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CONTACT THREE CR-SUBMANIFOLDS OF A $(4m+3)$-DIMENSIONAL UNIT SPHERE

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ABSTRACT. We study an $(n+3)(n \geq 7)$-dimensional real submanifold of a $(4m+3)$-unit sphere S^{4m+3} with Sasakian 3-structure induced from the canonical quaternionic Kähler structure of quaternionic $(m+1)$-number space Q^{m+1}, and especially determine contact three CR-submanifolds with $(p-1)$ contact three CR-dimension under the equality conditions given in (4.1), where $p = 4m - n$ denotes the codimension of the submanifold. Also we provide necessary conditions concerning sectional curvature in order that a compact contact three CR-submanifold of $(p-1)$ contact three CR-dimension in S^{4m+3} is the model space $S^{4n_1+3}(r_1) \times S^{4n_2+3}(r_2)$ for some portion (n_1, n_2) of $(n-3)/4$ and some r_1, r_2 with $r_1^2 + r_2^2 = 1$.

1. Introduction

Let us consider a $(4m+3)$-unit sphere S^{4m+3} as a real hypersurface of the real $4(m+1)$-dimensional quaternionic number space Q^{m+1}. For any point q in S^{4m+3}, we put

$$\xi = \bar{I}q, \quad \eta = \bar{J}q, \quad \zeta = \bar{K}q,$$

where $\{\bar{I}, \bar{J}, \bar{K}\}$ denotes the canonical quaternionic Kähler structure of Q^{m+1}. Then $\{\xi, \eta, \zeta\}$ becomes a Sasakian 3-structure, that is, ξ, η and ζ are mutually orthogonal unit Killing vector fields which satisfy

$$\begin{aligned} \bar{\nabla}_Y \bar{\nabla}_X \xi &= g(X,\xi)Y - g(Y,X)\xi, \\ \bar{\nabla}_Y \bar{\nabla}_X \eta &= g(X,\eta)Y - g(Y,X)\eta, \\ \bar{\nabla}_Y \bar{\nabla}_X \zeta &= g(X,\zeta)Y - g(Y,X)\zeta \end{aligned} \tag{1.1}$$

for any vector fields X, Y tangent to S^{4m+3}, where g denotes the canonical metric on S^{4m+3} induced from that of Q^{m+1} and $\bar{\nabla}$ the Riemannian connection

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with respect to g. In this case, putting

$$\phi X = \bar{\nabla}_X \xi, \quad \psi X = \bar{\nabla}_X \eta, \quad \theta X = \bar{\nabla}_X \zeta, \tag{1.2}$$

it follows that

$$\begin{aligned} &\phi\xi = 0, \ \psi\eta = 0, \ \theta\zeta = 0, \\ &\psi\zeta = -\theta\eta = \xi, \ \theta\xi = -\phi\zeta = \eta, \ \phi\eta = -\psi\xi = \zeta, \\ &[\eta, \zeta] = -2\xi, \ [\zeta, \xi] = -2\eta, \ [\xi, \eta] = -2\zeta \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} &\phi^2 = -I + f_\xi \otimes \xi, \quad \psi^2 = -I + f_\eta \otimes \eta, \quad \theta^2 = -I + f_\zeta \otimes \zeta, \\ &\psi\theta = \phi + f_\zeta \otimes \eta, \quad \theta\phi = \psi + f_\xi \otimes \zeta, \quad \phi\psi = \theta + f_\eta \otimes \xi, \\ &\theta\psi = -\phi + f_\eta \otimes \zeta, \quad \phi\theta = -\psi + f_\zeta \otimes \xi, \quad \psi\phi = -\theta + f_\xi \otimes \eta, \end{aligned} \tag{1.4}$$

where I denotes the identity transformation and

$$f_\xi(X) = g(X, \xi), \quad f_\eta(X) = g(X, \eta), \quad f_\zeta(X) = g(X, \zeta) \tag{1.5}$$

(cf. [4, 5, 6, 10]). Moreover, from (1.1) and (1.2), we have

$$\begin{aligned} (\bar{\nabla}_Y \phi)X &= g(X, \xi)Y - g(Y, X)\xi, \\ (\bar{\nabla}_Y \psi)X &= g(X, \eta)Y - g(Y, X)\eta, \\ (\bar{\nabla}_Y \theta)X &= g(X, \zeta)Y - g(Y, X)\zeta \end{aligned} \tag{1.6}$$

for any vector fields X, Y tangent to S^{4m+3}.

