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WEAK SOLUTIONS OF THE EQUATION OF MOTION OF MEMBRANE WITH STRONG VISCOSITY

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Abstract. We study the equation of a membrane with strong viscosity. Based on the variational formulation corresponding to the suitable function space setting, we have proved the fundamental results on existence, uniqueness and continuous dependence on data of weak solutions.

1. Introduction

A freely flexible stretched film is called a membrane. Let Ω be an open bounded set of $\mathbf{R}^n$ with the smooth boundary Γ. We set $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$ for $T > 0$. The nonlinear equation of the longitudinal motion of a vibrating membrane surrounding Ω is described by the following Dirichlet boundary value problem:

$$\frac{\partial^2 y}{\partial t^2} - \operatorname{div}\Big(\frac{\nabla y}{\sqrt{1+|\nabla y|^2}}\Big) = 0 \quad \text{in } Q, \tag{1.1}$$

with

$$y = 0 \quad \text{on } \Sigma, \tag{1.2}$$

$$y(0,x) = y_0(x), \qquad \frac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in } \Omega, \tag{1.3}$$

where y is the height of the membrane. Then the reasonable physical candidate for the potential energy is the surface area of $h = y(x)$, $x \in \Omega$, since energy is stored in the membrane when it is stretched. Equation (1.1) is derived as the Euler-Lagrange equation of the action integral

$$\int_0^T \Big(\int_\Omega \frac{1}{2}\Big|\frac{\partial y}{\partial t}\Big|^2 dx - J(y)\Big)dt,$$

where $J(y)$ is the surface area of the graph y. And it is well known that the nonlinear term in (1.1) appears in the minimal surface problems as a nonlinear elliptic operator. For the base of that we refer to Gilbarg and Trudinger [3].

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Recently, there are several authors related to this problem (1.1)-(1.3). Kikuchi [8] has treated this problem in the space of functions having bounded variation and constructed approximate solutions by Rothe's method. But it seems to be difficult to construct a solution in a Hilbert or reflexive Banach spaces not only for theoretical construction but also for any other applications. So we consider some modified but more realistic model given by the following problem with viscosity terms:

$$
(1.4)\qquad \begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \operatorname{div}\Big(\dfrac{\nabla y}{\sqrt{1+|\nabla y|^2}}\Big) - \mu\Delta\dfrac{\partial y}{\partial t} = f \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0,x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in } \Omega, \end{cases}
$$

where $\mu > 0$ and f is a forcing function. Damping mechanism appears extendly and naturally in physical situations and there are many factors of it. We classify it largely by air and structural factors. Among them the modified problem (1.4) seems to be structually damped case. In the linear and semilinear cases, for the research of damped systems, there are a lot of books and articles about the well posedness and the practical applications (cf. [2], [10], [1], etc.) with semigroup or unified variational treatments. However the quasilinear cases like as this case require more manipulations in the analysis of systems. Because the damped systems are very much model-dependent due to the strong nonlinearity. In fact, the proposal of this problem can be found in [4] and [5] as a model of quasilinear wave equations (see also Temam [10]). Especially, it is given in Kobayashi, Pecher and Shibata [9] the proof of the existence of solutions of (1.4). Using some regular data conditions, they used resolvent estimates to construct regular solutions in a modified Banach space. However, it seems that there are a little researches on the variational treatment of (1.4) and the related control problems. Our aim of this paper is to prove the basic results on existence and uniqueness of weak solutions of (1.4) in the framework of variational method in Dautray and Lions [2]. The most difficult part of the existence proof is to show the strong convergence of nonlinear terms, and the part is completed by using the argument in [6] (see also [2, p.569] for the linear case). Finally we note that the quadratic optimal control problems associated with the equation (1.4) are studied in Hwang and Nakagiri [7].

