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Common fixed point theorem for a sequence of mappings in intuitionistic fuzzy metric space

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Abstract

Park and Kim [4], Grabiec [1] studied a fixed point theorem in fuzzy metric space, and Vasuki [8] proved a common fixed point theorem in a fuzzy metric space. Park, Park and Kwun [6] defined the intuitionistic fuzzy metric space in which it is a little revised in Park's definition. Using this definition, Park, Kwun and Park [5] and Park, Park and Kwun [7] proved a fixed point theorem in intuitionistic fuzzy metric space. In this paper, we will prove a common fixed point theorem for a sequence of mappings in a intuitionistic fuzzy metric space. Our result offers a generalization of Vasuki's results [8].

Key words : Intuitionistic fuzzy metric space, Common fixed point, A sequence of mapping.

1. Introduction

Park and Kim [4] proved a fixed point theorem in fuzzy metric space, Grabiec [1] studied Banach contraction principle in fuzzy metric in the sense of Kramosil and Michalek [2]. Also, Vasuki [8] proved a common fixed point theorem in a fuzzy metric space.

Recently, Park, Park and Kwun [6] defined the intuitionistic fuzzy metric space in which it is a little revised in Park [3]. Using this definition, Park, Kwun and Park [5] proved a fixed point theorem of Banach for the contractive mapping of a complete intuitionistic fuzzy metric space. Also, Park, Park and Kwun [7] studied a fixed point theorems in the intuitionistic fuzzy metric space.

In this paper, we will prove a common fixed point theorem for a sequence of mappings in a intuitionistic fuzzy metric space. Our result offers a generalization of Vasuki [8].

We shall deal with intuitionistic fuzzy metric space introduced by Park, Park and Kwun [6].

2. Preliminaries

Now, we will give some definitions, properties and notation of the intuitionistic fuzzy metric space.

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Definition 2.1([9]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous $t-$norm if $*$ is satisfying the following conditions:

(a) $*$ is commutative and associative,

(b) $*$ is continuous,

(c) $a * 1 = a$ for all $a \in [0,1]$,

(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0,1])$.

Definition 2.2([9]). A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous $t-$conorm if $\diamond$ is satisfying the following conditions:

(a) $\diamond$ is commutative and associative,

(b) $\diamond$ is continuous,

(c) $a \diamond 1 = a$ for all $a \in [0,1]$,

(d) $a \diamond b \geq c \diamond d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0,1])$.

Definition 2.3([6]). The $5-$tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous $t-$norm, $\diamond$ is a continuous $t-$conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions; for all $x, y, z \in X$, such that

(a) $M(x, y, t) > 0$,

(b) $M(x, y, t) = 1 \iff x = y$,

(c) $M(x, y, t) = M(y, x, t)$,

(d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(e) $M(x,y,\cdot):(0,\infty)\to(0,1]$ is continuous,
(f) $N(x,y,t)>0$,
(g) $N(x,y,t)=0 \Longleftrightarrow x=y$,
(h) $N(x,y,t)=N(y,x,t)$,
(i) $N(x,y,t)\diamond N(y,z,s)\geq N(x,z,t+s)$,
(j) $N(x,y,\cdot):(0,\infty)\to(0,1]$ is continuous.

Then (M,N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

In all that follows **N** stands for the set of natural numbers and X stands for an intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ with the following properties:

$$\lim_{t\to\infty} M(x,y,t)=1,\quad \lim_{t\to\infty} N(x,y,t)=0 \tag{2.1}$$

for all $x,y\in X$.

Definition 2.4([6]). Let X be an intuitionistic fuzzy metric space.

(a) A sequence $\{x_n\}$ in a intuitionistic fuzzy metric space X is called Cauchy if $\lim_{n\to\infty} M(x_{n+p},x_n,t)=1$, $\lim_{n\to\infty} N(x_{n+p},x_n,t)=0$ for every $t>0$ and each $p>0$.

