

ON COMPLETE CONVERGENCE FOR ARRAYS OF ROWWISE INDEPENDENT RANDOM ELEMENTS

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ABSTRACT. A complete convergence theorem for arrays of rowwise independent random variables was proved by Sung, Volodin, and Hu [14]. In this paper, we extend this theorem to the Banach space without any geometric assumptions on the underlying Banach space. Our theorem also improves some known results from the literature.

1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [5] as follows. A sequence $\{U_n, n \geq 1\}$ of random variables *converges completely* to the constant θ if

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \epsilon) < \infty \text{ for all } \epsilon > 0.$$

In view of the Borel-Cantelli lemma, this implies that $U_n \rightarrow \theta$ almost surely. The converse is true if $\{U_n, n \geq 1\}$ are independent random variables. Hsu and Robbins [5] proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite. Erdős [2] proved the converse. We refer to Gut [3] for a survey on results on complete convergence related to strong laws and published before the nineties.

The result of Hsu-Robbins-Erdős has been generalized and extended in several directions. Some of these generalizations are in a Banach space setting. A sequence of Banach space valued random elements is said to *converge completely* to the 0 element of the Banach space if the corresponding sequence of norms converges completely to 0.

Recently, Sung et al. [14] proved the following complete convergence theorem for arrays of rowwise independent random variables.

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Theorem 1. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random variables, $\{k_n, n \geq 1\}$ a sequence of positive integers and $\{a_n, n \geq 1\}$ a sequence of positive constants. Suppose that for every $\epsilon > 0$ and some $\delta > 0$:*

- (i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|X_{ni}| > \epsilon) < \infty,$
- (ii) *there exists $J \geq 2$ such that*

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}| \leq \delta) \right)^J < \infty,$$

- (iii) $\sum_{i=1}^{k_n} EX_{ni} I(|X_{ni}| \leq \delta) \rightarrow 0$ as $n \rightarrow \infty.$

Then $\sum_{n=1}^{\infty} a_n P(|\sum_{i=1}^{k_n} X_{ni}| > \epsilon) < \infty$ for all $\epsilon > 0.$

Theorem 1 was first presented by Hu et al. [8]. Hu and Volodin [9] imposed one additional condition in addendum to the paper. Many people tried to prove Theorem 1 without the additional condition (for random variables, see Hu et al. [6] and Kuczmaszewska [11], and for random elements, see Hu et al. [7]).

The following theorem is a version of Banach space setting of Theorem 1 and is due to Hu et al. [7]. No assumptions are made concerning the geometry of the underlying Banach space.

Theorem 2. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements, $\{k_n, n \geq 1\}$ a sequence of positive integers and $\{a_n, n \geq 1\}$ a sequence of positive constants. Suppose that for every $\epsilon > 0$ and some $\delta > 0$:*

- (i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon) < \infty,$
- (ii) *there exists $J \geq 2$ such that*

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} E\|X_{ni}\|^2 I(\|X_{ni}\| \leq \delta) \right)^J < \infty,$$

- (iii) $\sum_{i=1}^{k_n} P(\|X_{ni}\| > \delta) = o(1),$
- (iv) $\|\sum_{i=1}^{k_n} X_{ni}\| \rightarrow 0$ in probability.

Then $\sum_{n=1}^{\infty} a_n P(\|\sum_{i=1}^{k_n} X_{ni}\| > \epsilon) < \infty$ for all $\epsilon > 0.$

In this paper, we extend Theorem 1 to the Banach space without any geometric assumptions on the underlying Banach space. Our result also improves Theorem 2. More precisely, Theorem 2 holds without condition (iii).

We state our first theorem which shows that $o(1)$ in condition (iii) of Theorem 2 can be replaced by $O(1)$. The proof will appear in Section 3.

Theorem 3. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements, $\{k_n, n \geq 1\}$ a sequence of positive integers and $\{a_n, n \geq 1\}$ a sequence of positive constants. Suppose that for every $\epsilon > 0$ and some $\delta > 0$:*

- (i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon) < \infty,$

(ii) there exists $J \geq 2$ such that

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} E \|X_{ni}\|^2 I(\|X_{ni}\| \leq \delta) \right)^J < \infty,$$

(iii) $\sum_{i=1}^{k_n} P(\|X_{ni}\| > \delta) = O(1)$,

(iv) $\|\sum_{i=1}^{k_n} X_{ni}\| \rightarrow 0$ in probability.

