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단조집합함수에 의해 정의된 구간치 쇼케이적분에 대한 르베그형태 정리에 관한 연구

On Lebesgue-type theorems for interval-valued Choquet integrals with respect to a monotone set function

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Abstract

In this paper, we consider Lebesgue-type theorems in non-additive measure theory and then investigate interval-valued Choquet integrals and interval-valued fuzzy integral with respect to a additive monotone set function. Furthermore, we discuss the equivalence among the Lebesgue's theorems, the monotone convergence theorems of interval-valued fuzzy integrals with respect to a monotone set function and find some sufficient condition that the monotone convergence theorem of interval-valued Choquet integrals with respect to a monotone set function holds.

Key words : monotone set functions, interval-valued functions, Choquet integrals, fuzzy integrals, Lebesgue's theorems, monotone convergence theorems.

1. Introduction

We consider both interval-valued Choquet integral [1,2,3,6] and interval-valued fuzzy integral [5] with respect to a monotone set function. Set-valued Choquet integrals was introduced by Jang and Kwon([1]) and restudied by Zhang, Guo and Lia([6]) and that the theory about set-valued integrals has drawn much attention due to numerous applications in mathematics, economics, theory of control and many other fields. Set-valued fuzzy integral was first defined by D. Zhang and Z. Wang[4]. we note that Lebesgue's theorems asserts that almost everywhere convergence implies convergence in measure on a measurable set of finite measure.

In this paper, we consider Lebesgue-type theorems for interval-valued functions in non-additive measure theory and then investigate interval-valued Choquet integrals and interval-valued fuzzy integral with respect to a additive monotone set function. Furthermore, we discuss the equivalence among the Lebesgue's theorems, the monotone convergence theorems of interval-valued fuzzy integral with respect to a monotone set function and find some sufficient condition that the monotone convergence theorem of interval-valued Choquet integrals with respect to a monotone set function holds.

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2. Preliminaries

Let X be a set, (X, Ω) a measurable space and F the class of all finite non-negative measurable functions on X. A set function $\mu:\Omega\to R^+=[0,+\infty)$ is said to be monotone if $\mu(A)\le\mu(B)$, whenever $A,B\in\Omega$ and $A\subset B$; null-additive if $\mu(A\cup F)=\mu(A)$ for any $A\in\Omega$ whenever $F\in\Omega$ and $\mu(F)=0$; continuous from below if $\lim_{n\to\infty}\mu(A_n)=\mu(A)$ whenever $\{A_n\}\subset\Omega$ and $A_n\nearrow A$; continuous from above if $\lim_{n\to\infty}\mu(A_n)=\mu(A)$ whenever $\{A_n\}\subset\Omega$, $A_n\searrow A$ and $\mu(A_1)<\infty$; strongly order continuous if $\lim_{n\to\infty}\mu(A_n)=0$ whenever $\{A_n\}\subset\Omega$, $A_n\searrow B$ and $\mu(B)=0$; pseudo-order continuous if $\lim_{n\to\infty}\mu(A_n)=0$ whenever $A\in\Omega$, $\{A_n\}\subset\Omega$, $A_n\searrow B$ and $\mu(A-B)=\mu(A)$. We note that if μ is both continuous from below and continuous from above, then it is continuous. In this paper, we always assume that μ is a monotone set function with $\mu(\phi)=0$.

Definition 2.1 Let $f\in F$ and $\{f_n\}\subset F$. $\{f_n\}$ is said to converge to f almost everywhere (resp. pseudo-almost everywhere) on A if there is a subset $E\subset A$ such that $\mu(E)=0$ (resp. $\mu(A-E)=\mu(A)$) and f_n converges to f on $A-E$

Definition 2.2 Let $f\in F$ and $\{f_n\}\subset F$. $\{f_n\}$is said to converge to f in measure μ(resp. pseudo-in measure μ) on A if for any $\varepsilon>0$,

$$\lim_{n\to\infty}\mu(\{x:|f_n(x)-f(x)|\geq\epsilon\}\cap A)=0$$

(resp. $\lim_{n\to\infty}\mu(\{x:|f_n(x)-f(x)|<\varepsilon\}\cap A)=\mu(A)$).