(b) A sequence $\{x_n\}$ in X is convergent to $x\in X$ if $\lim_{n\to\infty} M(x_n,x,t)=1$, $\lim_{n\to\infty} N(x_n,x,t)=0$ for each $t>0$.

(c) X is complete if every Cauchy sequence in X converges in X.

Remark 2.1([6]). The following conditions are satisfied :

(a) For any $r_1,r_2\in(0,1)$ with $r_1>r_2$, there exist $r_3,r_4\in(0,1)$ such that $r_1*r_3\geq r_2$ and $r_4\diamond r_2\leq r_1$.

(b)For any $r_5\in(0,1)$, there exist $r_6,r_7\in(0,1)$ such that $r_6*r_6\geq r_5$ and $r_7\diamond r_7\leq r_5$.

Lemma 2.1([7]). In an intuitionistic fuzzy metric space X, $M(x,y,\cdot)$ is nondecreasing and $N(x,y,\cdot)$ is nonincreasing for all $x,y\in X$.

3. Common fixed point

In this section, we will prove a common fixed point theorem for a sequence of mappings in a intuitionistic fuzzy metric space being an extension of Vasuki [8].

Theorem 3.1. Let $\{T_n\}_n$ be a sequence of mappings of a complete intuitionistic fuzzy metric space X into itself satisfying (2.1). If for any two mappings T_i,T_j, we have

$$\begin{aligned} &M(T_i^m x, T_i^m y, \alpha_{i,j}t)\geq M(x,y,t),\\ &N(T_i^m x, T_i^m y, \alpha_{i,j}t)\leq N(x,y,t)\end{aligned} \tag{3.1}$$

for some m and $0<\alpha_{i,j}<k<1$, $i,j=1,2,\cdots$, $x,y\in X$. Then the sequence $\{T_n\}_n$ has a unique fixed point in X.

Proof. Let $x_0\in X$ and $x_1=T_1^m x_0, x_2=T_2^m x_1,\cdots$. Then for all $p>0$,

$$\begin{aligned} M(x_1,x_2,t)&=M(T_1^m x_0, T_2^m x_1,t)\\ &\geq M(x_0,x_1,\frac{t}{\alpha_{1,2}})\\ M(x_2,x_3,t)&=M(T_2^m x_1, T_3^m x_2,t)\\ &\geq M(x_1,x_2,\frac{t}{\alpha_{2,3}})\\ &\geq M(x_0,x_1,\frac{t}{\alpha_{1,2}\alpha_{2,3}}),\\ N(x_1,x_2,t)&=N(T_1^m x_0, T_2^m x_1,t)\\ &\leq N(x_0,x_1,\frac{t}{\alpha_{1,2}})\\ N(x_2,x_3,t)&=N(T_2^m x_1, T_3^m x_2,t)\\ &\leq N(x_1,x_2,\frac{t}{\alpha_{2,3}})\\ &\leq N(x_0,x_1,\frac{t}{\alpha_{1,2}\alpha_{2,3}}).\end{aligned}$$

By simple induction, we have

$$\begin{aligned} M(x_n,x_{n+1},t)&=M(T_n^m x_{n-1}, T_{n+1}^m x_n,t)\\ &\geq M(x_{n-1},x_n,\frac{t}{\alpha_n,n+1})\\ &\cdots\cdots\\ &\geq M(x_0,x_1,\frac{t}{\Pi_{i=1}^n\alpha_{i,i+1}}),\\ N(x_n,x_{n+1},t)&=N(T_n^m x_{n-1}, T_{n+1}^m x_n,t)\\ &\leq N(x_{n-1},x_n,\frac{t}{\alpha_n,n+1})\\ &\cdots\cdots\\ &\leq N(x_0,x_1,\frac{t}{\Pi_{i=1}^n\alpha_{i,i+1}}).\end{aligned}$$