Then $\sum_{n=1}^{\infty} a_n P(\|\sum_{i=1}^{k_n} X_{ni}\| > \epsilon) < \infty$ for all $\epsilon > 0$.

The following theorem is our main result which shows that condition (iii) of Theorem 2 can be removed. The proof will appear in Section 3.

Theorem 4. Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements, $\{k_n, n \geq 1\}$ a sequence of positive integers and $\{a_n, n \geq 1\}$ a sequence of positive constants. Suppose that for every $\epsilon > 0$ and some $\delta > 0$:

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon) < \infty$,

(ii) there exists $J \geq 2$ such that

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} E \|X_{ni}\|^2 I(\|X_{ni}\| \leq \delta) \right)^J < \infty,$$

(iii) $\|\sum_{i=1}^{k_n} X_{ni}\| \rightarrow 0$ in probability.

Then $\sum_{n=1}^{\infty} a_n P(\|\sum_{i=1}^{k_n} X_{ni}\| > \epsilon) < \infty$ for all $\epsilon > 0$.

As an application of Theorem 4, we have the following corollary, which will be proved in Section 3.

Corollary 1. Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise independent random elements which are weakly mean dominated by a random variable X , that is, there exists a constant $C > 0$ such that $\frac{1}{n} \sum_{i=1}^n P(\|X_{ni}\| > x) \leq CP(|X| > x)$ for all $x \geq 0$ and $n \geq 1$. Let α and ϕ be nondecreasing functions defined on $(0, \infty)$ satisfying

$$0 < \alpha(x) \uparrow \infty \text{ and } 0 < \phi(2x) \leq D\phi(x) \text{ for all } x > 0,$$

where $D > 0$ is a constant. Suppose that $E\phi(|X|) < \infty, E|X|^s < \infty$ for some $1 \leq s \leq 2, \|\sum_{i=1}^n X_{ni}\|/\alpha(n) \rightarrow 0$ in probability, and there exists $J \geq 2$ such that

$$\sum_{n=1}^{\infty} \frac{\phi(\alpha(n)) - \phi(\alpha(n-1))}{n} \left(\frac{n}{\alpha^s(n)} \right)^J < \infty.$$

Then

$$\sum_{n=1}^{\infty} \frac{\phi(\alpha(n)) - \phi(\alpha(n-1))}{n} P\left(\left\|\sum_{i=1}^n X_{ni}\right\| > \epsilon\alpha(n)\right) < \infty \text{ for all } \epsilon > 0.$$

2. Preliminary lemmas

Let B be a real separable Banach space with norm $\|\cdot\|$. Let (Ω, \mathcal{F}, P) be a probability space. A random element (or B -valued random element) is defined to be a \mathcal{F} -measurable mapping from Ω to B equipped with the Borel σ -algebra (the σ -algebra generated by the open sets determined by $\|\cdot\|$). The expected value of a random element X is defined to be Bochner integral (when $E\|X\| < \infty$) and is denoted by EX .

The following lemma is an iterated form of Hoffmann-Jørgensen [4] inequality and is due to Jain [10].

Lemma 1. *If X_1, \dots, X_n are independent symmetric random elements, then for every integer $j \geq 1$ and every $t > 0$*

$$P(\|S_n\| > 3^j t) \leq C_j P(\max_{1 \leq i \leq n} \|X_i\| > t) + D_j \left(P(\|S_n\| > t) \right)^{2^j},$$

where C_j and D_j are positive constants depending only on j , and $S_n = \sum_{i=1}^n X_i$.

The following lemma gives us a useful contraction principle and can be found in Lemma 6.5 of Ledoux and Talagrand [13].

Lemma 2. *Let $\{X_i, i \geq 1\}$ be a sequence of symmetric random elements. Let further $\{\xi_i, i \geq 1\}$ and $\{\zeta_i, i \geq 1\}$ be real random variables such that $\xi_i = \phi_i(X_i)$, where $\phi_i : B \rightarrow \mathbb{R}$ is symmetric (even), and similarly for ζ_i . Then, if $|\xi_i| \leq |\zeta_i|$ almost surely for every i , for every $t > 0$*

$$P(\|\sum_i \xi_i X_i\| > t) \leq 2P(\|\sum_i \zeta_i X_i\| > t).$$

In particular, this inequality applies when $\xi_i = I_{\{X_i \in A_i\}} \leq 1 \equiv \zeta_i$ where the sets A_i are symmetric in B (in particular $A_i = \{\|x\| \leq a_i\}$).

