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Weighted average of fuzzy numbers under T_W(the weakest t-norm)-based fuzzy arithmetic operations

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Abstract

Many authors considered the computational aspect of sup-min convolution when applied to weighted average operations. They used a computational algorithm based on α-cut representation of fuzzy sets, nonlinear programming implementation of the extension principle, and interval analysis. It is well known that T_W(the weakest t-norm)-based addition and multiplication preserve the shape of L-R type fuzzy numbers. In this paper, we consider the computational aspect of the extension principle by the use of T_W when the principle is applied to fuzzy weighted average operations. We give the exact solution for the case where variables and coefficients are L-L fuzzy numbers without programming or the aid of computer resources.

Key words : Extended operations, Fuzzy weighted average, T_W-norm.

1. Introduction

The extension principle introduced by Zadeh[15] is one of the most basic ideas of fuzzy set theory. It provides a general method for extending nonfuzzy mathematical concept in order to deal with fuzzy quantities. The extension principle is systematically applied to real algebra. An algebraic operation encountered in risk and decision analysis is the weighted average operation. In [1], Dong and Wang considered the computational aspect of sup-min convolution when applied to weighted average operations. They used a computational algorithm based on α-cut representation of fuzzy sets, nonlinear programming implementation of the extension principle, and interval analysis. Improved methods devised for calculating fuzzy weighted averages are proposed in [2, 9, 11, 13]. In general, the implementation of the solution procedure is not trivial since the solution procedure corresponds to a nonlinear programming problem which is very complex except for the simplest mapping functions. The procedure is difficult to implement even on a computer. To overcome this difficulties, in this paper, we use T_W, the weakest t-norm, instead of 'min' for fuzzy arithmetic operations based on sup-t-norm convolution when applied to weighted average operations where variables and coefficients are L-L type fuzzy numbers. It is well known that T_W-based addition and multiplication preserves the shape of L-R fuzzy numbers [6, 7, 10, 14] and hence simplifies fuzzy arithmetic operations of fuzzy numbers. As applications of T_W-based fuzzy arithmetic operations, Hong et al. [4, 5] considered fuzzy regression analysis problem, Hong and Do [3] considered fuzzy system reliability analysis problem and Hong [8] considered correlation coefficients of fuzzy numbers. Using the shape preserving and analytic formula of division, we give the exact solution for the case where variables and coefficients are L-L fuzzy numbers without programming or the aid of computer resources.

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2. T_W-based algebraic operation of fuzzy numbers

A fuzzy number is a convex subset of the real line R with a normalized membership function.

A triangular fuzzy number $\tilde{a}$ denoted by (a, α, β) is defined by

$$\tilde{a}(t) = \begin{cases} 1 - \frac{|a-t|}{\alpha} & \text{if } a - \alpha \le t \le a, \\ 1 - \frac{|a-\beta|}{\beta} & \text{if } a \le t \le a + \beta, \\ 0 & \text{otherwise.} \end{cases}$$

where $a \in R$ is the center and $\alpha > 0$ is the left spread, $\beta > 0$ is the right spread of $\tilde{a}$.

If $\alpha = \beta$, then the triangular fuzzy number is called a symmetric triangular fuzzy number and denoted by (a, α).

A fuzzy number $\tilde{a} = (a, \alpha, \beta)_{LR}$ of type L-R is a function from the reals into the interval $[0, 1]$ satisfying

$$\tilde{a} = \begin{cases} R(\frac{t-a}{\beta}) & \text{for } a \le t \le a + \beta, \\ L(\frac{a-t}{\alpha}) & \text{for } a - \alpha \le t \le a, \\ 0 & \text{else.} \end{cases}$$

where L and R are non-increasing and continuous functions from $[0, 1]$ to $[0, 1]$ satisfying $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$.

A binary operation T on the unit interval is said to be a triangular norm [12] (t-norm for short) iff T is associative, commutative, non-decreasing and $T(x, 1) = x$ for each $x \in [0, 1]$. Moreover, every t-norm satisfies the inequality

$$T_W(a, b) \le T(a, b) \le \min(a, b) = T_M$$

where

$$T_W(a, b) = \begin{cases} a & \text{if } b = 1, \\ b & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The critical importance of $\min(a, b)$, $a \cdot b$, $\max(0, a + b - 1)$ and $T_W(a, b)$ is emphasized from a mathematical point of view in [13] among others.

