrix}\frac{1}{\sigma_m^2}+\frac{k\beta^2}{\sigma_2^2} & -\frac{k\beta}{\sigma_2^2}\\ -\frac{k\beta}{\sigma_2^2} & -\frac{k}{\sigma_2^2}\end{bmatrix},\quad \boldsymbol{I}_4^*=\mathrm{diag}\Big\{\frac{k\sigma_m^2}{\sigma_2^2},\frac{2}{\sigma_m^2},\frac{2(k-1)}{\sigma_1^2},\frac{2}{\sigma_2^2}\Big\}.$$

Thus, Jeffrey' prior is given by

$$\pi_J(\boldsymbol{\theta})\propto\sigma_m^{-1}\sigma_1^{-1}\sigma_2^{-3}\ .$$

REMARK 2.1. Following Bernardo (1979), if the parameter of interest is $\boldsymbol{\theta}$ (no nuisance parameter), then Jeffreys' prior is a reference prior.

2.2. Reference priors

In view of the form of the information matrix as given in (2.2), by taking rectangular compacts for $(\mu_m, \mu_s, \beta, \sigma_m, \sigma_1, \sigma_2)$ it follows from Berger and Bernardo (1992a) or Datta and Ghosh (1995) that the five group reference prior for $\{(\mu_m, \mu_s), \beta, \sigma_m, \sigma_1, \sigma_2\}$ is $\pi_{R5}(\boldsymbol{\theta}) \propto \sigma_m^{-1}\sigma_1^{-1}\sigma_2^{-1}$ while, the two group reference prior with the ordering $\{(\mu_m, \mu_s), (\beta, \sigma_m, \sigma_1, \sigma_2)\}$ or $\{(\beta, \sigma_m, \sigma_1, \sigma_2), (\mu_m, \mu_s)\}$ is $\pi_{R2}(\boldsymbol{\theta}) \propto \sigma_1^{-1}\sigma_2^{-2}$.

REMARK 2.2. Due to invariance of noninformative priors (Datta and Ghosh, 1996; Mukerjee and Ghosh, 1997), the five group reference prior, the two group reference prior and Jeffreys' prior in $(\mu_m, \mu_s, \sigma_m, \sigma_s, \rho_{ms}, \rho_{ss})$ parametrization are given respectively by

$$\begin{aligned}
\pi_{R5} &\propto \sigma_m^{-2}(1-\rho_{ss})^{-1}[1+2(k-1)\rho_{ss}][1+(k-1)\rho_{ss}-k\rho_{ms}^2]^{-1},\\
\pi_{R2} &\propto \sigma_m^{-1}\sigma_s^{-1}(1-\rho_{ss})^{-1}[1+2(k-1)\rho_{ss}][1+(k-1)\rho_{ss}-k\rho_{ms}^2]^{-\frac{3}{2}},\\
\pi_J &\propto \sigma_m^{-2}\sigma_s^{-2}(1-\rho_{ss})^{-1}[1+2(k-1)\rho_{ss}][1+(k-1)\rho_{ss}-k\rho_{ms}^2]^{-2}.
\end{aligned}$$

2.3. Probability matching priors

For the specific familial data, the parameter of interest is $\boldsymbol{\theta} \equiv (\theta_1, \ldots, \theta_6) = (\mu_m, \mu_s, \beta, \sigma_m, \sigma_1, \sigma_2)$. We find a prior which satisfies probability matching criterion separately for each component of the parameter vector $\boldsymbol{\theta}$. Such a prior is referred to as a simultaneously-marginally-probability-matching prior for the different components of $\boldsymbol{\theta}$. From Datta and Ghosh (1995), such a prior $\pi(\boldsymbol{\theta})$ for the parametric function $\boldsymbol{t}(\boldsymbol{\theta}) = (t_1(\boldsymbol{\theta}), \ldots, t_s(\boldsymbol{\theta}))^T$ is found as a solution of

$$\sum_{i=1}^{p} \frac{\partial}{\partial\theta_i}\{\eta_{ji}(\boldsymbol{\theta})\pi(\boldsymbol{\theta})\} = 0 \quad j = 1, \ldots, s, \tag{2.3}$$

where

$$\eta_j(\boldsymbol{\theta}) = \Big(\eta_{j1}(\boldsymbol{\theta}), \ldots, \eta_{jp}(\boldsymbol{\theta})\Big) = \Big\{\nabla_{t_j}^T(\boldsymbol{\theta})\boldsymbol{I}^{-1}(\boldsymbol{\theta})\nabla_{t_j}(\boldsymbol{\theta})\Big\}^{-1/2}\boldsymbol{I}^{-1}(\boldsymbol{\theta})\nabla_{t_j}(\boldsymbol{\theta})$$

and

$$\nabla_{t_j}(\boldsymbol{\theta}) = (\partial t_j(\boldsymbol{\theta})/\partial\theta_1, \ldots, \partial t_j(\boldsymbol{\theta})/\partial\theta_p) \quad j = 1, \ldots, s.$$

For the specific problem, $\boldsymbol{t}(\boldsymbol{\theta}) = (\theta_1, \ldots, \theta_6) = (\mu_m, \mu_s, \beta, \sigma_m, \sigma_1, \sigma_2)$. From Datta and Ghosh (1995), one has the following theorem.