The next lemma is a modification of a result of Kuelbs and Zinn [12] concerning the relationship between convergence in probability and mean convergence for sums of independent bounded random variables. We refer to Lemma 2.1 of Hu et al. [7] for the proof.

Lemma 3. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent symmetric random elements. Suppose there exists $\delta > 0$ such that $\|X_{ni}\| \leq \delta$ almost surely for all $1 \leq i \leq k_n, n \geq 1$. Put $S_n = \sum_{i=1}^{k_n} X_{ni}$. If $S_n \rightarrow 0$ in probability, then $E\|S_n\| \rightarrow 0$ as $n \rightarrow \infty$.*

The following inequality is a Banach space analogue of the classical Marcinkiewicz-Zygmund inequality and is due to de Acosta [1]. When $p = 2$, C_2 can be taken to be 4.

Lemma 4. *Let $\{X_i, 1 \leq i \leq n\}$ be a sequence of independent random elements. Then for $1 < p \leq 2$, there is a positive constant C_p depending only on p such*

that

$$E\|S_n\| - E\|S_n\|^p \leq C_p \sum_{i=1}^n E\|X_i\|^p,$$

where $S_n = \sum_{i=1}^n X_i$.

Finally, we need the following lemma. The proof is standard and is omitted.

Lemma 5. *If X and Y have the same distribution, then for every $t > 0$*

$$E\|X - Y\|^2 I(\|X - Y\| \leq t) \leq 8E\|X\|^2 I(\|X\| \leq \frac{t}{2}) + 2t^2 P(\|X\| > \frac{t}{2}).$$

3. Proofs

Proof of Theorem 3. Let $\{X_{ni}^s, 1 \leq i \leq k_n, n \geq 1\}$ be an array of the symmetrized version of $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$, i.e., $X_{ni}^s = X_{ni} - X_{ni}^*$, where X_{ni} and X_{ni}^* are independent and have the same distribution. Let μ_n be a median of $\|\sum_{i=1}^{k_n} X_{ni}\|$. By (iv), $\mu_n \rightarrow 0$ as $n \rightarrow \infty$. Then we have by the weak symmetrization inequality that for all large n

$$\begin{aligned} P(\|\sum_{i=1}^{k_n} X_{ni}\| > \epsilon) &\leq P(\|\sum_{i=1}^{k_n} X_{ni}\| - \mu_n > \frac{\epsilon}{2}) \leq 2P(\|\sum_{i=1}^{k_n} X_{ni}^s\| > \frac{\epsilon}{2}) \\ &\leq 2P(\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| > \frac{\epsilon}{4}) + 2P(\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| > 2\delta)\| > \frac{\epsilon}{4}) \\ &\leq 2P(\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| > \frac{\epsilon}{4}) + 2\sum_{i=1}^{k_n} P(\|X_{ni}^s\| > 2\delta) \\ &\leq 2P(\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| > \frac{\epsilon}{4}) + 4\sum_{i=1}^{k_n} P(\|X_{ni}\| > \delta). \end{aligned}$$

By (i), it is enough to prove that

$$\sum_{n=1}^{\infty} a_n P(\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| > \frac{\epsilon}{4}) < \infty.$$

By Lemma 2 and (iv), we have that

$$\begin{aligned} &P(\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| > \frac{\epsilon}{4}) \\ &\leq 2P(\|\sum_{i=1}^{k_n} X_{ni}^s\| > \frac{\epsilon}{4}) \end{aligned}$$

$$\begin{aligned} &\leq 2P(\|\sum_{i=1}^{k_n} X_{ni}\| + \|\sum_{i=1}^{k_n} X_{ni}^*\| > \frac{\epsilon}{4}) \\ &\leq 4P(\|\sum_{i=1}^{k_n} X_{ni}\| > \frac{\epsilon}{8}) \rightarrow 0. \end{aligned}$$