The usual arithmetical operation of reals can be extended to the arithmetical operations on fuzzy numbers by means of Zadeh's extension principle [15] based on a triangular norm T. Let $\tilde{A}, \tilde{B}$ be fuzzy numbers of the real line R. The fuzzy number arithmetic operations are summarized as follows:

Fuzzy number addition $\oplus$:

$$(\tilde{A} \oplus \tilde{B})(z) = \sup_{x+y=z} T(\tilde{A}(x), \tilde{B}(y)).$$

Fuzzy number multiplication $\otimes$:

$$(\tilde{A} \otimes \tilde{B})(z) = \sup_{x \cdot y=z} T(\tilde{A}(x), \tilde{B}(y)).$$

Fuzzy number division $\oslash$:

$$(\tilde{A} \oslash \tilde{B})(z) = \sup_{\frac{x}{y}=z} T(\tilde{A}(x), \tilde{B}(y)).$$

The addition (subtraction) rule for L- R fuzzy numbers is well known in the case of T_M-based addition and then the resulting sum is again on L-R fuzzy numbers, i.e., the shape is preserved. It is also known that T_W-based addition and multiplication preserves the shape of L-R fuzzy numbers [6, 7, 10, 14]. Of course, we know that T_M-based multiplication does not preserve the shape of L-R fuzzy numbers. In this section, we consider T_W-based division of L-R fuzzy numbers.

Let $T = T_W$ be the weakest t-norm and, let $\tilde{A} = (a, \alpha_A, \beta_A)_{LR}$, $\tilde{B} = (b, \alpha_B, \beta_B)_{LR}$ be two L-R fuzzy numbers. By [3, 6, 7, 10, 14],

$$\begin{aligned} \tilde{A} \oplus \tilde{B} &= (a, \alpha_A, \beta_A)_{LR} \oplus (b, \alpha_B, \beta_B)_{LR} \\ &= (a + b, \max(\alpha_A, \alpha_B), \max(\beta_A, \beta_B))_{LR}, \end{aligned} \quad (1)$$

$$\tilde{A} \otimes \tilde{B} = \begin{cases} (ab, \max(\alpha_A b, \alpha_B a), \max(\beta_A b, \beta_B a))_{LR} \\ \qquad \text{for } a, b > 0, \\ (ab, \max(\beta_A b, \beta_B a), \max(\alpha_A b, \alpha_B a))_{RL} \\ \qquad \text{for } a, b < 0, \\ (ab, \max(\alpha_A b - \beta_B a), \max(\beta_A b - \alpha_B a))_{RR} \\ \qquad \text{for } a < 0, b > 0, L = R, \\ (0, \alpha_A b, \beta_A b)_{LR} \\ \qquad \text{for } a = 0, b > 0, \\ (0, -\beta_A b, -\alpha_A b)_{RL} \\ \qquad \text{for } a = 0, b < 0, \\ (0, 0, 0)_{LR} \\ \qquad \text{for } a = 0, b = 0. \end{cases} \quad (2)$$

It $\tilde{A}$ and $\tilde{B}$ are symmetric fuzzy numbers, i.e., $L = R$ and $\alpha_A = \beta_A, \alpha_B = \beta_B$, the multiplication can be simplified as

$$\tilde{A} \otimes \tilde{B} = (ab, \max(\alpha_A|b|, \alpha_B|a|, \max(\alpha_A|b|, \alpha_B|a|)_{LL} \quad (3)$$

For the case of division, we have by [8], for $\tilde{A} = (a, \alpha_A, \beta_A)_{LR}$, $\tilde{B} = (b, \alpha_B, \beta_B)_{RL}$,

Case I : For $a, b > 0$

$$(\tilde{A} \oslash \tilde{B})(z) = \begin{cases} L\left[(a/b - z)/((1/b)\max(\alpha_A, \beta_B z))\right] \\ \qquad \text{if } \min \{(a - \alpha_A)/b, a/(\beta_B + b)\} \le z \le a/b, \\ R\left[(z - a/b)/((1/b)\max(\beta_A, \alpha_B z))\right] \\ \qquad \text{if } \max \{(a + \beta_A)/b, a/(b - \alpha_B)\} \ge z \ge a/b, \\ 0 \quad \text{otherwise.} \end{cases}$$

Case II : For $a < 0, b < 0$

$$(\tilde{A} \oslash \tilde{B})(z) = \begin{cases} R\left[(a/b - z)/((1/b)\max(\beta_B, \alpha_A z))\right] \\ \qquad \text{if } \min \{(a - \beta_B)/b, a/(\alpha_A + b)\} \le z \le a/b, \\ L\left[(z - a/b)/((1/b)\max(\alpha_B, \beta_A z))\right] \\ \qquad \text{if } \max \{(a + \alpha_B)/b, a/(b - \beta_A)\} \ge z \ge a/b, \\ 0 \quad \text{otherwise.} \end{cases}$$

