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FEYNMAN INTEGRAL, ASPECT OF DOBRAKOV INTEGRAL, I

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Abstract. This paper is the first in a series in which we consider bilinear integration with respect to measure-valued measure. We use the integration techniques to establish generalized Egorov theorem and Vitali theorem.

1. Introduction

The measure-valued measures V_φ were introduced in [13] and studied in relation to a measure-valued Feynman-Kac formula. For a given complex Borel measure $\varphi : \mathcal{B}(\mathbb{R}) \to \mathbb{C}$ on $\mathbb{R}$, the measure-valued measures V_φ is defined as follows. The space of all continuous functions $\omega : [0,t] \to \mathbb{R}$ is denoted by $C([0,t])$. It is given with the uniform norm. If $X_s : C([0,t]) \to \mathbb{R}$ denotes evolution at time $0 \le s \le t$, then for the cylinder set $E = \{X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n\}$ in $C([0,t])$ with $0 \le t_1 < \cdots < \cdots < t_n \le t$ and Borel sets $B_1, \ldots, B_n$, the complex Borel measure $V_\varphi(E)$ is defined by the formula

$$\begin{aligned}(V_\varphi(E))(B) &\\ = \frac{1}{\sqrt{(2\pi(t-t_n))\cdots(2\pi t_1)}} &\int_B \int_{B_n} \cdots \int_{B_1} \int_{\mathbb{R}} e^{-\frac{|\xi - x_n|^2}{2(t-t_n)}} \\ \times\, e^{-\frac{|x_n - x_{n-1}|^2}{2(t_n - t_{n-1})}} &\cdots e^{-\frac{|x_2 - x_1|^2}{2(t_2 - t_1)}} e^{-\frac{|x_1 - x|^2}{2t_1}}\, d\varphi(x) dx_1 \cdots dx_n d\xi \end{aligned} \tag{1}$$

for each Borel subset B of $\mathbb{R}$. Clearly V_φ is closely related to Wiener measure and the complex valued measure $V_\varphi(\cdot)(B)$ may be viewed as Wiener measure with an initial distribution φ subject to the condition $\{X_t \in B\}$. Another way of looking at the measure valued measure V_φ is to take the semigroup $S(t) : \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$, $t \ge 0$, defined on the space $\mathcal{M}(\mathbb{R})$ of Borel measures on

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$\mathbb{R}$ by $S(0) = Id$ and

$$[S(t)\mu](B) = \frac{1}{\sqrt{2\pi t}} \int_B \int_{\mathbb{R}} e^{-\frac{|\xi - x|^2}{2t}} d\mu(x) d\xi, \ B \in \mathcal{B}(\mathbb{R}), \ \mu \in \mathcal{M}(\mathbb{R})$$

for $t > 0$ and $Q(B)\mu = \chi_B \cdot \mu$ for all $B \in \mathcal{B}(\mathbb{R})$ and $\mu \in \mathcal{M}(\mathbb{R})$. If M^t denotes the operator valued set function associated with the (S, Q)-process [7], then for each cylinder set E we have $M^t(E)\varphi = V_\varphi(E \cap C([0, t]))$. If we replace $\mathcal{M}(\mathbb{R})$ by the Hilbert space $L^2(\mathbb{R})$ and S by

$$[S_F(t)](\psi)(\xi) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{i\frac{|\xi - x|^2}{2t}} \psi(x) dx, \ \psi \in L^2(\mathbb{R}), \ t > 0,$$

where the integral is understood in the sense of mean-square convergence, then we obtain the operator valued set functions M_F^t associated with the Feynman path integral. Roughly speaking, for each $t \geq 0$, the bounded linear operator $S_F(t)$ is equal to $S(it)$ applied to complex measures with square-integrable densities with respect to Lebesgue measure on $\mathbb{R}$.

2. Convergence theorem for measure-valued measures

In this paper we denote the inner product of two elements a, b in a Banach space to $\langle a, b \rangle$ or even ab.

Let $(\Sigma, \mathcal{E})$, $(\Omega, \mathcal{B})$ be measurable spaces. The space of all complex measures defined on $\mathcal{E}$ with the total variation norm is denoted by $\mathcal{M}(\mathcal{E})$. The space of nonnegative elements of $\mathcal{M}(\mathcal{E})$ is written as $\mathcal{M}_+(\mathcal{E})$. The variation of a scalar measure μ is written as $|\mu|$. Let $\mathcal{X}$ be a Banach space. For any $\mathcal{M}(\mathcal{E})$-valued measure $m : \mathcal{B} \to \mathcal{M}(\mathcal{E})$, the operator valued measure $m^{\mathcal{X}} : \mathcal{B} \to \mathcal{L}(\mathcal{X}, \mathcal{M}(\mathcal{E}, \mathcal{X}))$ be defined by

$$m^{\mathcal{X}}(B)x = xm(B), \quad x \in \mathcal{X}, \ B \in \mathcal{B}.$$

Here $\mathcal{M}(\mathcal{E}, \mathcal{X})$ is the space of $\mathcal{X}$-valued measures on $\mathcal{E}$ equipped with the semivariation norm defined by

$$||n|| = \sup\{|\langle n, x' \rangle|(\Sigma) : x' \in \mathcal{X}', ||x'|| \leq 1\}, \quad n \in \mathcal{M}(\mathcal{E}, \mathcal{X}).$$