2. Main results

We study the following Dirichlet boundary value problem for the equation of motion of a membrane with strong viscosity

$$
(2.1)\qquad \begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \operatorname{div}\Big(\dfrac{\nabla y}{\sqrt{1+|\nabla y|^2}}\Big) - \mu\Delta\dfrac{\partial y}{\partial t} = f \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0,x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in } \Omega, \end{cases}
$$

where f is a forcing function, y_0 and y_1 are initial data and $\mu > 0$ is a constant. In (2.1) we suppose $f \in L^2(0,T;H^{-1}(\Omega))$, $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$. The solution space $W(0,T)$ of (2.1) is defined by

$$\{g | g \in L^2(0,T;H_0^1(\Omega)),\ g' \in L^2(0,T;H_0^1(\Omega)),\ g'' \in L^2(0,T;H^{-1}(\Omega))\}$$

endowed with the norm

$$\|g\|_{W(0,T)} = \left(\|g\|^2_{L^2(0,T;H_0^1(\Omega))} + \|g'\|^2_{L^2(0,T;H_0^1(\Omega))} + \|g''\|^2_{L^2(0,T;H^{-1}(\Omega))}\right)^{\frac{1}{2}},$$

where $g' = \frac{dg}{dt}$ and $g'' = \frac{d^2g}{dt^2}$ (cf. Dautray and Lions [2, p.471]). We remark that $W(0,T)$ is continuously imbedded in $C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ (cf. [2, p.555]). The scalar products and norms on $L^2(\Omega)$ and $H_0^1(\Omega)$ are denoted by (ϕ,ψ), $|\phi|$ and $(\phi,\psi)_{H_0^1(\Omega)}$, $\|\phi\|$, respectively. The scalar product and norm on $[L^2(\Omega)]^n$ are also denoted by (ϕ,ψ) and $|\phi|$. Then the scalar product $(\phi,\psi)_{H_0^1(\Omega)}$ and the norm $\|\phi\|$ of $H_0^1(\Omega)$ are given by $(\nabla\phi,\nabla\psi)$ and $\|\phi\| = |(\nabla\phi,\nabla\phi)|^{\frac{1}{2}}$, respectively. The duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ is denoted by $\langle\phi,\psi\rangle$. Related to the nonlinear term in (2.1), we define the function $G : \mathbf{R}^n \to \mathbf{R}^n$ by $G(x) = \dfrac{x}{\sqrt{1+|x|^2}}, x \in \mathbf{R}^n$. Then it is easily verified that

$$|G(x) - G(y)| \le 2|x-y|, \quad \forall x,y \in \mathbf{R}^n. \tag{2.2}$$

The nonlinear operator $G(\nabla\cdot) : H_0^1(\Omega) \to [L^2(\Omega)]^n$ is introduced by

$$G(\nabla\phi)(x) = \frac{\nabla\phi(x)}{\sqrt{1+|\nabla\phi(x)|^2}}, \quad \text{a.e. } x \in \Omega, \quad \forall\phi \in H_0^1(\Omega). \tag{2.3}$$

By the definition of $G(\nabla\cdot)$ in (2.3), we have the following useful property on $G(\nabla\cdot)$:

$$|G(\nabla\phi)| \le |\nabla\phi|, \quad |G(\nabla\phi) - G(\nabla\psi)| \le 2|\nabla\phi - \nabla\psi|, \quad \forall\phi,\psi \in H_0^1(\Omega). \tag{2.4}$$

Definition 2.1. A function y is said to be a weak solution of (2.1) if $y \in W(0,T)$ and y satisfies

$$\begin{cases} \langle y''(\cdot),\phi\rangle + (G(\nabla y(\cdot)),\nabla\phi) + \mu(\nabla y'(\cdot),\nabla\phi) = \langle f(\cdot),\phi\rangle \\ \qquad\qquad \text{for all } \phi \in H_0^1(\Omega) \text{ in the sense of } \mathcal{D}'(0,T), \\ y(0) = y_0 \in H_0^1(\Omega), \quad y'(0) = y_1 \in L^2(\Omega). \end{cases} \tag{2.5}$$

The following theorem gives the result on existence, uniqueness and regularity of the weak solution of (2.1).

Theorem 2.2. *Assume that $\mu > 0$, $f \in L^2(0,T;H^{-1}(\Omega))$ and $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$. Then the problem (2.1) has a unique weak solution y in $W(0,T)$. Furthermore, y has the following estimate*

$$\begin{aligned} &|y'(t)|^2 + |\nabla y(t)|^2 + \int_0^t |\nabla y'(s)|^2 ds \\ &\le\ C(\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;H^{-1}(\Omega))}), \qquad \forall t \in [0,T], \end{aligned} \tag{2.6}$$

where C is a constant depending only on $\mu > 0$.