Case III : For $a = 0, b > 0$

$$(\tilde{A} \oslash \tilde{B})(z) = (0, \alpha_A/b, \beta_A/b)_{LR}.$$

Case IV : For $a = 0, b < 0$

$$(\tilde{A} \oslash \tilde{B})(z) = (0, -\beta_A/b, -\alpha_A/b)_{RL}.$$

Case V : For $a < 0, b > 0$, we assume $L = R$ additionally. Then

$$(\tilde{A} \oslash \tilde{B})(z) = \begin{cases} R\left[(a/b - z)/((1/b)\max(\alpha_A, \alpha_B z))\right] \\ \quad \text{if } \min \ \{(a - \alpha_A)/b, a/(b + \alpha_B)\} \le z \le a/b, \\ R\left[(z - a/b)/((1/b)\max(\beta_A, \beta_B z))\right] \\ \quad \text{if } \max \ \{(a + \beta_A)/b, a/(b - \beta_B)\} \ge z \ge a/b, \\ 0 \quad \text{otherwise.} \end{cases}$$

Case VI : For $a > 0, b < 0$, we assume $L = R$. Then

$$(\tilde{A} \oslash \tilde{B})(z) = \begin{cases} R\left[(a/b - z)/((1/b)\max(\beta_A, \beta_B z))\right] \\ \quad \text{if } \min \ \{(a - \beta_B)/b, a/(b + \beta_A)\} \le z \le a/b, \\ R\left[(z - a/b)/((1/b)\max(\alpha_A, \alpha_B z))\right] \\ \quad \text{if } \max \ \{(a + \alpha_A)/b, a/(b - \alpha_B)\} \ge z \ge a/b, \\ 0 \quad \text{otherwise.} \end{cases}$$

Here, we note that $\tilde{A} \oslash \tilde{B}$ is not exact L-R fuzzy number.

Example 2.1 Let $A = (4, 1, 1)$ and $B = (2, 1, 1)$. Then

$$(\tilde{A} \oslash \tilde{B})(z) = \begin{cases} \max\left(1 - \left(\frac{2-z}{1/2}\right), 1 - \left(\frac{2-z}{z/2}\right)\right) \\ \qquad \text{if } 4/3 \le z \le 2, \\ \max\left(1 - \left(\frac{z-2}{1/2}\right), 1 - \left(\frac{z-2}{z/2}\right)\right) \\ \qquad \text{if } 2 \le z \le 4, \\ 0 \qquad \text{otherwise.} \end{cases}$$

$$= \begin{cases} 3 - (4/z) & \text{if } 4/3 \le z \le 2, \\ -1 + (4/z) & \text{if } 2 \le z \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

3. Analysis of weighted average

An extended operation commonly encountered in risk and decision analysis is the weighted average operation. To be specific, suppose the various alternatives to be assessed are denoted by $B_1, B_2, \cdots, B_M$ and the criteria which enter into the evaluation of each alternative have been identified to be $\alpha_1, \alpha_2, \cdots, \alpha_N$. Then for a given alternative B_i, the relative merit of criterion α_j is assessed by a rating, denoted by r_{ij}. Furthermore, the relative importance of each criterion is assessed by a weighting coefficient, w_j for criterion α_j. Then alternative B_j will receive the weighted average rating

$$\bar{r}_i = \frac{\sum_{j=1}^{n} w_j r_{ij}}{\sum_{j=1}^{N} w_j}. \tag{4}$$

When w_j and r_{ij} are represented by fuzzy numbers, the weighted average $\bar{r}_i$ of Eq. (4) involves extended multiplication, addition and division.

This section addresses the computational aspect of the extension principle when applied to the weighted average operation. The algebraic operations based on the weakest t-norm T_W are used. The method provides an exact solution to extended weighted averages in a very efficient and simple manner. The same examples as in [1] are given to illustrate the method.

Consider the weighted average

$$\begin{aligned} Y =& ((W_1 \otimes X_1) \oplus (W_2 \otimes X_2) \oplus \cdots \oplus (W_n \otimes X_n)) \\ & \oslash (W_1 \oplus W_2 \oplus \cdots \oplus W_n) \end{aligned}$$

where $W_i = (w_i, \delta_i, \gamma_i)_{LR}, X_i = (x_i, \alpha_i, \beta_i)_{LR}, i = 1, 2, \cdots, n$.