Definition 2.3 ([3]) (1) The Choquet integral of a measurable function f with respect to a monotone set function μ on $A\in\Omega$ is defined by

$$(C)\int_A fd\mu-\int_0^\infty \mu(\{x|f(x)>r\}\cap A)dr$$

where the integrand on the right-hand side is an ordinary one.

(2) A measurable function f is called c-integrable if the Choquet integral of f can be defined and its value is finite.

Definition 2.4 ([7]) The fuzzy integral of a measurable function f with respect to a monotone set function μ on $A\in\Omega$ is defined by

$$(F)\int_A fd\mu=\sup_{\alpha\in[0,\infty)}[\alpha\wedge\mu(A\cap\{x|f(x)>\alpha\})]$$

Theorem 2.5 ([4]) The following are equivalent.
(1) μ is continuous from below;
(2) for any $A\in\Omega$, $f\in F$, $\{f_n\}\subset F$, $f_n\to f$ pseudo-almost everywhere on A imply $f_n\to f$ pseudo-in measure on A;
(3) for any $A\in\Omega$, $f\in F$, $\{f_n\}\subset F$, $f_n\nearrow f$ pseudo-almost everywhere on A imply

$$\lim_{n\to\infty}(c)\int_A f_n d\mu=(C)\int_A fd\mu;$$

(4) for any $A\in\Omega$, $f\in F$, $\{f_n\}\subset F$, $f_n\nearrow f$ pseudo-almost everywhere on A imply

$$\lim_{n\to\infty}(S)\int_A f_n d\mu=(S)\int_A fd\mu;$$

Theorem 2.6 ([4]) The following are equivalent.
(1) μ is null-additive and continuous from below;
(2) for any $A\in\Omega$, $f\in F$, $\{f_n\}\in F$, $f_n\to f$ almost everywhere on A imply $f_n\to f$ in measure on A;
(3) for any $A\in\Omega$, $f\in F$, $\{f_n\}\in F$, $f_n\nearrow f$ almost everywhere on A imply

$$\lim_{n\to\infty}(c)\int_A f_n d\mu=(C)\int_A fd\mu;$$

(4) for any $A\in\Omega$, $f\in F$, $\{f_n\}\in F$, $f_n\nearrow f$ almost everywhere on A imply

$$\lim_{n\to\infty}(S)\int_A f_n d\mu=(S)\int_A fd\mu;$$

3. Convergence of sequences of interval-valued functions

We denote $I(R^+)$ by

$$I(R^+)=\{\bar{a}=[a^-,a^+]|a^-\leqq a^+,a^-,a^+\in R^+\}.$$

For any $a\in R^+$, we define $a=[a,a]$. Obviously, $a\in I(R^+)$.

Definition 3.1 If $\bar{a},\bar{b}\in I(R^+)$, then we define
(1) $\bar{a}\wedge\bar{b}=[a^-\wedge b^-,a^+\wedge b^+]$,
(2) $\bar{a}\vee\bar{b}=[a^-\vee b^-,a^+\vee b^+]$,
(3) $\bar{a}\leq\bar{b}$ if and only if $a^-\leq b^-$ and $a^+\leq b^+$,
(4) $\bar{a}<\bar{b}$ if and only if $\bar{a}\leq\bar{b}$ and $\bar{a}\neq\bar{b}$
(5) $\bar{a}\subset\bar{b}$ if and only if $a^-\leq b^-$ and $a^+\leq b^+$.

It is easily to see that if we define

$$\bar{a}\cdot\bar{b}=\{x\cdot y|x\in\bar{a},y\in\bar{b}\}$$

for $\bar{a},\bar{b}\in I(R^+)$, then

$$\bar{a}\cdot\bar{b}=[a^-\cdot b^-,a^+\cdot b^+]$$

and that if $d_H: I(R^+)\times I(R^+)\to[0,\infty$ is a Hausdorff metric, then

$$d_H(\bar{a},\bar{b})=\max\{|a^--b^-|,|a^+-b^+|.$$

Definition 3.2 ([1,2,3,6]) (1) An interval-valued function $\bar{f}$ is said to be measurable if for each open set $O\subset R^+$,