Next we give the result on the continuous dependence of weak solutions of (2.1) on initial values y_0, y_1 and forcing terms f. Let P be a product space defined by

$$P = H_0^1(\Omega) \times L^2(\Omega) \times L^2(0,T;H^{-1}(\Omega)). \tag{2.7}$$

For each $p = (y_0, y_1, f) \in P$ we have a unique weak solution $y = y(p) \in W(0,T)$ of (2.1) by Theorem 2.2. Hence we can define the solution mapping $p = (y_0, y_1, f) \to y(q)$ of P into $W(0,T)$.

Theorem 2.3. *The solution mapping $p = (y_0, y_1, f) \to y(q)$ of P into $W(0,T)$ is strongly continuous. Further, for each $p_1 = (y_0^1, y_1^1, f_1) \in P$ and $p_2 = (y_0^2, y_1^2, f_2) \in P$ we have the inequality*

$$\begin{aligned} &|y'(p_1;t) - y'(p_2;t)|^2 + |\nabla y(p_1;t) - \nabla y(p_2;t)|^2 \\ &+ \int_0^t |\nabla y'(p_1;s) - \nabla y'(p_2;s)|^2 ds \\ \leq\ & C(\|y_0^1 - y_0^2\|^2 + |y_1^1 - y_1^2|^2 + \|f_1 - f_2\|^2_{L^2(0,T;H^{-1}(\Omega))}),\ \forall t \in [0,T]. \end{aligned} \tag{2.8}$$

3. Proof of main results

We will omit writing the integral variables in the definite integral without any confusion. For example, in (2.6) we will write $\int_0^t |\nabla y'|^2 ds$ instead of $\int_0^t |\nabla y'(s)|^2 ds$.

Proof of Theorem 2.2. We construct approximate solutions of (2.1) by the Galerkin's procedure. Since $H_0^1(\Omega)$ is separable, there exists a complete orthonormal system $\{w_m\}_{m=1}^\infty$ in $L^2(\Omega)$ such that $\{w_m\}_{m=1}^\infty$ is free and total in $H_0^1(\Omega)$. For each $m = 1, 2, \ldots$, we define an approximate solution $y_m(t)$ of the equation (2.1)

$$y_m(t) = \sum_{j=1}^m g_{jm}(t) w_j,$$

where $y_m(t)$ satisfies

$$\begin{cases} (y_m''(t), w_j) + (G(\nabla y_m(t)), \nabla w_j) + \mu(\nabla y_m'(t), \nabla w_j) \\ \qquad\qquad = \langle f(t), w_j \rangle, \quad t \in [0,T], \quad 1 \leq j \leq m, \\ y_m(0) = \sum_{i=1}^m (y_0, w_i) w_i, \qquad y_m'(0) = \sum_{i=1}^m (y_1, w_i) w_i. \end{cases} \tag{3.1}$$

Let V_m be m dimensional space spanned by $\{w_1, \ldots, w_m\}$. Then we can see that

$$y_{0m} = \sum_{i=1}^m (y_0, w_i) w_i \in V_m \to y_0 \quad \text{in} \quad H_0^1(\Omega) \quad \text{as} \quad m \to \infty, \tag{3.2}$$

$$(3.3)\qquad y_{1m}=\sum_{i=1}^{m}(y_1,w_i)w_i\in V_m\to y_1 \quad \text{in} \quad L^2(\Omega) \ \text{as} \ m\to\infty.$$