We assume $w_i, x_i > 0$ and $L = R$ and other cases can be computed in a similar manner, but the expression can be more complicated.

Since

$$W_i \otimes X_i = (w_i x_i, \max(w_i \alpha_i, x_i \delta_i), \max(w_i \beta_i, x_i \gamma_i))_{LL} \quad (i = 1, 2, \cdots, n),$$

we have

$$\begin{aligned} &(W_1 \otimes X_1) \oplus (W_2 \otimes X_2) \oplus \cdots \oplus (W_n \otimes X_n) \\ &= (\sum_{i=1}^{n} w_i x_i, \max_{1 \le i \le n} (w_i \alpha_i, x_i \delta_i), \max_{1 \le i \le n} (w_i \beta_i, x_i \gamma_i))_{LL} \end{aligned}$$

and

$$W_i \oplus \cdots \oplus W_n = (\sum_{i=1}^{n} w_i, \max_{1 \le i \le n} \delta_i, \max_{1 \le i \le n} \gamma_i)_{LL}.$$

Then, we have, by algebraic operations in Section 2,

$$Y(z) = ((W_1 \otimes X_1) \oplus (W_2 \otimes X_2) \oplus \cdots \oplus (W_n \otimes X_n)) \oslash (W_1 \oplus \cdots \oplus W_n)(z)$$

$$= \begin{cases} L\left[\dfrac{\bar{w} - z}{(1/\sum_{i=1}^{n} w_i)\max(\max_{1\le i\le n}(w_i\alpha_i, x_i\delta_i), (\max_{1\le i\le n}\gamma_i)z)}\right] \\ \quad \text{if} \min\left(\bar{w} - \max_{1\le i\le n}(w_i\alpha_i, x_i\delta_i)/\sum_{i=1}^{n} w_i, \dfrac{1}{1/\bar{w} + \max_{1\le i\le n}\gamma_i/\sum_{i=1}^{n} w_i x_i}\right) \le z \le \bar{w}, \\ L\left[\dfrac{z - \bar{w}}{(1/\sum_{i=1}^{n} w_i)\max(\max_{1\le i\le n}\gamma_i, z\max_{1\le i\le n}(w_i\alpha_i, x_i\delta_i))}\right] \\ \quad \text{if} \bar{w} \le z \le \max\left(\bar{w} - \max_{1\le i\le n}\gamma_i/\sum_{i=1}^{n} w_i, \dfrac{1}{1/\bar{w} + \max_{1\le i\le n}(w_i\alpha_i, x_i\delta_i)/\sum_{i=1}^{n} w_i x_i}\right), \\ 0 \quad \text{otherwise} \end{cases} \quad (5)$$

where $\bar{w} = \sum_{i=1}^{n} w_i x_i / \sum_{i=1}^{n} w_i$.

We now consider a two-term weighted average and a three-term weighted average operation. We use the same fuzzy numbers as in Dong and Wong [1].

Example 3.1 Consider the weighted average

$$Y = ((W_1 \otimes X_1) \oplus (W_2 \otimes X_2)) \oslash (W_1 \oplus W_2)$$

where $X_1 = (1, 1, 1), X_2 = (3, 1, 1), W_1 = (0.3, 0.3, 0.6)$ and $W_2 = (0.7, 0.3, 0.3)$. Then $W_1 \oplus W_2 = (1, 0.3, 0.6)$ and $(W_1 \otimes X_1) \oplus (W_2 \otimes X_2) = (2.4, 0.9, 0.9)$, and hence we have

$$((W_1 \otimes X_1) \oplus (W_2 \otimes X_2)) \oslash (W_1 \oplus W_2)(z) = \begin{cases} \frac{10}{9}z - \frac{5}{3} & \text{if } 1.5 \le z \le 2.4, \\ \frac{8}{3z} - \frac{1}{9} & \text{if } 2.4 \le z \le 24, \\ 0 & \text{otherwise. (See Fig.2)} \end{cases}$$

Example 3.2 Adding a term to the average operation introduces two more variables. The operation is now

$$Y = ((W_1 \otimes X_1) \oplus (W_2 \otimes X_2) \oplus (W_3 \otimes X_3)) \oslash (W_1 \oplus W_2 \oplus W_3)$$

where $X_3 = (5, 1, 1)$ and $W_3 = (0.8, 0.2, 0.2)$. By (5), we have

$$Y(z) = ((W_1 \otimes X_1) \oplus (W_2 \otimes X_2) \oplus (W_3 \otimes X_3)) \oslash (W_1 \oplus W_2 \oplus W_3)(z)$$
$$= (6.4, 1, 1) \oslash (1.8, 0.3, 0.6)(z)$$
$$= \begin{cases} 4 - \frac{3.2}{(0.3)z} & \text{if } \frac{8}{3} \le z \le 3.56, \\ -0.8 + \frac{64}{10z} & \text{if } 3.56 \le z \le 8, \\ 0 & \text{otherwise. (See Fig.3)} \end{cases}$$