Thus, for any positive integer p, we have

$$\begin{aligned} &M(x_n,x_{n+p},t)\\ &\geq M(x_n,x_{n+1},\frac{t}{p})*\cdots\\ &\qquad\cdots * M(x_{n+p-1},x_{n+p},\frac{t}{p})\\ &\geq M(x_0,x_1,\frac{t}{p\Pi_{i=1}^n\alpha_{i,i+1}})*\cdots\\ &\qquad\cdots * M(x_0,x_1,\frac{t}{p\Pi_{i=1}^n\alpha_{i,i+1}}),\end{aligned}$$

$$\geq M(x_0, x_1, \frac{t}{p\Pi_{i=1}^n \alpha_{i,i+1}}) * \cdots$$
$$\cdots * M(x_0, x_1, \frac{t}{p\Pi_{i=1}^n \alpha_{i,i+1}})$$
$$\geq M(x_0, x_1, \frac{t}{pk^n}) * \cdots * M(x_0, x_1, \frac{t}{pk^n})$$
$$\to 1 * \cdots * 1 = 1 \text{ as } n \to \infty,$$
$$N(x_n, x_{n+p}, t)$$
$$\leq N(x_n, x_{n+1}, \frac{t}{p}) \diamond \cdots$$
$$\cdots \diamond N(x_{n+p-1}, x_{n+p}, \frac{t}{p})$$
$$\leq N(x_0, x_1, \frac{t}{p\Pi_{i=1}^n \alpha_{i,i+1}}) \diamond \cdots$$
$$\cdots \diamond N(x_0, x_1, \frac{t}{p\Pi_{i=1}^n \alpha_{i,i+1}}),$$
$$\leq N(x_0, x_1, \frac{t}{p\Pi_{i=1}^n \alpha_{i,i+1}}) \diamond \cdots$$
$$\cdots \diamond N(x_0, x_1, \frac{t}{p\Pi_{i=1}^n \alpha_{i,i+1}})$$
$$\leq N(x_0, x_1, \frac{t}{pk^n}) \diamond \cdots \diamond N(x_0, x_1, \frac{t}{pk^n})$$
$$\to 0 \diamond \cdots \diamond 0 = 0 \text{ as } n \to \infty$$

from (2.1) and Remark 2.1. Therefore $\{x_n\}_n$ is a Cauchy sequence in X. Since X is complete, $\{x_n\}_n$ converges to some x in X.

Now we will prove that x is a periodic point of T_i. For some $m > 0$, by definition and Remark 2.1,

$$M(x, T_i^m x, t)$$
$$\geq M(x, x_n, t - kt) * M(x_n, T_i^m x, kt)$$
$$= M(x, x_n, t - kt) * M(T_n^m x_{n-1}, T_i^m x, kt)$$
$$\geq M(x, x_n, t - kt) * M(T_n^m x_{n-1}, T_i^m x, \alpha_{n,i} t)$$
$$\geq M(x, x_n, t(1-k)) * M(x_{n-1}, x, t)$$
$$\to 1 * 1 = 1 \text{ as } n \to \infty,$$
$$N(x, T_i^m x, t)$$
$$\leq N(x, x_n, t - kt) \diamond N(x_n, T_i^m x, kt)$$
$$= N(x, x_n, t - kt) \diamond N(T_n^m x_{n-1}, T_i^m x, kt)$$
$$\leq N(x, x_n, t - kt) \diamond N(T_n^m x_{n-1}, T_i^m x, \alpha_{n,i} t)$$
$$\leq N(x, x_n, t(1-k)) \diamond N(x_{n-1}, x, t)$$
$$\to 0 \diamond 0 = 0 \text{ as } n \to \infty$$

Hence $x = T_i^m x$.

Now, suppose that $y (x \neq y)$ be another periodic point of T_i, then there is $t > 0$ such that $M(x, y, t) < 1$ and $N(x, y, t) > 0$.