THEOREM 2.1. *A simultaneously-marginally-probability-matching prior for each component of* $\boldsymbol{\theta}$ *is given by*

$$\pi(\boldsymbol{\theta}) \propto \sigma_m^{-1}\sigma_1^{-1}\sigma_2^{-1},$$

which is same as the five group reference prior π_{R5}.

PROOF. Let $t_j(\boldsymbol{\theta}) = \theta_j$, $j = 1, \ldots, 6$ and $\pi(\boldsymbol{\theta}) \propto \sigma_m^{-1}\sigma_1^{-1}\sigma_2^{-1}$. Then

$$\begin{aligned}
&\eta_1(\boldsymbol{\theta}) = \sigma_m(1,\ \beta,\ 0,\ 0,\ 0,\ 0)^T,\\
&\eta_2(\boldsymbol{\theta}) = (\beta\sigma_m^2,\ (\sigma_2^2 + k\beta^2\sigma_m^2)/k,\ 0,\ 0,\ 0,\ 0)^T(\sigma_2^2 + k\beta^2\sigma_m^2)^{-\frac{1}{2}}\sqrt{k},\\
&\eta_3(\boldsymbol{\theta}) = \left(0,\ 0,\ \frac{\sigma_2}{\sigma_m},\ 0,\ 0,\ 0,\ 0\right)^T/\sqrt{k},\\
&\eta_4(\boldsymbol{\theta}) = (0,\ 0,\ 0,\ \sigma_m,\ 0,\ 0)^T/\sqrt{2},\\
&\eta_5(\boldsymbol{\theta}) = (0,\ 0,\ 0,\ 0,\ \sigma_1, 0)^T/\sqrt{2(k-1)},\\
\text{and}\quad &\eta_6(\boldsymbol{\theta}) = (0,\ 0,\ 0,\ 0,\ 0,\ \sigma_2)^T/\sqrt{2}.
\end{aligned}$$

Thus (2.3) holds and $\pi(\boldsymbol{\theta}) \propto \sigma_m^{-1}\sigma_1^{-1}\sigma_2^{-1}$ is a simultaneously-marginally-probabilit y-matching prior. □

3. PROPRIETY OF THE POSTERIOR DISTRIBUTIONS

First we find the joint posterior distribution of $(\mu_m, \mu_s, \sigma_m, \sigma_s, \rho_{ms}, \rho_{ss})$ under the five group reference prior. Suppose $(Y_1, \boldsymbol{X}_1)^T, (Y_2, \boldsymbol{X}_2)^T, \ldots, (Y_n, \boldsymbol{X}_n)^T$ are *i.i.d.* random vectors from the above familial data. Then the likelihood function is given by

$$\begin{aligned}
&L(\mu_m, \mu_s, \beta, \sigma_m, \sigma_1, \sigma_2)\\
&\propto \left\{\sigma_m^{-n} \exp\left[-(2\sigma_m^2)^{-1}\sum_{j=1}^{n}(y_j - \mu_m)^2\right]\right\}\\
&\quad\times\left\{\sigma_1^{-n(k-1)} \exp\left[-(2\sigma_1^2)^{-1}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{x}_j)^2\right]\right\}\\
&\quad\times\left\{\sigma_2^{-n} \exp\left[-k(2\sigma_2^2)^{-1}\sum_{j=1}^{n}\Big((\bar{x}_j - \mu_s) - \beta(y_j - \mu_m)\Big)^2\right]\right\}.
\end{aligned}$$