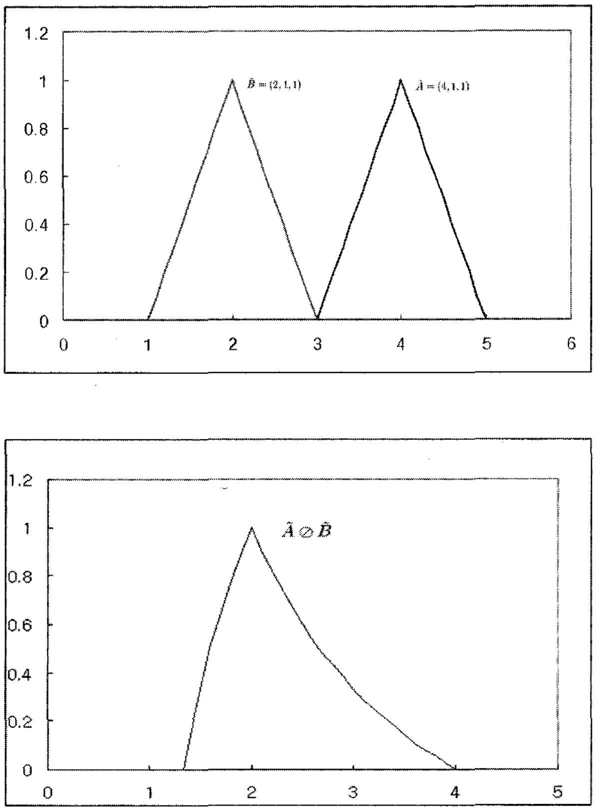

Figure 1. Fuzzy number $\tilde{A}, \tilde{B}, \tilde{A} \oslash \tilde{B}$.

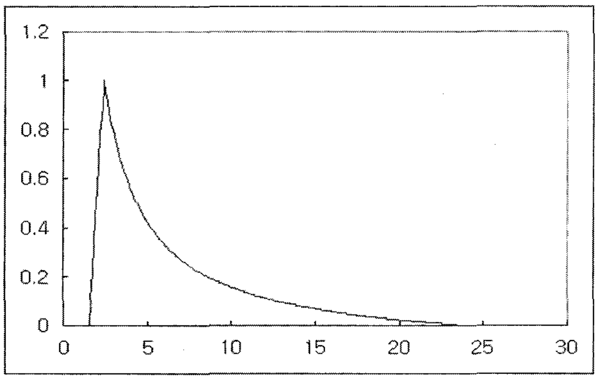

Figure 2. Two-term weighted average.

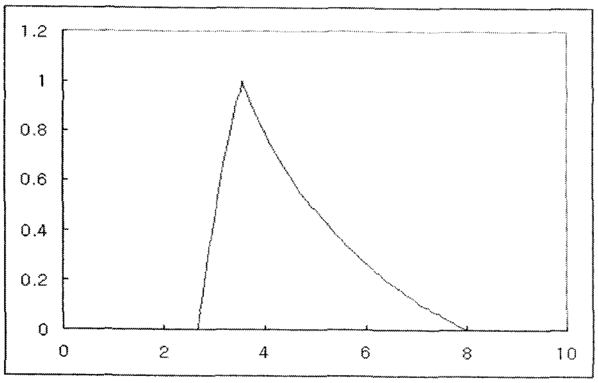

Figure 3. Three-term weighted average.

4. Conclusion

In general, the sup-min convolution by means of extension principle of Zadeh has been used for fuzzy arithmetic in risk and decision analysis. Many authors considered the computational aspect of sup-min convolution when applied to weighted average operations. But "min"-based multiplication does not preserve the shape of fuzzy numbers. They used a computational algorithm based on α-cut representation of fuzzy sets, nonlinear programming implementation of the extension principle, and interval analysis. In this paper, using the shape preserving property of T_W-based addition and multiplication and analytic formula of T_W-based division, we give the exact solution of weighted averages of fuzzy numbers for the case where variables and coefficients are L-L fuzzy numbers without programming or the aid of computer resources. The same examples that Dong and Wong [1] used are shown to illustrate the method.

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