We also write this as $||n||_{\mathcal{M}(\mathcal{E},\mathcal{X})}$. Although the semivariation of an $\mathcal{X}$-valued measure is always finite, for every infinite dimensional Banach space $\mathcal{X}$, there is an $\mathcal{X}$-valued measure n whose total variation

$$||n||_v = \sup\{\sum_j ||n(E_j)||\}$$

is infinite. The supremum is over all finite partitions $\{E_j\}$ of Σ by elements of $\mathcal{E}$. The space of $\mathcal{X}$-valued measures on $\mathcal{E}$ with finite variation and equipped with the variation norm is denoted by $\mathcal{M}_v(\mathcal{E}, \mathcal{X})$. The variation of $n \in \mathcal{M}_v(\mathcal{E}, \mathcal{X})$ is written as $v_{\mathcal{X}}(n) : \mathcal{E} \to [0, \infty)$. Our aim is to integrate $\mathcal{X}$-valued functions with respect to the $\mathcal{M}(\mathcal{E})$-valued measure m; the integral takes its values in $\mathcal{M}(\mathcal{E},\mathcal{X})$. We also need to consider the $\mathcal{X}$-semivariation of m on $\mathcal{B}$:

$$\beta_{\mathcal{X}}(m)(B) = \sup\{||\sum_{j=1}^{n} x_j m(B_j \cap B)||_{\mathcal{M}(\mathcal{E},\mathcal{X})}\}.$$

The supremum is taken over all $x_j \in \mathcal{X}$ with $||x_j|| \leq 1$ and all finite partitions $\{B_j\}$ of Ω. The $\mathcal{X}$-semivariation $\beta_{\mathcal{X}}(m)$ of m is identical to the semivariation of the operator valued measure $m^{\mathcal{X}}$ in the sense of Dobrakov. It can happen that $\beta_{\mathcal{X}}(m)$ has only the values 0 or ∞. An $\mathcal{X}$-valued function $f : \Omega \to \mathcal{X}$ is called a *simplefunction* if for some $n \in \mathbb{N}$, there exist vectors $x_j \in \mathcal{X}$ and sets $B_j \in \mathcal{B}$ for $j = 1, 2, \ldots, n$ such that $f = \sum_{j=1}^{n} x_j \chi_{B_j}$. For an $\mathcal{X}$-valued simple function $f = \sum_{j=1}^{n} x_j \chi_{B_j}$ and for B in $\mathcal{B}$, we define the integral

$$\int_B f \otimes dm = \sum_{j=1}^{n} x_j m(B \cap B_j) \in \mathcal{M}(\mathcal{E},\mathcal{X}).$$

We also write this $\int_B f dm^{\mathcal{X}}$. A standard argument ensures that the $\mathcal{X}$-valued measure $\int_B f \otimes dm$ is well-defined. The $\mathcal{X}$-semivariation $\beta_{\mathcal{X}}(m)$ of m is extended from sets to functions $f : \Omega \to \mathcal{X}$ by setting

$$\beta_{\mathcal{X}}(m)(f) = \beta_{\mathcal{X}}(m)(|f|) = \sup || \int_B s \otimes dm||_{\mathcal{M}(\mathcal{E},\mathcal{X})},$$

where the supremum is taken for all $\mathcal{X}$-valued $\mathcal{B}$-simple functions s with $||s(\omega)||_{\mathcal{X}} \leq ||f(\omega)||_{\mathcal{X}}$ for m-almost all $\omega \in \Omega$. In the notation of Dobrakov, we have $\beta_{\mathcal{X}}(m) = \hat{m}^{\mathcal{X}}$. We set

$$\begin{aligned}
\mathcal{L}_1(\beta_{\mathcal{X}}(m)) &= \{f|\ f : \Omega \to \mathcal{X} \text{ is } m\text{-measurable and } \beta_{\mathcal{X}}(m)(f) < \infty\} \\
B_{\mathcal{X}}(\mathcal{B}) &= \{\ f \in \mathcal{L}_1(\beta_{\mathcal{X}}(m))|f \text{ is uniformly bounded}\} \\
\mathcal{L}_1(B_{\mathcal{X}}, \beta_{\mathcal{X}}(m)) &= \overline{B_{\mathcal{X}}(\mathcal{B})}.
\end{aligned}$$