$$\bar{f}^{-1}(O)=\{x\in X|\bar{f}(x)\cap)\neq\phi\}\in\Omega.$$

(2) An interval-valued function $\bar{f}$ is said to be finite if $\|\bar{f}\|=$

We denote IF by the class of all finite measurable interval-valued functions

$$\bar{f}=[f^-,f^+]:X\to I(R^+)-\{\phi\}$$

on X

Definition 3.3 Let $\bar{f}\in IF$ and $\{\bar{f}_n\}\subset IF$. $\{\bar{f}_n\}$ is said to d_H-converge to $\bar{f}$ almost everywhere(resp. pseudo-almost everywhere) on A if there is a subset $E\subset A$ such that $\mu(E)=0$ (resp. $\mu(A-E)=\mu(A)$) and $\bar{f}_n$ d_H-converges to $\bar{f}$ on $A-E$ that is,

$$\lim_{n\to\infty}d_H(\bar{f}_n(x),\bar{f}(x))=0,$$

for all $x\in A-E$.

Definition 3.4 Let $\overline{f}\in \overline{IF}$ and $\{\overline{f_n}\}\subset \overline{IF}$. $\{\overline{f_n}\}$ is said to d_H-converge to $\overline{f}$ in measure μ(resp. pseudo-in measure μ) on A if for any $\varepsilon>0$,

$$\lim_{n\to\infty}\mu(\{x : d_H(\overline{f_n}(x),\overline{f}(x)) \geq \epsilon\}\cap A) = 0$$

$$\text{(resp.} \lim_{n\to\infty}\mu(\{x : d_H(\overline{f_n}(x),\overline{f}(x)) < \epsilon\}\cap A) = \mu(A)).$$

Definition 3.5 ([3]) (1) Let $A\in\Omega$. The Choquet integral of an interval-valued $\overline{f}$ on A is defined by

$$(C)\int_A \overline{f}d\mu = \left\{(C)\int_A fd\mu \mid f\in S(\overline{f})\right\}$$

where $S(\overline{f})$ is the family of measurable selections of $\overline{f}$.

(2) $\overline{f}$ is said to be c-integrable if

$$(C)\int \overline{f}d\mu \neq \phi.$$

(3) $\overline{f}$ is said to be Choquet integrably bounded if there is a c-integrable function g such that

$$\|\overline{f}\| = ,$$

for all $x\in X$.

Definition 3.6 ([5]) (1) Let $A\in\Omega$. The fuzzy integral of an interval-valued $\overline{f}$ on A is defined by

$$(S)\int_A \overline{f}d\mu = \left\{(S)\int_A fd\mu \mid f\in S(\overline{f})\right\}$$

where $S(\overline{f})$ is the family of measurable selections of $\overline{f}$.

(2) $\overline{f}$ is said to be f-integrable if

$$(S)\int \overline{f}d\mu\neq\Phi.$$

Theorem 3.7 ([6]) If a fuzzy measure μ is continuous and an interval-valued function $\overline{f}=[f^-,f^+]$ is Choquet integrably bounded, then

$$(C)\int_A \overline{f}d\mu = [(C)\int f^- d\mu, (C)\int f^+ d\mu].$$

Theorem 3.8 ([5]) If $\overline{f}=[f^-,f^+]\in\overline{IF}$, then $\overline{f}$ is f-integrable and

$$(S)\int \overline{f}d\mu = [(S)\int f^- d\mu, (S)\int f^+ d\mu]$$

We denote $\overline{IF}^*$ by the class of all Choquet integrably bounded interval-valued functions in $\overline{IF}$.

Lemma 3.9 Let $\overline{f}=[f^-,f^+]\in \overline{IF}^*$ and $\{\overline{f_n}\}=\{[f_n^-,f_n^+]\}\subset \overline{IF}^*$. Assume that μ is subadditive.

(1) $\{\overline{f_n}\}$ d_H-converges to $\overline{f}$ almost everywhere (resp. pseudo-almost everywhere) on A if and only if $\{f_n^-\}$ converges to f^- almost everywhere (resp. pseudo-almost everywhere) on A and $\{f_n^+\}$ converges to f^+ almost everywhere (resp. pseudo-almost everywhere) on A.

(2) $\{\overline{f_n}\}$ d_H-converges to $\overline{f}$ in measure μ(resp. pseudo-in measure μ) on A if and only if $\{f_n^-\}$ converges to f^- in measure μ(resp. pseudo-in measure μ) on A and $\{f_n^+\}$ converges to f^+ in measure μ(resp. pseudo-in measure μ) on A.