Writing $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ and $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$, under the five group reference prior, the joint posterior density is given by

$$\begin{aligned}
&\pi_{R5}(\mu_m,\mu_s,\beta,\sigma_m,\sigma_1,\sigma_2|\boldsymbol{y},\boldsymbol{x})\\
&\propto \Big\{\sigma_m^{-n-1}\exp\Big[-(2\sigma_m^2)^{-1}\sum_{j=1}^{n}(y_j-\mu_m)^2\Big]\Big\}\\
&\times\Big\{\sigma_1^{-n(k-1)-1}\exp\Big[-(2\sigma_1^2)^{-1}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\bar{x}_j)^2\Big]\Big\}\\
&\times\Big\{\sigma_2^{-n-1}\exp\Big[-k(2\sigma_2^2)^{-1}\sum_{j=1}^{n}\{(\bar{x}_j-\mu_s)-\beta(y_j-\mu_m)\}^2\Big]\Big\}\,.
\end{aligned}$$

The following theorem proves the propriety of $\pi_{R5}(\mu_m,\mu_s,\beta,\sigma_m,\sigma_1,\sigma_2 \mid \boldsymbol{y},\boldsymbol{x})$.

THEOREM 3.1. *$\pi_{R5}(\mu_m,\mu_s,\beta,\sigma_m,\sigma_1,\sigma_2|\boldsymbol{y},\boldsymbol{x})$ is proper.*

PROOF. First we write

$$\begin{aligned}
&\sum_{j=1}^{n}\Big[(\bar{x}_j-\mu_s)-\beta(y_j-\mu_m)\Big]^2\\
&=n\mu_s^2-2\mu_s\sum_{j=1}^{n}\Big[\bar{x}_j-\beta(y_j-\mu_m)\Big]+\sum_{j=1}^{n}\Big[\bar{x}_j-\beta(y_j-\mu_m)\Big]^2\\
&=n\Big\{\mu_s-\frac{1}{n}\sum_{j=1}^{n}\Big[\bar{x}_j-\beta(y_j-\mu_m)\Big]\Big\}^2+\sum_{j=1}^{n}\Big[\bar{x}_j-\beta(y_j-\mu_m)\Big]^2\\
&\quad-\frac{1}{n}\Big\{\sum_{j=1}^{n}\Big[\bar{x}_j-\beta(y_j-\mu_m)\Big]\Big\}^2.
\end{aligned}$$

Integration with respect to μ_s, the joint posterior of $(\mu_m,\beta,\sigma_m,\sigma_1,\sigma_2)$ is given by

$$\begin{aligned}
&\pi_{R5}(\mu_m,\beta,\sigma_m,\sigma_1,\sigma_2|\boldsymbol{y},\boldsymbol{x})\\
&\propto \Big\{\sigma_m^{-n-1}\exp\Big[-(2\sigma_m^2)^{-1}\sum_{j=1}^{n}(y_j-\mu_m)^2\Big]\Big\}\\
&\times\Big\{\sigma_1^{-n(k-1)-1}\exp\Big[-(2\sigma_1^2)^{-1}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\bar{x}_j)^2\Big]\Big\}\\
&\times\Big\{\sigma_2^{-n}\exp\Big[-k(2\sigma_2^2)^{-1}\Big\{\sum_{j=1}^{n}[\bar{x}_j-\beta(y_j-\mu_m)]^2\\
&\quad-\frac{1}{n}\Big[\sum_{j=1}^{n}[\bar{x}_j-\beta(y_j-\mu_m)]\Big]^2\Big\}\Big]\Big\}\,.
\end{aligned}$$

Now,

$$\sum_{j=1}^{n}\Big[\bar{x}_j-\beta(y_j-\mu_m)\Big]^2-\frac{1}{n}\Big[\sum_{j=1}^{n}[\bar{x}_j-\beta(y_j-\mu_m)]\Big]^2$$
$$=\sum_{j=1}^{n}\bar{x}_j^2-2\beta\sum_{j=1}^{n}\bar{x}_j(y_j-\mu_m)+\beta^2\sum_{j=1}^{n}(y_j-\mu_m)^2-n\Big[\bar{x}-\beta(\bar{y}-\mu_m)\Big]^2$$
$$=\beta^2\Big[\sum_{j=1}^{n}(y_j-\mu_m)^2-n(\bar{y}-\mu_m)^2\Big]$$
$$-2\beta\Big[\sum_{j=1}^{n}\bar{x}_j(y_j-\mu_m)-n\bar{x}(\bar{y}-\mu_m)\Big]+\sum_{j=1}^{n}(\bar{x}_j-\bar{x})^2$$
$$=\sum_{j=1}^{n}(y_j-\bar{y})^2\Big[\beta-\frac{\sum_{j=1}^{n}(\bar{x}_j-\bar{x})(y_j-\bar{y})}{\sum_{j=1}^{n}(y_j-\bar{y})^2}\Big]^2$$
$$-\frac{\Big[\sum_{j=1}^{n}(\bar{x}_j-\bar{x})(y_j-\bar{y})\Big]^2}{\sum_{j=1}^{n}(y_j-\bar{y})^2}+\sum_{j=1}^{n}(\bar{x}_j-\bar{x})^2.$$