The closure is taken in the norm $\beta_{\mathcal{X}}(m)$ defined on $\mathcal{L}_1(\beta_{\mathcal{X}}(m))$. With modulo m-null functions, $\mathcal{L}_1(\beta_{\mathcal{X}}(m))$ becomes a Banach space $L_1(\beta_{\mathcal{X}}\ (m))$. As noted above, the $\mathcal{X}$-semivariation $\beta_{\mathcal{X}}(m) : \mathcal{B} \to [0,\infty]$ may take only the values 0 and ∞, so it can happen that $\mathcal{L}_1(\beta_{\mathcal{X}}(m))$ just consists of the zero function. Let $M \subset \mathcal{M}(\mathcal{E},\mathcal{X})^*$. Let M_1 denote the set of all elements μ of M such that $||\mu|| \leq 1$ and suppose that M_1 is norming for $\mathcal{M}(\mathcal{E},\mathcal{X})$. Then for any $\mu \in M$ we define $\mu \circ m^{\mathcal{X}} : \mathcal{B} \to \mathcal{X}^*$ by $\langle x, \mu \circ m^{\mathcal{X}}(B)\rangle = \langle m^{\mathcal{X}}(B)x, \mu\rangle$ for $x \in \mathcal{X}$ and $B \in \mathcal{B}$. The set

$$\mathcal{M}_1(m^{\mathcal{X}}) = \{v_{\mathcal{X}^*}(\mu \circ m^{\mathcal{X}}) : \mu \in M_1\}$$

consists of measures on $\mathcal{B}$ with values in $[0,\infty]$. Because

$$|\sum_{j=1}^{n}\langle x_j m(B_j \cap B), \mu\rangle| \leq ||\sum_{j=1}^{n} x_j m(B_j \cap B)||_{\mathcal{M}(\mathcal{E},\mathcal{X})}$$

for each $\mu \in M_1$, it follows that $\mathcal{M}_1(m^{\mathcal{X}})$ consists of finite measures if the $\mathcal{X}$-semivariation $\beta_{\mathcal{X}}(m)(\Omega)$ on Ω is finite. Because M_1 is a norming set for $\mathcal{M}(\mathcal{E}, \mathcal{X})$, we have

$$(2) \qquad \beta_{\mathcal{X}}(m)(B) = \sup_{\nu \in \mathcal{M}_1(m^{\mathcal{X}})} \nu(B), \qquad B \in \mathcal{B}.$$

By Hahn-Banach theorem, $M = sim(\mathcal{E}) \otimes \mathcal{X}^*$ is a dense subset of $\mathcal{M}(\mathcal{E}, \mathcal{X})^*$, so M_1 is norming for $\mathcal{M}(\mathcal{E}, \mathcal{X})$. In this section we take $M = sim(\mathcal{E}) \otimes \mathcal{X}^*$. In the case that Σ is a locally compact Hausdorff space and $\mathcal{E}$ is the Borel σ-algebra of Σ, another choice is $M = C_0(\Sigma) \otimes \mathcal{X}^*$ where $C_0(\Sigma)$ is the set of all continuous functions on Σ vanishing at infinity. We note that for any infinite dimensional Banach space $\mathcal{X}$, there exist measurable spaces $(\Sigma, \mathcal{E})$, $(\Omega, \mathcal{B})$ and a measure-valued measure $m : \mathcal{B} \to \mathcal{M}(\mathcal{E})$ such that $\beta_{\mathcal{X}}(m)(\Omega) = \infty$. Suppose that $\beta_{\mathcal{X}}(m)(\Omega) < \infty$. By virtue of the equality (2), the condition that $\mathcal{M}_1(m^{\mathcal{X}})$ is uniformly countable additive is equivalent to Dobrakov's condition that $\beta_{\mathcal{X}}(m)$ is continuous, that is, if $B_n \downarrow \emptyset$, then $\beta_{\mathcal{X}}(m)(B_n) \to 0$ as $n \to \infty$. For $\mathcal{X} = c_0$, the classical Banach space, there exist vector measures m for which $\beta_{c_0}(m)(\Omega) < \infty$ but $\beta_{c_0}(m)$ is not continuous.

Definition 2.1. ([9, Definition 1.5]) Let $m : \mathcal{B} \to \mathcal{M}(\mathcal{E})$ be a vector measure. A function $f : \Omega \to \mathcal{X}$ is said to be *m-integrable* in $\mathcal{M}(\mathcal{E}, \mathcal{X})$ if there exist $\mathcal{X}$-valued $\mathcal{B}$-simple function f_j, $j \in \mathbb{N}$, such that $f_j \to f$ pointwisely m-almost everywhere as $j \to \infty$ and $\{\int_B f_j \otimes dm\}_{j=1}^{\infty}$ converges in $\mathcal{M}(\mathcal{E}, \mathcal{X})$ for each $B \in \mathcal{B}$. Let $\int_B f \otimes dm$ denote this limit.