Proof. (1) ($\Rightarrow$) If $\{\overline{f_n}\}$ d_H-converges to $\overline{f}$ almost everywhere, then there is a measurable set $E\subset A$ such that $\mu(E)=0$ and

$$d_H(\overline{f_n}(x),\overline{f}(x)) = 0$$

for all $x\in A-E$. Thus,

$$\lim_{n\to\infty}|f_n^-(x)-f^-(x)|=0$$

and

$$\lim_{n\to\infty}|f_n^-(x)-f^+(x)|=0$$

for all $x\in A-E$, that is, $\{f_n^-\}$ converges to f^- almost everywhere on A and $\{f_n^+\}$ converges to f^+ almost everywhere on A.

($\Leftarrow$) If $\{f_n^-\}$ converges to f^- almost everywhere on A and $\{f_n^+\}$ converges to f^+ almost everywhere on A, then there are measurable sets $E_1, E_2\subset A$ such that $\mu(E_1)=0$ and $\mu(E_2)=0$

$$\lim_{n\to\infty}|f_n^-(x)-f^-(x)|=0$$

for all $x\in A-E_1$ and

$$\lim_{n\to\infty}|f_n^-(x)-f^+(x)|=0$$

for all $x\in A-E_2$. If we put $E=E_1\cup E_2$, then E is measurable and $\mu(E)=0$ since μ is subadditive. Hence for all $x\in A-E$,

$$\lim_{n\to\infty}d_H(\overline{f_n}(x),\overline{f}(x))$$

$$=\lim_{n\to\infty}\max\{|f_n^-(x)-f^-(x)|,|f_n^+(x)-f^+(x)|\}$$

$$=0.$$

That is, $\{\overline{f_n}\}$ d_H-converges to $\overline{f}$ almost everywhere on A. We note that the proof of the case of pseudo-almost everywhere is similar to the proof of the case of almost everywhere. Similarly, we can prove the converse of (1). So, we omit the prove the converse.

(2) If $\{\overline{f_n}\}$ d_H-converges to $\overline{f}$ in measure μ on A, then for all $\varepsilon>0$,

$$\lim_{n\to\infty}\mu(\{x|d_H(\overline{f_n}(x),\overline{f}(x))\geq\epsilon\}\cap A)=0.$$

Since

$$d_H(\overline{f_n}(x),\overline{f}(x))$$

$$=\max\{|f_n^-(x)-f(x)|,|f_n^+(x)-f(x)|\},$$

we have

$$\lim_{n\to\infty}\mu(\{x|d_H(f_n^-(x),f^-(x))\geq\epsilon\}\cap A)=0$$

and

$$\lim_{n\to\infty}\mu(\{x|d_H(f_n^+(x),f^+(x))\geq\epsilon\}\cap A)=0.$$

That is, $\{f_n^-\}$ converges to f^- in measure μ on A and $\{f_n^+\}$ converges to f^+ in measure μ on A. We note that the proof of the case of pseudo-in measure is similar to the proof of the case of in measure. Similarly, we can prove the converse of (2). So, we omit the prove of the converse.

We discuss the equivalence among the Lebesgue's type theorems, the monotone convergence theorems of interval-valued fuzzy integrals with respect to a monotone set function.

Theorem 3.10 Assume that μ is continuous from below and subadditive. The following two statements are equivalent.

(1) for any $A\in\Omega$, $\bar{f}\in IF^*$ and $\{\bar{f_n}\}\subset IF^*$, $\{\bar{f_n}\}$ d_H-converge to $\bar{f}$ pseudo-almost everywhere on A imply $\{\bar{f_n}\}$ d_H-converge to $\bar{f}$ pseudo-in measure μ on A.

(2) for any $A\in\Omega$, $\bar{f}\in IF^*$ and $\{\bar{f_n}\}\subset IF^*$, $\bar{f}_n\nearrow\bar{f}$ pseudo-almost everywhere on A on A imply

$$d_H-\lim_{n\to\infty}(C)\int_A \bar{f}_n\, d\mu=(C)\int_A \bar{f}\, d\mu.$$

Proof. (1)⇒(2) Assume that (2) holds. By Lemma 3.9, Theorem 2.5 (2) holds. Thus, by Theorem 2.5,

$$\lim_{n\to\infty}(C)\int_A f_n^- d\mu=(C)\int_A f^- d\mu$$

and

$$\lim_{n\to\infty}(C)\int_A f_n^+ d\mu=(C)\int_A f^+ d\mu$$

Thus, we have

$$\lim_{n\to\infty} d_H((C)\int_A \bar{f}_n d\mu,\ (C)\int_A \bar{f} d\mu)$$
$$= \lim_{n\to\infty} \max\,(|(C)\int_A f_n^- d\mu-(C)\int_A f^- d\mu|,$$
$$|(C)\int_A f_n^+ d\mu-(C)\int_A f^+ d\mu|)$$
$$=0$$

That is, (2) holds.