Thus, after integrating with respect to β, the joint posterior of $(\mu_m,\sigma_m,\sigma_1,\sigma_2)$ is obtained as

$$\pi_{R5}(\mu_m,\sigma_m,\sigma_1,\sigma_2|\boldsymbol{y},\boldsymbol{x})$$
$$\propto\Big\{\sigma_m^{-n-1}\exp\Big[-(2\sigma_m^2)^{-1}\sum_{j=1}^{n}(y_j-\mu_m)^2\Big]\Big\}$$
$$\times\Big\{\sigma_1^{-n(k-1)-1}\exp\Big[-(2\sigma_1^2)^{-1}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\bar{x}_j)^2\Big]\Big\}$$
$$\times\Big\{\sigma_2^{-n+1}\exp\Big[-k(2\sigma_2^2)^{-1}\Big(S_{xx}-\frac{S_{xy}^2}{S_{yy}}\Big)\Big]\Big\},$$

where $S_{xx}=\sum_{j=1}^{n}(\bar{x}_j-\bar{x})^2$, $S_{xy}=\sum_{j=1}^{n}(\bar{x}_j-\bar{x})(y_j-\bar{y})$, and $S_{yy}=\sum_{j=1}^{n}(y_j-\bar{y})^2$. Integration with respect to μ_m yields the joint posterior of $(\sigma_m,\sigma_1,\sigma_2)$ as

$$\pi_{R5}(\sigma_m,\sigma_1,\sigma_2|\boldsymbol{y},\boldsymbol{x})$$
$$\propto\Big\{\sigma_m^{-n}\exp\Big[-(2\sigma_m^2)^{-1}S_{yy}\Big]\Big\}\Big\{\sigma_1^{-n(k-1)}\exp\Big[-(2\sigma_1^2)^{-1}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\bar{x}_j)^2\Big]\Big\}$$
$$\times\Big\{\sigma_2^{-(n-1)}\exp\Big[-k(2\sigma_2^2)^{-1}\Big(S_{xx}-\frac{S_{xy}^2}{S_{yy}}\Big)\Big]\Big\}.$$

Note now that σ_m, σ_1, and σ_2 have independent inverse gamma posteriors. Hence, the joint posterior density of $(\mu_m, \mu_s, \beta, \sigma_m, \sigma_1, \sigma_2)$ under five group reference prior is integrable and the result follows. □

Second, we consider the joint posterior *pdf* of $(\mu_m, \mu_s, \beta, \sigma_m, \sigma_1, \sigma_2)$ under two group reference prior. Here the joint posterior density is given by

$$\begin{aligned}
&\pi_{R2}(\mu_m, \mu_s, \beta, \sigma_m, \sigma_1, \sigma_2 | \boldsymbol{y}, \boldsymbol{x}) \\
&\propto \left\{\sigma_m^{-n} \exp\left[-(2\sigma_m^2)^{-1} \sum_{j=1}^{n} (y_j - \mu_m)^2\right]\right\} \\
&\quad \times \left\{\sigma_1^{-n(k-1)-1} \exp\left[-(2\sigma_1^2)^{-1} \sum_{i=1}^{k}\sum_{j=1}^{n} (x_{ij} - \bar{x}_j)^2\right]\right\} \\
&\quad \times \left\{\sigma_2^{-n-2} \exp\left[-k(2\sigma_2^2)^{-1} \sum_{j=1}^{n} \left[(\bar{x}_j - \mu_s) - \beta(y_j - \mu_m)\right]^2\right]\right\}.
\end{aligned}$$

As in Theorem 3.1, it can be checked that

$$\begin{aligned}
&\pi_{R2}(\sigma_m, \sigma_1, \sigma_2 | \boldsymbol{y}, \boldsymbol{x}) \\
&\propto \left\{\sigma_m^{-n+1} \exp\left[-(2\sigma_m^2)^{-1} S_{yy}\right]\right\} \\
&\quad \times \left\{\sigma_1^{-n(k-1)-1} \exp\left[-(2\sigma_1^2)^{-1} \sum_{i=1}^{k}\sum_{j=1}^{n} (x_{ij} - \bar{x}_j)^2\right]\right\} \\
&\quad \times \left\{\sigma_2^{-n} \exp\left[-k(2\sigma_2^2)^{-1} \left(S_{xx} - \frac{S_{xy}^2}{S_{yy}}\right)\right]\right\}.
\end{aligned}$$

The propriety of $\pi_{R2}(\mu_m, \mu_s, \beta, \sigma_m, \sigma_1, \sigma_2)$ immediately follows.