The above limit is well defined and independent of the approximating sequence ([7, Lemma 4.1.4]). The set function $B \to \int_B f \otimes dm$, $B \in \mathcal{B}$, is σ-additive in $\mathcal{M}(\mathcal{E}, \mathcal{X})$ by the Vitali-Hahn-Sake theorem ([1, Theorem 1.5.6]). Clearly the map $(f, m) \to \int f \otimes dm$ is bilinear in the obvious sense. Also, for the case $\mathcal{X} = \mathbb{C}$, a function $f : \Omega \to \mathbb{C}$ is m-integrable (as defined above) if and only if it is m-integrable in the sense of vector measures defined in I. Kluvánek and G. Knowles [12]. If the $\mathcal{X}$-semivariation $\beta_{\mathcal{X}}(m)$ of m is σ-finite on the set $\{f \neq 0\}$, then f is m-integrable if and only if it is $m^{\mathcal{X}}$-integrable in the sense of Dobrakov and in this case,

$$\int_B f \otimes dm = \int_B f dm^{\mathcal{X}}$$

for every $B \in \mathcal{B}$ [9, Definition 1.5]. If $\beta_{\mathcal{X}}(m)$ is finite and continuous, then integral coincide with the Bartle bilinear integral.

We are now in a position to state our general convergence theorem for measure-valued measures. In the next section, we see how they can be simplified under the additional assumption of order boundedness.

Theorem 2.2. (Egorov) *Let $(\Sigma, \mathcal{E})$, $(\Omega, \mathcal{B})$ be measurable spaces and $\mathcal{X}$ a Banach space. Suppose that $m : \mathcal{B} \to \mathcal{M}(\mathcal{E})$ is an $\mathcal{M}(\mathcal{E})$-valued measure for which $\beta_{\mathcal{X}}(m)(\Omega) < \infty$ and $\beta_{\mathcal{X}}(m)$ is continuous. Let f_n, $f : \Omega \to \mathcal{X}$, $n \in \mathbb{N}$ be m-measurable functions such that $f_n \to f$, m-a.e. Then*

a) *for any* $\varepsilon > 0$, *there is a set* $B \in \mathcal{B}$ *such that* $\beta_{\mathcal{X}}(m)(B^c) < \varepsilon$ *and* $f_n \to f$ *uniformly on* B.

b) $f_n \to f$, $\beta_{\mathcal{X}}(m)$*-measure.*

Proof. Since $\beta_{\mathcal{X}}(m)$ is continuous, the set $\mathcal{M}_1(m^{\mathcal{X}})$ is uniformly σ-additive on $\mathcal{B}$. The Bartle-Dunford-Schwartz theorem and equation (2) shows that there is a positive, finite and σ-additive measure λ on $\mathcal{B}$ such that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $F \in \mathcal{B}$ with $\lambda(F) < \delta$, $\beta_{\mathcal{X}}(m(F)) < \varepsilon$. Hence by the Egorov theorem for λ, for this $\delta > 0$, there is $B \in \mathcal{B}$ such that $\lambda(B^c) < \delta$ and $f_n \to f$ uniformly on B, and $f_n \to f$ in $\beta_{\mathcal{X}}(m)$-measure. Thus we have $\beta_{\mathcal{X}}(m)(B^c) < \varepsilon$ and $f_n \to f$ in $\beta_{\mathcal{X}}(m)$-measure. □

Theorem 2.3. (Vitali) *Let* $(\Sigma, \mathcal{E})$, $(\Omega, \mathcal{B})$ *be measurable spaces and* $\mathcal{X}$ *a Banach space. Suppose that* $m : \mathcal{B} \to \mathcal{M}(\mathcal{E})$ *is an* $\mathcal{M}(\mathcal{E})$*-valued measure for which* $\beta_{\mathcal{X}}(m)(\Omega) < \infty$ *and* $\beta_{\mathcal{X}}(m)$ *is continuous. Let* $\langle f_n \rangle$ *be a sequence from* $\mathcal{L}_1(\beta_{\mathcal{X}}(m))$ *with* f_n *m-integrable functions and let* $f : \Omega \to \mathcal{X}$ *be m-measurable. Assume that*

a) $f_n \to f$ *in* $\beta_{\mathcal{X}}(m)$*-measure or*

a') $f_n \to f$, *m-a.e.*

b) $\lim_{\beta_{\mathcal{X}}(m)(A) \to 0} \beta_{\mathcal{X}}(m)(f_n \chi_A) = 0$, *uniformly for* $n \in \mathbb{N}$.

c) *For all* $\varepsilon > 0$, *there is a set* $A_\varepsilon \in \mathcal{B}$ *with* $\beta_{\mathcal{X}}(m)(A_\varepsilon) < \infty$, *such that* $\beta_{\mathcal{X}}(m)(f_n \chi_{\Omega - A_\varepsilon}) < \varepsilon$, *for all* $n \in \mathbb{N}$.