Let M be an $(n+3)$-dimensional submanifold tangent to the structure vectors ξ, η and ζ of S^{4m+3} and denote by TM and $TM^\perp$ the tangent and normal bundle of M, respectively. If there exists a subbundle ν of $TM^\perp$ such that

$$\phi\nu_x \subset \nu_x, \quad \psi\nu_x \subset \nu_x, \quad \theta\nu_x \subset \nu_x, \tag{1.7}$$

$$\phi\nu_x^\perp \subset T_xM, \quad \psi\nu_x^\perp \subset T_xM, \quad \theta\nu_x^\perp \subset T_xM \tag{1.8}$$

for each point $x \in M$, where $\nu^\perp$ is the complementary orthogonal subbundle to ν in $TM^\perp$, then the submanifold is called a *contact three CR submanifold* of S^{4m+3} and the dimension of ν *contact three CR-dimension* (for details, see [7]). A real hypersurface is a typical example of contact three CR-submanifold with zero contact three CR- dimension.

In this paper we shall study $(n+3)$-dimensional contact three CR-submanifold with $(p-1)$ contact three CR-dimension of S^{4m+3}, where $p = 4m - n$ is the codimension of submanifold. In this case the $\{\phi, \psi, \theta\}$-invariant subspace

$$\mathscr{D}_x = T_xM \cap \phi T_xM \cap \psi T_xM \cap \theta T_xM$$

of T_xM has constant dimension $n-3$ because the orthogonal complement $\mathscr{D}_x^\perp$ to $\mathscr{D}_x$ in T_xM has constant dimension 6 at every point $x \in M$ (for details, see [7]).

In this paper we shall investigate some geometric characterizations of

$$S^{4n_1+3}(r_1) \times S^{4n_2+3}(r_2) \quad (r_1^2 + r_2^2 = 1,, \ n_1 + n_2 = (n-3)/4)$$

as a contact three CR-submanifold of a $(4m+3)$-dimensional unit sphere (see Theorem 4.3, Theorem 5.3 and Theorem 6.3).

2. Preliminaries

Let M be an $(n+3)$-dimensional contact three CR-submanifold of $(p-1)$ contact three CR-dimension in a $(4m+3)$-dimensional Riemannian manifold $\bar{M}$ with Sasakian 3-structure $\{\xi, \eta, \zeta\}$ which satisfies (1.1), where $p = 4m - n$. Then, by definition, we may set $\nu^{\perp} = \text{Span}\ \{N\}$ for a unit normal vector field N to M. Here and in the sequel we use the same notations as shown in section 1. Put

$$U = -\phi N, \quad V = -\psi N, \quad W = -\theta N. \tag{2.1}$$

Then from (1.3), (1.4) and (1.8) we can see that U, V, W are mutually orthogonal unit tangent vector fields to M and satisfy

$$\begin{aligned} &g(\xi, U) = 0, \quad g(\xi, V) = 0, \quad g(\xi, W) = 0,\\ &g(\eta, U) = 0, \quad g(\eta, V) = 0, \quad g(\eta, W) = 0,\\ &g(\zeta, U) = 0, \quad g(\zeta, V) = 0, \quad g(\zeta, W) = 0. \end{aligned} \tag{2.2}$$

Moreover ξ, η, ζ, U, V and W are all contained in $\mathscr{D}_x^{\perp}$ and consequently $\dim \mathscr{D}_x^{\perp} \geq 6$ at any point $x \in M$. But, already mentioned in section 1, $\dim \mathscr{D}_x^{\perp} = 6$ at any point $x \in M$. Therefore, for any tangent vector field X and for a local orthonormal basis $\{N_\alpha\}_{\alpha=1,\ldots,p}$ $(N_1 := N)$ of normal vectors to M, we have the following decomposition in tangential and normal components:

$$\begin{aligned} \phi X = FX + u^1(X)N, \quad \psi X = GX + v^1(X)N,\\ \theta X = HX + w^1(X)N, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \phi N_\alpha = -U_\alpha + P_\phi N_\alpha, \quad \psi N_\alpha = -V_\alpha + P_\psi N_\alpha,\\ \theta N_\alpha = -W_\alpha + P_\theta N_\alpha, \quad \alpha = 1, \ldots, p. \end{aligned} \tag{2.4}$$

It follows easily from (1.4) that $\{F, G, H\}$ and $\{P_\phi, P_\psi, P_\theta\}$ are skew-symmetric linear endomorphisms acting on T_xM and $T_xM^{\perp}$, respectively.