Finally, we consider the joint posterior *pdf* of $(\mu_m, \mu_s, \beta, \sigma_m, \sigma_1, \sigma_2)$ under Jeffreys' prior. Under the Jeffreys' prior, the joint posterior *pdf* is given by

$$\begin{aligned}
&\pi_J(\mu_m, \mu_s, \beta, \sigma_m, \sigma_1, \sigma_2 | \boldsymbol{y}, \boldsymbol{x}) \\
&\propto \left\{\sigma_m^{-n-1} \exp\left[-(2\sigma_m^2)^{-1} \sum_{j=1}^{n} (y_j - \mu_m)^2\right]\right\} \\
&\quad \times \left\{\sigma_1^{-n(k-1)-1} \exp\left[-(2\sigma_1^2)^{-1} \sum_{i=1}^{k}\sum_{j=1}^{n} (x_{ij} - \bar{x}_j)^2\right]\right\} \\
&\quad \times \left\{\sigma_2^{-n-3} \exp\left[-k(2\sigma_2^2)^{-1} \sum_{j=1}^{n} [(\bar{x}_j - \mu_s) - \beta(y_j - \mu_m)]^2\right]\right\}.
\end{aligned}$$

As before,

$$
\begin{aligned}
&\pi_J(\sigma_m,\sigma_1,\sigma_2|\boldsymbol{y},\boldsymbol{x})\\
&\propto \Big\{\sigma_m^{-n}\exp\Big[-(2\sigma_m^2)^{-1}S_{yy}\Big]\Big\}\\
&\quad\times\Big\{\sigma_1^{-n(k-1)-1}\exp\Big[-(2\sigma_1^2)^{-1}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\bar{x}_j)^2\Big]\Big\}\\
&\quad\times\Big\{\sigma_2^{-n-1}\exp\Big[-k(2\sigma_2^2)^{-1}\Big(S_{xx}-\frac{S_{xy}^2}{S_{yy}}\Big)\Big]\Big\}
\end{aligned}
$$

and the propriety of $\pi_J(\mu_m,\mu_s,\beta,\sigma_m,\sigma_1,\sigma_2 \mid \boldsymbol{y},\boldsymbol{x})$ follows.

4. Simulation Study

4.1. Method

In this section we compare two group (π_{R2}) and five group (π_{R5}) reference priors along with Jeffreys' prior (π_J). We accomplish this by calculating the frequentist coverage probability of the posterior tail probabilities of each component of the parameter vector $\boldsymbol{\theta}_1 = (\mu_m,\mu_s,\sigma_m,\sigma_s,\rho_{ms},\rho_{ss})$. For example, we consider the parameter ρ_{ms} which is often of great interest to biological and medical researchers.

The computing work is accomplished in three stages. In the first stage, we generate 1,000 random samples of size $n = 20$ (or) 50 from the familial distributions. The second stage consists of computation of posterior α-quantile of the parameter for each of 1,000 sets of random samples using the Gibbs sampler. In the third stage, we compute the coverage probability.

As discussed above, the Gibbs sampler is used to compute the posterior α-quantiles of the parameters given $(Y_1,\boldsymbol{X}_1)^T,(Y_2,\boldsymbol{X}_2)^T,\ldots,(Y_n,\boldsymbol{X}_n)^T$. To this end, we need to generate random variables from the marginal posterior distribution of each component of the parameter vector $\boldsymbol{\theta}$. Such distributions are analytically intractable and requires high-dimensional numerical integration. Instead, we adopt Monte Carlo integration and use Gibbs sampling. Gibbs sampling, originally introduced by Geman and Geman (1984) and more recently popularized by Gelfand and Smith (1990), is a Markovian updating scheme that requires sampling from full conditional distributions. In implementing the Gibbs sampler, we follow the recommendation of Gelman and Rubin (1992) and run $n(\geq 2)$ parallel chains, each for $2d$ iterations with starting points drawn from an over dispersed distribution. But to diminish the effects of the starting distributions, the first d iterations of each chain are discarded. After d iterations, all subsequent iterates are retained for finding the desired posterior distributions as well as for monitoring the convergence of the Gibbs sampler.

For the given familial data, to implement and monitor the convergence of the Gibbs sampler, we consider $n = 10$ parallel chains, each for $2d = 400$ iterations with starting point drawn from an over-dispersed distribution. The implementation requires generation of samples from the following full conditional distributions.