Then $f \in \mathcal{L}_1(\beta_{\mathcal{X}}(m))$ *and* $\beta_{\mathcal{X}}(m)(f_n - f) \to 0$. *Furthermore, the function* f *is m-integrable in* $\mathcal{M}(\mathcal{E}, \mathcal{X})$ *and* $\int_B f_n \otimes dm \to \int_B f \otimes dm$ *in* $\mathcal{M}(\mathcal{E}, \mathcal{X})$, *uniformly for* $B \in \mathcal{B}$ *as* $n \to \infty$. *Conversely, if* $f \in \mathcal{L}_1(\beta_{\mathcal{X}}(m))$ *and* $\beta_{\mathcal{X}}(m)(f_n - f) \to 0$ *as* $n \to \infty$, *then conditions* a) *and* b) *are satisfied.*

Proof. By Egorov theorem, a') implies a). Assume conditions a), b) and c) satisfied. To show that $\langle f_n \rangle$ is a Cauchy sequence in $\mathcal{L}_1(\beta_{\mathcal{X}}(m))$, let $\varepsilon > 0$ and let $A_\varepsilon \in \mathcal{B}$ be a set satisfying condition c). By condition b) there is a $\delta > 0$ for all $A \in \mathcal{B}$ if $\beta_{\mathcal{X}}(m)(A) < \delta$ then $\beta_{\mathcal{X}}(m)(f_n \chi_A) < \varepsilon$ for all $n \in \mathbb{N}$. By a), there exists N_ε such that if

$$B_{n,m} = \{\, s \in A_\varepsilon : \|f_n(s) - f_m(s)\|_{\mathcal{X}} > \varepsilon / \beta_{\mathcal{X}}(m)(A_\varepsilon)\},$$

then for all $n, m \geq N_\varepsilon$, $B_{n,m} \in \mathcal{B}$ and $\beta_{\mathcal{X}}(m)(B_{n,m}) < \delta$. Then for $n, m \geq N_\varepsilon$

$$\begin{aligned}
&\beta_{\mathcal{X}}(m)(f_n - f_m) \\
&\leq \beta_{\mathcal{X}}(m)((f_n - f_m)\chi_{B_{n,m}}) + \beta_{\mathcal{X}}(m)((f_n - f_m)\chi_{A_\varepsilon - B_{n,m}}) \\
&\quad + \beta_{\mathcal{X}}(m)((f_n - f_m)\chi_{\Omega - A_\varepsilon}) \\
&\leq \beta_{\mathcal{X}}(m)(f_n \chi_{B_{n,m}}) + \beta_{\mathcal{X}}(m)(f_m \chi_{B_{n,m}}) + \beta_{\mathcal{X}}(m)((f_n - f_m)\chi_{A_\varepsilon - B_{n,m}}) \\
&\quad + \beta_{\mathcal{X}}(m)(f_n \chi_{\Omega - A_\varepsilon}) + \beta_{\mathcal{X}}(m)(f_m \chi_{\Omega - A_\varepsilon}) \\
&\leq 5\varepsilon.
\end{aligned}$$

Hence $\langle f_n \rangle$ is a Cauchy sequence in $\mathcal{L}_1(\beta_{\mathcal{X}}(m))$. In particular, $\int_B f_n \otimes dm$, $n = 1, 2, \ldots$, converges for each $B \in \mathcal{B}$. Since $\mathcal{L}_1(\beta_{\mathcal{X}}(m))$ is complete, there a

m-measurable function g in $\mathcal{L}_1(\beta_{\mathcal{X}}(m))$ such that $\beta_{\mathcal{X}}(m)(f_n - g) \to 0$. Then $f_n \to g$ in $\beta_{\mathcal{X}}(m)$-measure. From a) $f = g$ $\beta_{\mathcal{X}}(m)$-*a.e.*, $f \in \mathcal{L}_1(\beta_{\mathcal{X}}(m))$ and $\beta_{\mathcal{X}}(m)(f_n - f) \to 0$. To check that f is m-integrable in $\mathcal{M}(\mathcal{E}, \mathcal{X})$, we need to exhibit a sequence $\langle s_k \rangle$ of $\mathcal{X}$-valued $\mathcal{E}$-simple functions such that $s_k \to f$ m-*a.e.* as $k \to \infty$ and the indefinite integrals $\int s_k \otimes dm$, $k = 1, 2, \ldots$, are uniformly bounded and uniformly countably additive in $\mathcal{M}(\mathcal{E}, \mathcal{X})$. This follows from ([9, Theorem 2.6]), where it is also shown that $\int_B f_n \otimes dm \to \int_B f \otimes dm$ in $\mathcal{M}(\mathcal{E}, \mathcal{X})$, uniformly for $B \in \mathcal{B}$ as $n \to \infty$. Conversely, assume that $f \in \mathcal{L}_1(\beta_{\mathcal{X}}(m))$ and $\beta_{\mathcal{X}}(m)(f_n - f) \to 0$. Then $f_n \to f$ in $\beta_{\mathcal{X}}(m)$-measure, so a) is satisfied. To prove b), assume $f \in \mathcal{L}_1(\beta_{\mathcal{X}}(m))$. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that for every $n \geq N$ $\beta_{\mathcal{X}}(m)(f_n - f) < \varepsilon/2$. Then for all $A \in \mathcal{B}$ and $n \geq N$

$$\beta_{\mathcal{X}}(m)(f_n\chi_A - f\chi_A) < \varepsilon/2$$

that is,

$$\beta_{\mathcal{X}}(m)(f_n\chi_A) \leq \beta_{\mathcal{X}}(m)(f\chi_A) + \varepsilon/2.$$