Further

$$M(x, y, t)$$
$$= M(T_i^m x, T_i^m y, t) \geq M(x, y, \frac{t}{\alpha_{i,j}})$$
$$\geq M(x, y, \frac{t}{k}),$$
$$N(x, y, t)$$
$$= N(T_i^m x, T_i^m y, t) \leq N(x, y, \frac{t}{\alpha_{i,j}})$$
$$\leq N(x, y, \frac{t}{k}).$$

Similarly,

$$M(x, y, t)$$
$$= M(T_i^m x, T_i^m y, t) \geq M(T_i^m x, T_i^m y, \frac{t}{k})$$
$$\geq M(x, y, \frac{t}{k^2}),$$
$$N(x, y, t)$$
$$= N(T_i^m x, T_i^m y, t) \leq N(T_i^m x, T_i^m y, \frac{t}{k})$$
$$\leq N(x, y, \frac{t}{k^2}).$$

Hence, by induction,

$$M(x, y, t) \geq M(x, y, \frac{t}{k^n}),$$
$$N(x, y, t) \leq N(x, y, \frac{t}{k^n}).$$

Therefore

$$1 > M(x, y, t) \geq \lim_{n \to \infty} M(x, y, \frac{t}{k^n}) = 1,$$
$$0 < N(x, y, t) \leq \lim_{n \to \infty} N(x, y, \frac{t}{k^n}) = 0,$$

which is a contradiction. Hence $x = y$. That is, x is a unique periodic point of T_i.

Also,

$$T_i x = T_i(T_i^m x) = T_i^m(T_i x).$$

Hence $T_i x$ is also a periodic point of T_i. Therefore $x = T_i x$. That is, x is a unique common fixed point of the sequence $\{T_n\}_n$.

Example 3.1. Let (X, d) be a metric space. Denote $a * b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$ and let M_d, N_d be fuzzy sets on $X^2 \times (0, \infty)$ defines as follows:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)},$$
$$N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)} \quad \text{if } x, y \in X.$$

Then (M_d, N_d) is an intuitionistic fuzzy metric on X and $(X, M_d, N_d, *, \diamond)$ is an intuitionistic fuzzy metric space.

In this case, let $X = \{\frac{1}{n} : n \in \mathbf{N}\} \cup \{0\}$ with the metric d defined by $d(x, y) = |x-y|$, and define the sequence $\{T_n\}_n$ of mappings from X to X by $T_n(x) = \frac{1}{2}x$ for $m = 2$, $\alpha_{i,j} = \frac{1}{4}$ and all $n \in \mathbf{N}$. Then

$$\begin{aligned}
&M_d(T_i^2 x, T_i^2 y, t) \\
&= M(\frac{x}{4}, \frac{y}{4}, t) = \frac{t}{t + |\frac{x}{4} - \frac{y}{4}|} \\
&= \frac{4t}{4t + |x-y|} = M(x, y, \frac{t}{\alpha_{i,j}}), \\
&N_d(T_i^2 x, T_i^2 y, t) \\
&= N(\frac{x}{4}, \frac{y}{4}, t) = \frac{|\frac{x}{4} - \frac{y}{4}|}{t + |\frac{x}{4} - \frac{y}{4}|} \\
&= \frac{|x-y|}{4t + |x-y|} = N(x, y, \frac{t}{\alpha_{i,j}}),
\end{aligned}$$

where $\alpha_{i,j} = \frac{1}{4} < 1$. Clearly, all conditions of the above theorem are satisfied, and 0 is a unique common fixed point of the sequence $\{T_n\}_n$.

Corollary 3.2([7]). (Intuitionistic fuzzy Banach contraction theorem) Let X be a complete intuitionistic fuzzy metric space satisfying (2.1). Let $T : X \to X$ be a mapping such that

$$M(Tx, Ty, \alpha t) \geq M(x, y, t),$$
$$N(Tx, Ty, \alpha t) \leq N(x, y, t)$$

for all $x, y \in X$, $t > 0$ and $\alpha \in (0, 1)$. Then T has a unique fixed point in X.

Proof. By the above theorem, putting $T_n = T$ for all $n = 1, 2, \cdots\cdots, m = 1$ and $\alpha_{i,j} = \alpha$, then the proof follows.

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