(2)⇒(1) Assume that (2) holds. By Lemma 3.9, Theorem 2.5(3) holds. By Theorem 2.5, for any $A\in\Omega$, $f\in F$, $\{f_n\}\in F$, $f_n\to f$ pseudo-almost everywhere on A imply $f_n\to f$ pseudo-in measure on A. For any $A\in\Omega$, $\bar{f}\in IF^*$ and $\{\bar{f_n}\}\subset IF^*$, $\{\bar{f_n}\}$ d_H-converge to $\bar{f}$ pseudo-almost everywhere on A by Lemma 3.9, $f_n^-\to f^-$ pseudo-almost everywhere on A and $f_n^-\to f^-$ pseudo-almost everywhere on A. By Theorem 2.5, $f_n^-\to f^-$ pseudo-in measure on A and $f_n^-\to f^-$ pseudo-in measure on A. By Lemma 3.9, $\{\bar{f_n}\}$ d_H-converge to $\bar{f}$ pseudo-in measure μ on A. That is, (1) holds.

By using Definition 3.6, Theorem 3.8, and the same method of Theorem 3.10, clearly, we obtain the following theorem.

Theorem 3.11 Assume that μ is null additive and continuous from below. The following two statements are equivalent.

(1) for any $A\in\Omega$, $\bar{f}\in IF$ and $\{\bar{f_n}\}\subset IF$, $\{\bar{f_n}\}$ d_H-converge to $\bar{f}$ pseudo-almost everywhere on A imply $\{\bar{f_n}\}$ d_H-converge to $\bar{f}$ pseudo-in measure μ on A.

(2) for any $A\in\Omega$, $\bar{f}\in IF$ and $\{\bar{f_n}\}\subset IF$, $\bar{f}_n\nearrow\bar{f}$ on A imply

$$d_H-\lim_{n\to\infty}(S)\int_A \bar{f}_n\, d\mu=(S)\int_A \bar{f}\, d\mu.$$

Finally, clearly, we have the following theorems for interval-valued Choquet integrals and interval-valued fuzzy integrals with respect to a monotone set function.

Theorem 3.11 Let μ be continuous. Then the following two statements are equivalent.

(1) For any $A\in\Omega$, $f\in F^*$ and $\{f_n\}\in F^*$, $f_n\nearrow f$ on A then

$$\lim_{n\to\infty}(C)\int_A f_n d\mu=(C)\int_A f\, d\mu;$$

(2) For any $A\in\Omega$, $\bar{f}\in IF^*$ and $\{\bar{f_n}\}\subset IF^*$, $\bar{f_n}\nearrow\bar{f}$ on A then

$$d_H-\lim_{n\to\infty}(C)\int_A \bar{f_n}\, d\mu=(C)\int_A \bar{f}\, d\mu.$$

Theorem 3.12 Let μ be continuous from below. Then the following two statements are equivalent.

(1) For any $A\in\Omega$, $f\in F$ and $\{f_n\}\in F$, $f_n\nearrow f$ on A then

$$\lim_{n\to\infty}(S)\int_A f_n d\mu=(S)\int_A f\, d\mu;$$

(2) For any $A\in\Omega$, $\bar{f}\in IF$ and $\{\bar{f_n}\}\subset IF$, $\bar{f_n}\nearrow\bar{f}$ on A then

$$d_H-\lim_{n\to\infty}(S)\int_A \bar{f_n}\, d\mu=(S)\int_A \bar{f}\, d\mu.$$

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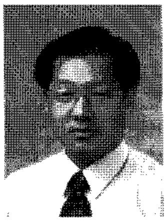

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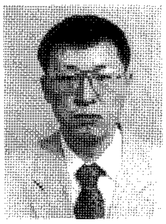

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