But since $f \in \mathcal{L}_1(\beta_{\mathcal{X}}(m))$, there is a $\delta_0 > 0$ such that $A \in \mathcal{B}$ and $\beta_{\mathcal{X}}(m)(A) < \delta_0$ then $\beta_{\mathcal{X}}(m)(f\chi_A) < \varepsilon/2$. Then for $n \geq N$ and $\beta_{\mathcal{X}}(m)$ $(A) < \delta_0$, $\beta_{\mathcal{X}}(m)(f_n\chi_A) < \varepsilon$. For $n \leq N$, we can find $\delta_1 > 0$ such that $A \in \mathcal{B}$ and $\beta_{\mathcal{X}}(m)(A) < \delta_1$ then $\beta_{\mathcal{X}}(m)(f_n\chi_A) < \varepsilon$ for all $n \leq N$. If we take $\delta = \inf\{\delta_0, \delta_1\}$, then for any $A \in \mathcal{B}$ with $\beta_{\mathcal{X}}(m)(A) < \delta$, $\beta_{\mathcal{X}}(m)(f_n\chi_A) < \varepsilon$ for all $n \in \mathbb{N}$. □

3. Order bounded measures

Let $(\Sigma, \mathcal{E})$, $(\Omega, \mathcal{B})$ be measurable spaces. A measure-valued measure $m : \mathcal{B} \to \mathcal{M}(\mathcal{E})$ is said to be *positive* if it takes its values in the space $\mathcal{M}_+(\mathcal{E})$ of nonnegative measures. We say that m is *order bounded* if there exists a positive vector measure, n, for which $n(A) \geq |m(A)|$ holds for all $A \in \mathcal{B}$. In the present context, this is equivalent to saying that m has order bounded range in the Banach lattice $\mathcal{M}(\mathcal{E})$, and see ([7, Lemma 4.4.4]). The smallest positive measure $|m|_{\mathcal{M}}$ satisfying this requirement is called the *modulus* of m. It is associated with the modulus of the regular linear map from the Banach lattice $\mathcal{L}_\infty(\Omega, \mathcal{B})$ to the Banach lattice $\mathcal{M}(\mathcal{E})$ defined by integration with respect to m. We write $|m|_{\mathcal{M}} \geq |n|_{\mathcal{M}}$ and say that m *dominates* n if $|m|_{\mathcal{M}}(A) \geq |n|_{\mathcal{M}}(A)$ for all $A \in \mathcal{B}$. As for the case of L^p-valued measure [10], the convergence theorems described in Section 2 can be simplified considerably for order bounded measure-valued measures, such as the measures V_φ described in Section 1.

Example 3.1. Let $\phi \in \mathcal{M}(\mathcal{B}(\mathbb{R}))$. Then $|V_\phi(B)| \leq V_{|\phi|}(B)$ for all $B \in \mathcal{B}$. Moreover, $|V_\phi|_{\mathcal{M}} = V_{|\phi|}$.

The modulus $|m|_{\mathcal{M}}$ is easily described.

Lemma 3.2. *Let $m : \mathcal{B} \to \mathcal{M}(\mathcal{E})$ be a measure-valued measure and let μ be the variation of the additive set function*

$$B \times E \longmapsto (m(B))(E), \qquad B \in \mathcal{B}, E \in \mathcal{E}.$$

Then m is order bounded if and only if $\mu(\Omega \times \Sigma) < \infty$. If m is order bounded, then the modulus $|m|_{\mathcal{M}} : \mathcal{B} \to \mathcal{M}_+(\mathcal{E})$ is given by $(|m|_{\mathcal{M}}(B))(E) = \mu(B \times E)$, for all $B \in \mathcal{B}$, $E \in \mathcal{E}$.

Another way of viewing a measure-valued measure is as a bimeasure. In general, the variation μ defined in Lemma 3.2 need not be σ-additive. A sufficient condition guaranteeing the σ-additive of μ is that m is regular in each variable. We denote the scalar measure $B \longmapsto (m(B))(E)$, $B \in \mathcal{B}$, by m_E for each $E \in \mathcal{E}$.