Hence the equation (3.1) in V_m induces the system of nonlinear second order equations for $g_{jm}(t)$ with initial conditions $g_{jm}(0)=(y_0,w_j)$, $g'_{jm}(0)=(y_1,w_j)j=1,\dots,m$. Since the nonlinear term $G(\nabla y_m(t))$ in (3.1) is Lipschitz continuous by (2.4), i.e., for fixed $j\in\{1,\dots,m\}$

$$|(G(\nabla y_m^1(t))-G(\nabla y_m^2(t)),\nabla w_j)|\le 2|\nabla w_j||\nabla y_m^1(t)-\nabla y_m^2(t)|,$$

the system is also Lipschitz continuous in g_{im}. Hence, it is verified that the system admits a unique solution $g_{jm}(t), j=1,\dots,m$ over $[0,T]$. This allows us to construct the approximate solution $y_m(t)$ of (3.1). Now we will derive a priori estimates of $y_m(t)$. At first, we multiply both sides of the equation (3.1) by $g'_{jm}(t)$ and sum over j to have

$$(3.4)\qquad (y''_m(t),y'_m(t))+(G(\nabla y_m(t)),\nabla y'_m(t))+\mu|\nabla y'_m(t)|^2=\langle f(t),y'_m(t)\rangle.$$

Secondly, we also multiply both sides of the equation (3.1) by $g_{jm}(t)$ and sum over j to have

$$\begin{aligned}(3.5)\qquad &(y''_m(t),y_m(t))+(G(\nabla y_m(t)),\nabla y_m(t))+\mu(\nabla y'_m(t),\nabla y_m(t))\\ =\;&\langle f(t),y_m(t)\rangle.\end{aligned}$$

We sum (3.4) and (3.5) to have

$$\begin{aligned}(3.6)\qquad &(y''_m(t),y'_m(t))+\mu|\nabla y'_m(t)|^2+\mu(\nabla y'_m(t),\nabla y_m(t))\\ =\;&-(y''_m(t),y_m(t))+\langle f(t),y'_m(t)+y_m(t)\rangle\\ &-(G(\nabla y_m(t)),\nabla y'_m(t)+\nabla y_m(t)).\end{aligned}$$

Since

$$(y''_m(t),y'_m(t)) = \frac{1}{2}\frac{d}{dt}|y'_m(t)|^2,\quad \mu(\nabla y'_m(t),\nabla y_m(t))=\frac{\mu}{2}\frac{d}{dt}|\nabla y_m(t)|^2,$$

(3.6) can be rewritten as

$$\begin{aligned}(3.7)\qquad &\frac{1}{2}\frac{d}{dt}|y'_m(t)|^2+\mu|\nabla y'_m(t)|^2+\frac{\mu}{2}\frac{d}{dt}|\nabla y_m(t)|^2\\ =\;&-(y''_m(t),y_m(t))+\langle f(t),y'_m(t)+y_m(t)\rangle\\ &-(G(\nabla y_m(t)),\nabla y'_m(t)+\nabla y_m(t)).\end{aligned}$$

By integrating (3.7) over $[0,t]$, we obtain

$$\begin{aligned}(3.8)\qquad &|y'_m(t)|^2+2\mu\int_0^t|\nabla y'_m|^2ds+\mu|\nabla y_m(t)|^2\\ =\;&|y_{1m}|^2+\mu|\nabla y_{0m}|^2-\;2(y'_m(t),y_m(t))+2(y_{1m},y_{0m})\\ &+2\int_0^t|y'_m|^2ds+2\int_0^t\langle f,y'_m+y_m\rangle ds\\ &-2\int_0^t(G(\nabla y_m),\nabla y'_m+\nabla y_m)ds.\end{aligned}$$

Let $\epsilon > 0$ be an arbitrary real number. Then, we have by (2.4) and the Schwarz inequality

$$\left|2\int_0^t (G(\nabla y_m), \nabla y_m')ds\right| \le \frac{1}{\epsilon}\int_0^t |\nabla y_m|^2 ds + \epsilon\int_0^t |\nabla y_m'|^2 ds, \tag{3.9}$$

$$\left|2\int_0^t (G(\nabla y_m), \nabla y_m)ds\right| \le 2\int_0^t |\nabla y_m|^2 ds, \tag{3.10}$$

$$\left|2\int_0^t \langle f, y_m'\rangle ds\right| \le \frac{1}{\epsilon}\|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + \epsilon\int_0^t |\nabla y_m'|^2 ds, \tag{3.11}$$

$$\left|2\int_0^t \langle f, y_m\rangle ds\right| \le \|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + \int_0^t |\nabla y_m|^2 ds. \tag{3.12}$$