(I) Full conditionals under the two group reference prior:

$$\mu_m | \boldsymbol{y}, \mu_s, \beta, \sigma_m, \sigma_1, \sigma_2$$
$$\sim N\left[\frac{\sum_{j=1}^n \left(\frac{y_j}{\sigma_m^2} + \frac{k\beta}{\sigma_2^2} y_j - \frac{k\beta}{\sigma_2^2}(\bar{x}_j - \mu_s)\right)}{n/\sigma_m^2 + nk\beta^2/\sigma_2^2}, \left(\frac{n}{\sigma_m^2} + \frac{nk\beta^2}{\sigma_2^2}\right)^{-1}\right],$$

$$\mu_s | \boldsymbol{y}, \mu_m, \beta, \sigma_m, \sigma_1, \sigma_2$$
$$\sim N\left[n^{-1}\sum_{j=1}^n \{\bar{x}_j - \beta(y_j - \mu_m)\}, \frac{\sigma_2^2}{nk}\right],$$

$$\beta | \boldsymbol{y}, \mu_m, \mu_s, \sigma_m, \sigma_1, \sigma_2$$
$$\sim N\left[\frac{\sum_{j=1}^n (y_j - \mu_m)(\bar{x}_j - \mu_s)}{\sum_{j=1}^n (y_j - \mu_m)^2}, \left\{\frac{k}{\sigma_2^2}\sum_{j=1}^n (y_j - \mu_m)^2\right\}^{-1}\right],$$

$$\sigma_m^2 | \boldsymbol{y}, \mu_m, \mu_s, \beta, \sigma_1, \sigma_2$$
$$\sim \text{Inverse Gamma}\left[\frac{n+1}{2}, \frac{1}{2}\sum_{j=1}^n (y_j - \mu_m)^2\right],$$

$$\sigma_1^2 | \boldsymbol{y}, \mu_m, \mu_s, \beta, \sigma_m, \sigma_2$$
$$\sim \text{Inverse Gamma}\left[\frac{n(k-1)}{2}, \frac{1}{2}\sum_{i=1}^k \sum_{j=1}^n (x_{ij} - \bar{x}_j)^2\right],$$

$$\sigma_2^2 | \boldsymbol{y}, \mu_m, \mu_s, \beta, \sigma_m, \sigma_1$$
$$\sim \text{Inverse Gamma}\left[\frac{n-1}{2}, \frac{k}{2}\sum_{j=1}^n \{(\bar{x}_j - \mu_s) - \beta(y_j - \mu_m)\}^2\right].$$

(II) Full conditionals under the five group reference prior:

$$\mu_m|\boldsymbol{y},\mu_s,\beta,\sigma_m,\sigma_1,\sigma_2$$
$$\sim N\left[\frac{\sum_{j=1}^{n}\left(\frac{y_j}{\sigma_m^2}+\frac{k\beta}{\sigma_2^2}y_j-\frac{k\beta}{\sigma_2^2}(\bar{x}_j-\mu_s)\right)}{n/\sigma_m^2+nk\beta^2/\sigma_2^2}\ ,\ \left(\frac{n}{\sigma_m^2}+\frac{nk\beta^2}{\sigma_2^2}\right)^{-1}\right],$$

$$\mu_s|\boldsymbol{y},\mu_m,\beta,\sigma_m,\sigma_1,\sigma_2$$
$$\sim N\left[n^{-1}\sum_{j=1}^{n}\{\bar{x}_j-\beta(y_j-\mu_m)\}\ ,\ \frac{\sigma_2^2}{nk}\right],$$

$$\beta|\boldsymbol{y},\mu_m,\mu_s,\sigma_m,\sigma_1,\sigma_2$$
$$\sim N\left[\frac{\sum_{j=1}^{n}(y_j-\mu_m)(\bar{x}_j-\mu_s)}{\sum_{j=1}^{n}(y_j-\mu_m)^2}\ ,\ \left\{\frac{k}{\sigma_2^2}\sum_{j=1}^{n}(y_j-\mu_m)^2\right\}^{-1}\right],$$

$$\sigma_m^2|\boldsymbol{y},\mu_m,\mu_s,\beta,\sigma_1,\sigma_2$$
$$\sim \text{Inverse Gamma}\left[\frac{n}{2}\ ,\ \frac{1}{2}\sum_{j=1}^{n}(y_j-\mu_m)^2\right],$$

$$\sigma_1^2|\boldsymbol{y},\mu_m,\mu_s,\beta,\sigma_m,\sigma_2$$
$$\sim \text{Inverse Gamma}\left[\frac{n(k-1)}{2}\ ,\ \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij}-\bar{x}_j)^2\right],$$

$$\sigma_2^2|\boldsymbol{y},\mu_m,\mu_s,\beta,\sigma_m,\sigma_1$$
$$\sim \text{Inverse Gamma}\left[\frac{n}{2}\ ,\ \frac{k}{2}\sum_{j=1}^{n}\{(\bar{x}_j-\mu_s)-\beta(y_j-\mu_m)\}^2\right].$$