Proposition 3.3. *Let $m : \mathcal{B} \to \mathcal{M}_+(\mathcal{E})$ be a positive measure-valued measure. Then*

$$v_{\mathcal{M}(\mathcal{E})}(m)(B) = \beta_{\mathcal{X}}(m)(B) = v_{\mathcal{L}(\mathcal{X},\mathcal{M}(\mathcal{E},\mathcal{X}))}(m^{\mathcal{X}})(B) = m_\Sigma(B)$$

for all $B \in \mathcal{B}$.

Proof. i)

$$\begin{aligned} v(m)(B) &= \sup \sum_{j=1}^{n} \|m(B_j \cap B)\| \\ &= \sup(\sum_{j=1}^{n} m(B_j \cap B)(\Sigma)) \\ &= m(B)(\Sigma) \\ &= m_\Sigma(B). \end{aligned}$$

ii)

$$\begin{aligned} \beta_{\mathcal{X}}(m)(B) &= \sup_{\{x_j\}\{B_j\}} \|\sum_{j=1}^{n} x_j m(B_j \cap B)\|_{\mathcal{M}(\mathcal{E},\mathcal{X})} \\ &= \sup_{\{x_j\}\{B_j\}} \sup_{\substack{\{x^*\}\{E_k\} \\ |c_k|\le 1}} |\sum_{j,k} c_k \langle x_j, x^*\rangle (m(B_j \cap B))(E_k)| \\ &= \sup_{\substack{\{x^*\}\{E_k\} \\ \{x_j\}\{B_j\}|c_k|\le 1}} |\sum_{j,k} c_k \langle x_j, x^*\rangle \mu((B_j \cap B) \times E_k)| \\ &= \sup \sum_{j,k} |\langle x_j, x^*\rangle| \mu((B_j \cap B) \times E_k) \\ &= \sup \sum_{j} |\langle x_j, x^*\rangle| \mu((B_j \cap B) \times \Sigma) \\ &= \sup \sum_{j} |\langle x_j, x^*\rangle| m_\Sigma(B_j \cap B) \\ &= m_\Sigma(B) \end{aligned}$$

where $c_k \in \mathbb{C}$.

$$\begin{aligned} v(m^{\mathcal{X}})(B) &= \sup \sum_{j=1}^{n} \|m(B_j \cap B)\|_{\mathcal{L}(\mathcal{X},\mathcal{M}(\mathcal{E},\mathcal{X}))} \\ &= \sup \sum_{j=1}^{n} \sup_{\|x\|\le 1} \|xm(B_j \cap B)\|_{\mathcal{M}(\mathcal{E},\mathcal{X})} \\ &= \sup \sum_{j=1}^{n} \sup_{\substack{\|x\|\le 1, \|x^*\|\le 1 \\ |c_k|\le 1}} |\sum_k \langle x, x^* \rangle c_k m(B_j \cap B)(E_k)| \\ &= m_\Sigma(B). \end{aligned}$$

□

Corollary 3.4. *Let $m : \mathcal{B} \to \mathcal{M}(\mathcal{E})$ be an order bounded measure-valued measure with modulus $|m|_{\mathcal{M}}$. Let $f : \Omega \to \mathcal{X}$ be an $(|m|_{\mathcal{M}})_\Sigma$-Bochner integrable function. Then f is m_E-Bochner integrable for each $E \in \mathcal{E}$ and m-integrable in $\mathcal{M}_v(\mathcal{E}, \mathcal{X})$. The equality*

$$\Big(\int_B f \otimes dm\Big)(E) = \int_B f dm_E$$

holds for all $B \in \mathcal{B}$ and $E \in \mathcal{E}$. Moreover,

$$\big(v_{\mathcal{M}_v(\mathcal{E},\mathcal{X})}\big(\int f \otimes dm\big)\big)(B) \le \int_B \|f\|_{\mathcal{X}} d(|m|_{\mathcal{M}})_\Sigma.$$

Proof. [3, Theorem 6]. □

For order bounded measure-valued measures m, the m-integrability of strongly measurable $\mathcal{X}$-valued functions is equivalent to their Pettis integrability with respect to an associated scalar measure.

Proposition 3.5. *Let $m : \mathcal{B} \to \mathcal{M}(\mathcal{E})$ be an order bounded measure-valued measure with modulus $|m|_{\mathcal{M}}$. A strongly m-measurable function $f : \Omega \to \mathcal{X}$ is m-integrable in $\mathcal{M}(\mathcal{E}, \mathcal{X})$ if and only if it is Pettis $(|m|_{\mathcal{M}})_\Sigma$-integrable in $\mathcal{X}$. In this case, f is Pettis m_E-integrable in $\mathcal{X}$ for each $E \in \mathcal{E}$ and the equality*

$$\Big(\int_B f \otimes dm\Big)(E) = \int_B f dm_E$$

holds for all $B \in \mathcal{B}$ and $E \in \mathcal{E}$.