Also we have

$$\begin{aligned} 2|(y_m'(t), y_m(t))| &\le \epsilon|y_m'(t)|^2 + \frac{1}{\epsilon}\left|y_{0m} + \int_0^t y_m'(s)ds\right|^2 \\ &\le \epsilon|y_m'(t)|^2 + \frac{2}{\epsilon}|y_{0m}|^2 + \frac{2T}{\epsilon}\int_0^t |y_m'|^2 ds. \end{aligned} \tag{3.13}$$

We note by (3.2), (3.3) that $|y_{1m}| \le C'|y_1|$, $|\nabla y_{0m}| \le C'\|y_0\|$ for some $C' > 0$ independent of m. Therefore from (3.8) to (3.13), we can obtain the following inequality

$$\begin{aligned} &(1-\epsilon)|y_m'(t)|^2 + \mu|\nabla y_m(t)|^2 + (2\mu - 2\epsilon)\int_0^t |\nabla y_m'|^2 ds \\ \le\ & C_\epsilon(\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;H^{-1}(\Omega))}) \\ &+ (5 + \frac{1+2T}{\epsilon})\int_0^t (|y_m'|^2 + |\nabla y_m|^2)ds, \end{aligned} \tag{3.14}$$

for some $C_\epsilon > 0$. If we choose $\epsilon = \min\{\frac{1}{2}, \frac{\mu}{2}\}$, then by Gronwall's inequality it follows that

$$\begin{aligned} &|y_m'(t)|^2 + |\nabla y_m(t)|^2 + \int_0^t |\nabla y_m'|^2 ds \\ \le\ & C(\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;H^{-1}(\Omega))}), \end{aligned} \tag{3.15}$$

where C is some positive constant independent of m. Therefore y_m and y_m' remain in a bounded sets of $L^\infty(0,T;H_0^1(\Omega))$ and $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$, respectively. And the nonlinear term $G(\nabla y_m)$ is uniformly bounded. Hence by the extraction theorem of Rellich's, we can extract a subsequence $\{y_{m_k}\}$ of $\{y_m\}$ and find $z \in L^\infty(0,T;H_0^1(\Omega))$, $z' \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;$

$H_0^1(\Omega))$ and also $F(\cdot) \in L^2(0,T;L^2(\Omega))$ such that

$$(3.16) \quad \begin{aligned} &\text{i)} \quad y_{m_k} \to z \ \text{ weakly-star in } \ L^\infty(0,T;H_0^1(\Omega)), \\ &\qquad\qquad \text{and weakly in } L^2(0,T;H_0^1(\Omega)), \\ &\text{ii)} \quad y'_{m_k} \to z' \ \text{ weakly-star in } \ L^\infty(0,T;L^2(\Omega)), \\ &\qquad\qquad \text{and weakly in } L^2(0,T;H_0^1(\Omega)), \\ &\text{iii)} \quad \mu\Delta y'_{m_k} \to \mu\Delta z' \ \text{ weakly in } \ L^2(0,T;H^{-1}(\Omega)), \\ &\text{iv)} \quad G(\nabla y_m) \ \to \ F(\cdot) \text{ weakly in } \ L^2(0,T;L^2(\Omega)), \end{aligned}$$

as $k \to \infty$. By the standard argument of Dautray and Lions [2, pp. 564-566], in view of (3.16), it can be verified that the limit z of $\{y_{m_k}\}$ belong to $W(0,T)$ and is a weak solution of the linear problem

$$(3.17) \quad \begin{cases} \dfrac{\partial^2 z}{\partial t^2} - \text{div}F(t) - \mu\Delta\dfrac{\partial z}{\partial t} = f \quad \text{in } \ Q, \\ z = 0 \quad \text{ on } \Sigma, \\ z(0,x) = y_0, \quad \dfrac{\partial z}{\partial t}(0,x) = y_1 \ \text{ in } \Omega. \end{cases}$$

Thus, to prove that z is a weak solution of (2.1), we need to show that $F(\cdot) = G(\nabla z(\cdot))$. For the purpose we shall show $\nabla y_m(t) \to \nabla z(t)$ strongly in $[L^2(\Omega)]^n$. To this end we use the strong convergence arguments in [6]. For notational simplicity, we denote y_{m_k} by y_m again. The approximate solution $y_m(t)$ satisfies (3.8). For the weak solution z of (3.17), we can obtain the following equality similarly as for $y_m(t)$ as in [2, p.567].