Noting that $\|X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| \leq 2\delta$, it follows by Lemma 3 that

$$E\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| \rightarrow 0$$

as $n \rightarrow \infty$. Take an integer j such that $2^j \geq J$. Then we have by Lemma 1 that

$$\begin{aligned} &P(\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| > \frac{\epsilon}{4}) \\ &\leq C_j P(\max_{1 \leq i \leq k_n} \|X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| > \frac{\epsilon}{4 \cdot 3^j}) \\ &\quad + D_j P\left(\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| > \frac{\epsilon}{4 \cdot 3^j}\right)^{2^j} \\ &\leq C_j P(\max_{1 \leq i \leq k_n} \|X_{ni}^s\| > \frac{\epsilon}{4 \cdot 3^j}) \\ &\quad + D_j P\left(\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| > \frac{\epsilon}{4 \cdot 3^j}\right)^J \\ &\leq 2C_j \sum_{i=1}^{k_n} P(\|X_{ni}\| > \frac{\epsilon}{8 \cdot 3^j}) \\ &\quad + D_j P\left(\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| > \frac{\epsilon}{4 \cdot 3^j}\right)^J. \end{aligned}$$

Thus, by (i), it suffices to prove that

$$\sum_{n=1}^{\infty} a_n P\left(\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| > \frac{\epsilon}{4 \cdot 3^j}\right)^J < \infty.$$

Since $E\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| \rightarrow 0$, it follows by Lemma 4 and Lemma 5 that for all large n

$$\begin{aligned} &P(\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| > \frac{\epsilon}{4 \cdot 3^j}) \\ &\leq P(\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| - E\|\sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta)\| > \frac{\epsilon}{8 \cdot 3^j}) \end{aligned}$$

$$\begin{aligned} &\leq \left(\frac{8 \cdot 3^j}{\epsilon}\right)^2 E \left| \left\| \sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta) \right\| - E \left\| \sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta) \right\| \right|^2 \\ &\leq 4 \left(\frac{8 \cdot 3^j}{\epsilon}\right)^2 \sum_{i=1}^{k_n} E \|X_{ni}^s\|^2 I(\|X_{ni}^s\| \leq 2\delta) \\ &\leq 4 \left(\frac{8 \cdot 3^j}{\epsilon}\right)^2 \sum_{i=1}^{k_n} \left\{ 8E \|X_{ni}\|^2 I(\|X_{ni}\| \leq \delta) + 8\delta^2 P(\|X_{ni}\| > \delta) \right\}. \end{aligned}$$

Noting that $\left(\sum_{i=1}^{k_n} P(\|X_{ni}\| > \delta)\right)^J \leq O(1) \sum_{i=1}^{k_n} P(\|X_{ni}\| > \delta)$ by (iii), the c_r -inequality implies that $\sum_{n=1}^{\infty} a_n P\left(\left\| \sum_{i=1}^{k_n} X_{ni}^s I(\|X_{ni}^s\| \leq 2\delta) \right\| > \frac{\epsilon}{4 \cdot 3^j}\right)^J < \infty$ by (i) and (ii). Thus the proof is complete. \square

Proof of Theorem 4. Let $A = \{n \mid \sum_{i=1}^{k_n} P(\|X_{ni}\| > \delta) \leq 1\}$. Define a sequence $\{u_n, n \geq 1\}$ of positive integers by

$$u_n = \begin{cases} k_n, & \text{if } n \in A, \\ 1, & \text{if } n \notin A. \end{cases}$$

Define an array $\{Y_{ni}, 1 \leq i \leq u_n, n \geq 1\}$ of random elements by

$$Y_{ni} = \begin{cases} X_{ni}, & \text{if } n \in A, \\ 0, & \text{if } n \notin A. \end{cases}$$

Then $\{Y_{ni}, 1 \leq i \leq u_n, n \geq 1\}$ satisfies all conditions of Theorem 3 and so we have that

$$\sum_{n \in A} a_n P\left(\left\| \sum_{i=1}^{k_n} X_{ni} \right\| > \epsilon\right) = \sum_{n=1}^{\infty} a_n P\left(\left\| \sum_{i=1}^{u_n} Y_{ni} \right\| > \epsilon\right) < \infty.$$

Observe that

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \delta) &= \sum_{n \in A} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \delta) \\ &\quad + \sum_{n \notin A} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \delta) \\ &\geq \sum_{n \in A} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \delta) + \sum_{n \notin A} a_n. \end{aligned}$$