Proof. To show this proposition, it is sufficient to show that

$$\Big(\int_B f \otimes dm\Big)(E) = \int_B f(t)\mu(dt \times E).$$

Since all simple functions are $m^{\mathcal{X}}$-integrable, above equality is true for simple f_n. As general proof method of integral, it is satisfying the equality for the function f such that simple functions f_n converges to f $m^{\mathcal{X}}$-*a.e.* And $\{(f_n m^{\mathcal{X}})(\cdot)(E) : E \in \mathcal{E}, n = 1, 2, \ldots\}$ is uniformly countably additive if and only if $\{\int |\langle f_n, x^*\rangle|\mu(dt \times E) : \|x^*\| \leq 1, n = 1, 2, \ldots\}$ is uniformly countably additive if and only if f is m_Σ-Pettis integrable. □

Remark 3.6. If m is positive and $\beta_{\mathcal{X}}(m)(f) < \infty$, then $\int_\Omega \|f\|_{\mathcal{X}} dm_\Sigma < \infty$. See [4].

Proposition 3.7. *Let $m : \mathcal{B} \to \mathcal{M}(\mathcal{E})$ be an order bounded measure-valued measure. Then for any Banach space $\mathcal{X}$, we have $\beta_{\mathcal{X}}(m)(\Omega) < \infty$ and $\beta_{\mathcal{X}}(m)$ is continuous.*

Proof. We have

$$\begin{aligned}\beta_{\mathcal{X}}(m)(E) &= \sup\{|\sum_{j,k}\langle x_k m(B_k \cap E)(B_j), x_j^*\rangle| : \|x_k\|, \|x_j^*\| \leq 1\} \\ &\leq (|m|_{\mathcal{M}}(E))(\Sigma) \\ &\leq (|m|_{\mathcal{M}})_\Sigma(E),\end{aligned}$$

so $\beta_{\mathcal{X}}(m)(\Omega)$ is finite. Because $|m|_{\mathcal{M}} : \mathcal{B} \to \mathcal{M}_+(\mathcal{E})$ is a measure, it follows that $\beta_{\mathcal{X}}(m)$ is continuous. □

For order bounded measure-valued measures, we have the following simplified versions of the convergence theorem of Section 2. We state them without proof.

Theorem 3.8. (Egorov) *Let $(\Sigma, \mathcal{E})$, $(\Omega, \mathcal{B})$ be measurable spaces and $\mathcal{X}$ a Banach space. Suppose that $m : \mathcal{B} \to \mathcal{M}(\mathcal{E})$ is an order bounded $\mathcal{M}(\mathcal{E})$-valued measure. Let f_n, $f : \Omega \to \mathcal{X}$, $n \in \mathbb{N}$ be m-measurable functions such that $f_n \to f$, m-a.e. Then* a) *for any $\varepsilon > 0$, there is a set $B \in \mathcal{B}$ such that $\beta_{\mathcal{X}}(m)(B^c) < \varepsilon$ and $f_n \to f$ uniformly on B.* b) *$f_n \to f$, $\beta_{\mathcal{X}}(m)$-measure.*

Theorem 3.9. (Vitali) *Let $(\Sigma, \mathcal{E})$, $(\Omega, \mathcal{B})$ be measurable spaces and $\mathcal{X}$ a Banach space. Suppose that $m : \mathcal{B} \to \mathcal{M}(\mathcal{E})$ is an order bounded $\mathcal{M}(\mathcal{E})$-valued measure with modulus $|m|_{\mathcal{M}}$. Let $\langle f_n\rangle$ be a sequence from $\mathcal{L}_1(\beta_{\mathcal{X}}(m))$ with f_n Pettis $(|m|_{\mathcal{M}})_\Sigma$-integrable functions and let $f : \Omega \to \mathcal{X}$ be m-measurable. Assume that* a) *$f_n \to f$ in $\beta_{\mathcal{X}}(m)$-measure or* a') *$f_n \to f$, m-a.e.* b) *$\lim_{\beta_{\mathcal{X}}(m)(A)\to 0} \beta_{\mathcal{X}}(m)(f_n\chi_A) = 0$, uniformly for $n \in \mathbb{N}$. Then $f \in \mathcal{L}_1(\beta_{\mathcal{X}}(m))$ and $\beta_{\mathcal{X}}(m)(f_n - f) \to 0$. Furthermore, the function f is m-integrable in $\mathcal{M}(\mathcal{E}, \mathcal{X})$ and $\int_B f_n \otimes dm \to \int_B f \otimes dm$ in $\mathcal{M}(\mathcal{E}, \mathcal{X})$, uniformly for $B \in \mathcal{B}$ as $n \to \infty$. Conversely, if $f \in \mathcal{L}_1(\beta_{\mathcal{X}}(m))$ and $\beta_{\mathcal{X}}(m)(f_n - f) \to 0$ as $n \to \infty$, then conditions* a) *and* b) *are satisfied.*

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