(III) Full conditionals under the Jeffreys' prior:

$$\mu_m|\boldsymbol{y},\mu_s,\beta,\sigma_m,\sigma_1,\sigma_2$$
$$\sim N\left[\frac{\sum_{j=1}^{n}\left(\frac{y_j}{\sigma_m^2}+\frac{k\beta}{\sigma_2^2}y_j-\frac{k\beta}{\sigma_2^2}(\bar{x}_j-\mu_s)\right)}{n/\sigma_m^2+nk\beta^2/\sigma_2^2}\ ,\ \left(\frac{n}{\sigma_m^2}+\frac{nk\beta^2}{\sigma_2^2}\right)^{-1}\right],$$

$$\mu_s|\boldsymbol{y},\mu_m,\beta,\sigma_m,\sigma_1,\sigma_2$$
$$\sim N\left[n^{-1}\sum_{j=1}^{n}\{\bar{x}_j-\beta(y_j-\mu_m)\}\ ,\ \frac{\sigma_2^2}{nk}\right],$$

$$\beta|\boldsymbol{y}, \mu_m, \mu_s, \sigma_m, \sigma_1, \sigma_2$$
$$\sim N\left[\frac{\sum_{j=1}^n (y_j-\mu_m)(\bar{x}_j-\mu_s)}{\sum_{j=1}^n (y_j-\mu_m)^2}\,,\ \left\{\frac{k}{\sigma_2^2}\sum_{j=1}^n (y_j-\mu_m)^2\right\}^{-1}\right],$$
$$\sigma_m^2|\boldsymbol{y}, \mu_m, \mu_s, \beta, \sigma_1, \sigma_2$$
$$\sim \text{Inverse Gamma}\left[\frac{n}{2}\,,\ \frac{1}{2}\sum_{j=1}^n (y_j-\mu_m)^2\right],$$
$$\sigma_1^2|\boldsymbol{y}, \mu_m, \mu_s, \beta, \sigma_m, \sigma_2$$
$$\sim \text{Inverse Gamma}\left[\frac{n(k-1)}{2}\,,\ \frac{1}{2}\sum_{i=1}^k\sum_{j=1}^n (x_{ij}-\bar{x}_j)^2\right],$$
$$\sigma_2^2|\boldsymbol{y}, \mu_m, \mu_s, \beta, \sigma_m, \sigma_1$$
$$\sim \text{Inverse Gamma}\left[\frac{n-2}{2}\,,\ \frac{k}{2}\sum_{j=1}^n \{(\bar{x}_j-\mu_s)-\beta(y_j-\mu_m)\}^2\right].$$

4.2. Results

The following 3 tables provide the estimated tail probabilities of the posterior distributions of θ_1 under the two-group and five-group reference prior and Jeffreys' prior when the frequentist tail probability is 0.95. We generate random samples of sizes $n = 20$, and 50 from the familial distribution. Throughout, we take $\mu_m = \mu_s = 0$ and $\sigma_m = \sigma_s = 1.0$, but take different values of $(\rho_{ms}, \rho_{ss}) = \{(0.3, 0.7), (0.1, 0.9), (0.5, 0.5)\}$.

The results in tables are shown for three different values of (ρ_{ms}, ρ_{ss}) that two group reference prior is slightly edge over the others in terms of the coverage probability satisfying $0 \le \rho_{ss} < 1$ and $\rho_{ms}^2 \le \rho_{ss}$.

5. CONCLUDING REMARKS

In this paper, we have developed noninformative priors for the familial data when families have the same number of offspring. Two- and five-group reference priors have been derived along with Jeffreys' prior. A five group reference prior is derived which is the same as a simultaneously-marginally-probability matching prior.