$$(3.18) \quad \begin{aligned} & |z'(t)|^2 + 2\mu\int_0^t |\nabla z'|^2 ds + \mu|\nabla z(t)|^2 \\ = \ & |y_1|^2 + \mu|\nabla y_0|^2 - 2(z'(t), z(t)) + 2(y_1, y_0) \\ & + 2\int_0^t |z'|^2 ds + 2\int_0^t \langle f, z' + z\rangle ds - 2\int_0^t (F, \nabla z' + \nabla z) ds. \end{aligned}$$

Moreover the following equalities hold:

$$\begin{aligned} & (y'_m(t), y_m(t)) + (z'(t), z(t)) \\ = \ & (y'_m(t) - z'(t), y_m(t) - z(t)) + (y'_m(t), z(t)) + (z'(t), y_m(t)); \\ & |\phi_m(t)|^2 + |\psi(t)|^2 = |\phi_m(t) - \psi(t)|^2 + 2(\phi_m(t), \psi(t)); \\ & |\nabla\phi_m(t)|^2 + |\nabla\psi(t)|^2 = |\nabla(\phi_m(t) - \psi(t))|^2 + 2(\nabla\phi_m(t), \nabla\psi(t)); \end{aligned}$$

$$\begin{aligned} & (G(\nabla y_m(t)), \nabla\phi_m(t)) + (F(t), \nabla\psi(t)) \\ = \ & (G(\nabla y_m(t)) - G(\nabla\psi(t)), \nabla(\phi_m(t) - \psi(t))) + (G(\nabla y_m(t)), \nabla\psi(t)) \\ & + (G(\nabla\psi(t)), \nabla(\phi_m(t) - \psi(t))) + (F(t), \nabla\psi(t)), \end{aligned}$$

where $\phi_m(t) = y_m(t)$ or $y'_m(t)$ and $\psi(t) = z(t)$ or $z'(t)$. Adding (3.8) to (3.18) and using the above equalities, we have

$$\begin{aligned}(3.19)\qquad & |y'_m(t) - z'(t)|^2 + 2\mu \int_0^t |\nabla(y'_m - z')|^2 ds + \mu|\nabla(y_m(t) - z(t))|^2 \\ = \; & \Phi_m^0 + 2\sum_{i=1}^{3} \Phi_m^i(t) - \; 2(y'_m(t) - z'(t), y_m(t) - z(t)) \\ & + 2\int_0^t |y'_m - z'|^2 ds - 2\int_0^t (G(\nabla y_m) - G(\nabla z), \nabla y_m - \nabla z) ds \\ & - 2\int_0^t (G(\nabla y_m) - G(\nabla z), \nabla y'_m - \nabla z') ds,\end{aligned}$$

where

$$\begin{aligned}\Phi_m^0 \; &= \; |y_{1m}|^2 + |y_1|^2 + \mu(|\nabla y_{0m}|^2 + |\nabla y_0|^2) \\ & \quad +2((y_{1m}, y_{0m}) + (y_1, y_0)), \\ \Phi_m^1(t) \; &= \; -(y'_m(t), z'(t)) - \mu(\nabla y_m(t), \nabla z(t)) - 2\mu\int_0^t (\nabla y'_m, \nabla z') ds, \\ \Phi_m^2(t) \; &= \; 2\int_0^t (y'_m, z') ds + \int_0^t \langle f, z + z' \rangle ds \\ & \quad + \int_0^t \langle f, y_m + y'_m \rangle ds - (y'_m(t), z(t)) - (z'(t), y_m(t)), \\ \Phi_m^3(t) \; &= \; -\int_0^t (G(\nabla y_m), \nabla(z + z')) ds - \int_0^t (F, \nabla(z + z')) ds \\ & \quad - \int_0^t (G(\nabla z), \nabla((y_m - z) + (y'_m - z'))) ds.\end{aligned}$$

For simplicity we set

$$S_m(t) = \Phi_m^0 + 2\sum_{i=1}^{3} \Phi_m^i(t).$$

It is clear from (3.2) and (3.3) that

$$(3.20) \qquad \Phi_m^0 \; \rightarrow \; 2|y_1|^2 + 2\mu|\nabla y_0|^2 + 4(y_1, y_0).$$