It follows by (i) that $\sum_{n \notin A} a_n < \infty$. Thus we obtain that

$$\begin{aligned} \sum_{n=1}^{\infty} a_n P\left(\left\|\sum_{i=1}^{k_n} X_{ni}\right\| > \epsilon\right) &= \sum_{n \in A} a_n P\left(\left\|\sum_{i=1}^{k_n} X_{ni}\right\| > \epsilon\right) \\ &\quad + \sum_{n \notin A} a_n P\left(\left\|\sum_{i=1}^{k_n} X_{ni}\right\| > \epsilon\right) \\ &\leq \sum_{n \in A} a_n P\left(\left\|\sum_{i=1}^{k_n} X_{ni}\right\| > \epsilon\right) + \sum_{n \notin A} a_n < \infty. \end{aligned}$$

□

Proof of Corollary 1. We will apply Theorem 4 with $a_n = (\phi(\alpha(n)) - \phi(\alpha(n-1)))/n, n \geq 1$ and X_{ni} replaced by $X_{ni}/\alpha(n), 1 \leq i \leq n, n \geq 1$. We only need to verify that conditions (i) and (ii) of Theorem 4 hold. By the weak mean domination hypothesis, we have that

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \sum_{i=1}^n P\left(\left\|\frac{X_{ni}}{\alpha(n)}\right\| > \epsilon\right) &\leq C \sum_{n=1}^{\infty} a_n n P\left(\frac{|X|}{\epsilon} > \alpha(n)\right) \\ &= C \sum_{i=1}^{\infty} P\left(\alpha(i) < \frac{|X|}{\epsilon} \leq \alpha(i+1)\right) \sum_{n=1}^i n a_n \\ &\leq CE\phi\left(\frac{|X|}{\epsilon}\right) < \infty, \end{aligned}$$

since $E\phi(|X|) < \infty$ and $\phi(2x) \leq D\phi(x)$. Hence condition (i) holds.

To establish condition (ii), note that

$$\begin{aligned} &\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^n E\left\|\frac{X_{ni}}{\alpha(n)}\right\|^2 I\left(\left\|\frac{X_{ni}}{\alpha(n)}\right\| \leq \delta\right)\right)^J \\ &\leq \delta^{J(2-s)} \sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^n E\left\|\frac{X_{ni}}{\alpha(n)}\right\|^s\right)^J \\ &\leq \delta^{J(2-s)} \sum_{n=1}^{\infty} a_n \left(\frac{nCE|X|^s}{\alpha^s(n)}\right)^J, \end{aligned}$$

since $\sum_{i=1}^n E\|X_{ni}\|^s \leq nCE|X|^s$ by the weak mean domination. Hence condition (ii) holds. □

A sequence $\{U_n, n \geq 1\}$ of random variables is *bounded in probability* if for every $\epsilon > 0$ there exists a constant $C > 0$ such that $P(|U_n| > C) < \epsilon$ for all $n \geq 1$.

Remark 1. Let $\{U_n, n \geq 1\}$ be a bounded in probability. Let $\{\beta_n, n \geq 1\}$ be a sequence of positive real numbers such that $\beta_n \rightarrow 0$ as $n \rightarrow \infty$. Then $\beta_n U_n \rightarrow 0$ in probability.

Proof. Let $\epsilon > 0$ and $\delta > 0$ be given. Since $\{U_n, n \geq 1\}$ is bounded in probability, there exists a constant $C > 0$ such that $P(|U_n| > C) < \epsilon$ for all $n \geq 1$. Since $\beta_n \rightarrow 0$ as $n \rightarrow \infty$, there exists a positive integer N such that $\beta_n < \delta/C$ if $n > N$. For $n > N$, $P(|\beta_n U_n| > \delta) \leq P(|U_n| > C) < \epsilon$. Hence the proof is complete. \square

Remark 2. Let $\{\gamma_n, n \geq 1\}$ be a sequence of positive real numbers such that $\lim_{n \rightarrow \infty} \alpha(n)/\gamma_n = \infty$, and $\{\|\sum_{i=1}^n X_{ni}\|/\gamma_n, n \geq 1\}$ is bounded in probability. By Remark 1, $\|\sum_{i=1}^n X_{ni}\|/\alpha(n) \rightarrow 0$ in probability. When $\theta(x) = |x|^s$ for some $s > 0$,

$$\frac{n}{\alpha^s(n)} < \frac{1}{r} \left(\frac{rn + \theta(\gamma_n)}{\theta(\alpha(n))} \right) \text{ for any } r > 0.$$

It follows that Corollary 1 improves Theorem 3.1 of Tómacs [15] when $\theta(x) = |x|^s$ for some $1 \leq s \leq 2$.

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