TABLE 4.1 *Estimated frequentist coverage probability of the posterior tail probabilities of each component of* $\boldsymbol{\theta}_1$*, when* ρ_{ms}*=0.1 and* ρ_{ss}*=0.9*

	$n = 20$			$n = 50$		
	π_{R2}	π_{R5}	π_J	π_{R2}	π_{R5}	π_J
μ_m	0.958	0.957	0.950	0.940	0.955	0.948
μ_s	0.964	0.963	0.977	0.957	0.950	0.969
σ_m	0.957	0.955	0.945	0.942	0.951	0.942
σ_s	0.919	0.936	0.905	0.956	0.951	0.941
ρ_{ms}	0.954	0.961	0.962	0.959	0.949	0.954
ρ_{ss}	0.937	0.935	0.925	0.957	0.946	0.946

TABLE 4.2 *Estimated frequentist coverage probability of the Posterior tail probabilities of each component of* $\boldsymbol{\theta}_1$*, when* ρ_{ms}*=0.3 and* ρ_{ss}*=0.7*

	$n = 20$			$n = 50$		
	π_{R2}	π_{R5}	π_J	π_{R2}	π_{R5}	π_J
μ_m	0.906	0.917	0.908	0.884	0.864	0.883
μ_s	0.951	0.953	0.959	0.941	0.958	0.961
σ_m	0.932	0.959	0.951	0.940	0.945	0.956
σ_s	0.938	0.938	0.915	0.942	0.935	0.924
ρ_{ms}	0.959	0.949	0.958	0.955	0.945	0.955
ρ_{ss}	0.950	0.956	0.928	0.955	0.939	0.949

TABLE 4.3 *Estimated frequentist coverage probability of the posterior tail probabilities of each component of* $\boldsymbol{\theta}_1$*, when* ρ_{ms}*=0.5 and* ρ_{ss}*=0.5*

	$n = 20$			$n = 50$		
	π_{R2}	π_{R5}	π_J	π_{R2}	π_{R5}	π_J
μ_m	0.805	0.833	0.811	0.807	0.778	0.789
μ_s	0.891	0.896	0.908	0.909	0.864	0.900
σ_m	0.940	0.926	0.935	0.955	0.958	0.958
σ_s	0.945	0.936	0.926	0.942	0.950	0.944
ρ_{ms}	0.945	0.929	0.960	0.955	0.967	0.957
ρ_{ss}	0.959	0.940	0.934	0.948	0.963	0.948

REFERENCES

BERGER, J. O. AND BERNARDO, J. M. (1989). "Estimating a product of means: Bayesian analysis with reference priors", *Journal of the American Statistical Association*, **84**, 200–207.

BERGER, J. O. AND BERNARDO, J. M. (1992a). "On the development of reference priors", *Bayesian Statistics 4*, 35–60, Oxford University Press, New York.

BERGER, J. O. AND BERNARDO, J. M. (1992b). *Reference Priors in a Variance Components Problem*, Springer-Verlag, New York.

BERNARDO, J. M. (1979). "Reference posterior distributions for Bayesian inference", *Journal of the Royal Statistical Society*, Ser B, **41**, 113–147.

DATTA, G. S. AND GHOSH, J. K.(1995). "On priors providing frequentist validity for Bayesian inference", *Biometrika*, **82**, 37–45.

DATTA, G. S. AND GHOSH, M. (1995). "Some remarks on noninformative priors", *Journal of the American Statistical Association*, **90**, 1357–1363.

DATTA, G. S. AND GHOSH, M. (1996). "On the invariance of noninformative priors", *The Annals of Statistics*, **24**, 141–159.

GELFAND, A. E. AND SMITH, A. F. M. (1990). "Sampling-based approaches to calculating marginal densities", *Journal of the American Statistical Association*, **85**, 398–409.

GELMAN, A. AND RUBIN, D. B. (1992). "Inference from iterative simulation using multiple sequences", *Statistical Science*, **7**, 457–472.

GEMAN, S. AND GEMAN, D. (1984). "Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images", *IEEE Transactions on Pattern Analysis and Machine Intelligence*, **6**, 721–741.

GLESER, L. J. (1992). "A note on the analysis of familial data", *Biometrika*, **79**, 412–415.

KEMPTHORNE, O. AND TANDON, O. B. (1953). "The estimation of heritability by regression of offspring on parent", *Biometrics*, **9**, 90–100.

MAK, T. K. AND NG, K. W. (1981). "Analysis of familial data: linear model approach", *Biometrika*, **68**, 457–461.

MUKERJEE, R. AND GHOSH, M. (1997). "Second-order probability matching priors", *Biometrika*, **84**, 970–975.

ROSNER, B. (1979). "Maximum likelihood estimation of interclass correlation", *Biometrika*, **66**, 533–538.

ROSNER, B., DONER, A. AND HENNEKENS, C. H. (1977). "Estimation of interclass correlation from familial data", *Applied Statistics*, **26**, 179–187.

SRIVASTAVA, M. S. (1984). "Estimation of interclass correlations in familial data", *Biometrika*, **71**, 177–185.

SRIVASTAVA, M. S. AND KEEN, K. J. (1988). "Estimation of the interclass correlation coefficient", *Biometrika*, **75**, 731–739.