By (3.16) we have $\nabla y_m(t) \rightarrow \nabla z(t)$ weakly in $[L^2(\Omega)]^n$, $y'_m(t) \rightarrow z'(t)$ weakly in $L^2(\Omega)$ and $\nabla y'_m \rightarrow \nabla z'$ weakly in $L^2(0, T; [L^2(\Omega)]^n)$, and moreover $G(\nabla y_m) \rightarrow F$ weakly in $L^2(0, T; [L^2(\Omega)]^n)$, so that

$$(3.21) \qquad \Phi_m^1(t) \; \rightarrow \; -|z'(t)|^2 - \mu|\nabla z(t)|^2 - 2\mu\int_0^t |\nabla z'|^2 ds,$$

$$(3.22) \qquad \Phi_m^2(t) \; \rightarrow \; 2\int_0^t |z'|^2 ds + 2\int_0^t \langle f, z + z' \rangle ds - 2(z'(t), z(t)),$$

$$\Phi_m^3(t) \quad \to \quad -2\int_0^t (F, \nabla(z+z'))ds. \tag{3.23}$$

Whence by (3.20)-(3.23) and by the equality (3.18) for z, we have

$$S_m(t) \to 0 \qquad \text{as } m \to \infty. \tag{3.24}$$

The right hand side of (3.19) is estimated as follows:

$$\begin{aligned}
&\text{(r.h.s. of (3.19))} \\
\le\ & S_m(t) + \frac{1}{2}|y_m'(t) - z'(t)|^2 + 4|y_{0m} - y_0|^2 \\
&+ 4\int_0^t |\nabla(y_m - z)|^2 ds + (4T+2)\int_0^t |y_m' - z'|^2 ds \\
&+ 4\int_0^t |\nabla(y_m' - z')||\nabla(y_m - z)|ds \\
\le\ & S_m(t) + \frac{1}{2}|y_m'(t) - z'(t)|^2 + 4|y_{0m} - y_0|^2 + \mu\int_0^t |\nabla(y_m' - z')|^2 ds \\
&+ K\int_0^t \Big(|\nabla(y_m - z)|^2 + |y_m' - z'|^2\Big)ds,
\end{aligned} \tag{3.25}$$

where $K = (2 + 4T + 4 + \frac{16}{\mu})$. By (3.19) and (3.25) we can obtain the following inequality

$$\begin{aligned}
&\frac{1}{2}|y_m'(t) - z'(t)|^2 + \mu|\nabla(y_m(t) - z(t))|^2 + \mu\int_0^t |\nabla(y_m' - z')|^2 ds \\
\le\ & S_m(t) + 4|y_{0m} - y_0|^2 + K\int_0^t \Big(|\nabla(y_m - z)|^2 + |y_m' - z'|^2\Big)ds.
\end{aligned} \tag{3.26}$$

We divide (3.26) by $\alpha = \min\{\frac{1}{2}, \mu\} > 0$ and if we set $C = \frac{1}{\alpha}K$ and

$$M_m(t) = |\nabla(y_m(t) - z(t))|^2 + |y_m'(t) - z'(t)|^2, \tag{3.27}$$

$$\Psi_m(t) = \frac{1}{\alpha}(S_m(t) + 4|y_{0m} - y_0|^2), \tag{3.28}$$

then we can have

$$M_m(t) \le \Psi_m(t) + C\int_0^t M_m(s)ds. \tag{3.29}$$

Here we note, thanks to (3.2) and (3.24) that

$$\Psi_m(t) \to 0 \ \text{ when } m \ \to \ \infty, \ \text{ for all } \ t \in [0,T]. \tag{3.30}$$

We apply Gronwall's inequality to (3.29), then we have

$$M_m(t) \le \Psi_m(t) + C\exp(CT)\int_0^T \Psi_m(s)ds. \tag{3.31}$$

By the equality (3.18), we see that $\Psi_m(t)$ is uniformly bounded. Then it follows from (3.30) and (3.31) that

$$\lim_{m\to\infty} M_m(t) = 0. \tag{3.32}$$

Therefore we can verify that

$$y_m(t) \to z(t) \text{ strongly in } H_0^1(\Omega), \tag{3.33}$$

$$y'_m(t) \to z'(t) \text{ strongly in } L^2(\Omega). \tag{3.34}$$

Finally, by (3.33) and Lipschitz property of $G(\nabla z)$ in (2.4) we can deduce

$$F(t) = G(\nabla z(t)) \quad \text{in } [L^2(\Omega)]^n \quad \forall t \in [0,T]. \tag{3.35}$$

The energy inequality (2.6) follows from (3.15) and the strong convergence of each terms in (3.15) by (3.33), (3.34) and (3.26). □

The uniqueness of weak solutions follows directly from the continuous dependence (2.8) given in Theorem 2.3.

Proof of Theorem 2.3. Let $y(p_1)$ and $y(p_2)$ be the weak solutions of (2.1) corresponding to $p_1 \in P$ and $p_2 \in P$, respectively. Set $\varphi = y(q_1) - y(q_2)$. Then φ satisfies

$$\begin{cases} \dfrac{\partial^2\varphi}{\partial t^2} - \operatorname{div}\Big(G(\nabla y(p_1)) - G(\nabla y(p_2))\Big) \\ \qquad\qquad\qquad -\mu\Delta\dfrac{\partial\varphi}{\partial t} = f_1 - f_2 \quad \text{in } Q, \\ \varphi = 0 \quad \text{on } \Sigma, \\ \varphi(0,x) = y_0^1(x) - y_0^2(x), \quad \dfrac{\partial\varphi}{\partial t}(0,x) = y_1^1(x) - y_1^2(x) \quad \text{in } \Omega \end{cases} \tag{3.36}$$

in the weak sense. Since

$$|G(\nabla y(p_1)) - G(\nabla y(p_2))| \le 2|\nabla\varphi|,$$

we can repeat the same calculations as for (2.6) to have the estimates

$$\begin{aligned} & |\nabla\varphi(t)|^2 + |\varphi'(t)|^2 + \int_0^t |\nabla\varphi'|^2 ds \\ \le\ & C(\|y_0^1 - y_0^2\|^2 + |y_1^1 - y_1^2|^2 + \|f_1 - f_2\|^2_{L^2(0,T;H^{-1}(\Omega))}). \end{aligned}$$

This proves the strong Lipshitz continuity of solution mapping $p = (y_0, y_1, f) \to y(q)$ of P into $W(0,T)$. □

References

[1] H. T. Banks, R. C. Smith, and Y. Wang, *Smart Material Structures, Modeling, Estimation and Control*, RAM, John Wiley and Sons, Masson, 1996.

[2] R. Dautray and J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 5*, Evolution Problems I, Springer-Verlag, 1992.

[3] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[4] J. Greenberg, *On the existence, uniqueness and stability of equation* $\rho_0 X_{tt} = E(X_x) X_{xx} + X_{xxt}$, J. Math. Anal. Appl. **25** (1969), 575–591.

[5] J. Greenberg, R. MacCamy, and V. Mizel, *On the existence, uniqueness and stability of equation* $\sigma(u_x)u_{tt} + \lambda u_{xxt} = \rho_0 u_{tt}$, J. Math. Mech. **17** (1967/68), 707–728.

[6] J. Ha and S. Nakagiri, *Existence and regularity of weak solutions for semilinear second order evolution equations*, Funcial. Ekvac. **41** (1998), no. 1, 1–24.

[7] J. S. Hwang and S. Nakagiri, *Optimal control problems for the equation of motion of membrane with strong viscosity*, J. Math. Anal. Appl. **321** (2006), no. 1, 327–342.

[8] K. Kikuchi, *An analysis of the nonlinear equation of motion of a vibrating membrane in the space of BV functions*, J. Math. Soc. Japan **52** (2000), no. 4, 741–766.

[9] T. Kobayashi, H. Pecher, and Y. Shibata, *On a global in time existence theorem of smooth solutions to nonlinear wave equation with viscosity*, Math. Ann. **296** (1993), no. 2, 215–234.

[10] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physis, Second Edition*, Applied Mathematical Sciences. Vol. 68, Springer-Verlag, Berlin-Heidelberg-New York, 1997.

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