Since the structure vectors ξ, η and ζ are tangent to M, the equations (1.4), (2.3) and (2.4) imply

$$\begin{aligned} &F^2X = -X + f_\xi(X)\xi + u^1(X)U_1, \quad u^1(FX) = 0,\\ &G^2X = -X + f_\eta(X)\eta + v^1(X)V_1, \quad v^1(GX) = 0,\\ &H^2X = -X + f_\zeta(X)\zeta + w^1(X)W_1, \quad w^1(HX) = 0, \end{aligned} \tag{2.5}$$

$$
\begin{aligned}
&GFX = -HX + f_\xi(X)\eta + u^1(X)V_1, \quad v^1(FX) = -w^1(X),\\
&HFX = GX + f_\xi(X)\zeta + u^1(X)W_1, \quad w^1(FX) = v^1(X),\\
&FGX = HX + f_\eta(X)\xi + v^1(X)U_1, \quad u^1(GX) = w^1(X),\\
&HGX = -FX + f_\eta(X)\zeta + v^1(X)W_1, \quad w^1(GX) = -u^1(X),\\
&FHX = -GX + f_\zeta(X)\xi + w^1(X)U_1, \quad u^1(HX) = -v^1(X),\\
&GHX = FX + f_\zeta(X)\eta + w^1(X)V_1, \quad v^1(HX) = u^1(X),
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
g(U_\alpha, X) =&u^1(X)\delta_{1\alpha}, \quad g(V_\alpha, X) = v^1(X)\delta_{1\alpha},\\
&g(W_\alpha, X) = w^1(X)\delta_{1\alpha}, \quad \alpha = 1, \dots, p,
\end{aligned}
\tag{2.7}
$$

which yields

$$
\begin{aligned}
g(U_1, X) =u^1(X), \ g(V_1, X) = v^1(X), \ g(W_1, X) = w^1(X),\\
U_\alpha = 0, \ V_\alpha = 0, \ W_\alpha = 0, \ \alpha = 2, \dots, p,
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - g(P_\phi N_\alpha, P_\phi N_\beta),\\
g(V_\alpha, V_\beta) = \delta_{\alpha\beta} - g(P_\psi N_\alpha, P_\psi N_\beta),\\
g(W_\alpha, W_\beta) = \delta_{\alpha\beta} - g(P_\theta N_\alpha, P_\theta N_\beta).
\end{aligned}
\tag{2.9}
$$

From (1.3) and (2.3), it follows that

$$
\begin{aligned}
&F\xi = 0, \ G\eta = 0, \ H\zeta = 0,\\
&F\eta = \zeta, \ F\zeta = -\eta, \ G\xi = -\zeta, \ G\zeta = \xi, \ H\xi = \eta, \ H\eta = -\xi,\\
&u^1(\xi) = 0, \ u^1(\eta) = 0, \ u^1(\zeta) = 0, \ v^1(\xi) = 0, \ v^1(\eta) = 0,\\
&v^1(\zeta) = 0, \ w^1(\xi) = 0, \ w^1(\eta) = 0, \ w^1(\zeta) = 0.
\end{aligned}
\tag{2.10}
$$

Using (1.4) and (2.1)-(2.4), we have

$$
\begin{aligned}
&FU_1 = 0, \ GV_1 = 0, \ HW_1 = 0, \ FV_1 = W_1, \ FW_1 = -V_1,\\
&GU_1 = -W_1, \ GW_1 = U_1, \ HU_1 = V_1, \ HV_1 = -U_1,
\end{aligned}
\tag{2.11}
$$

$$
P_\phi N_1 = 0, \ P_\psi N_1 = 0, \ P_\theta N_1 = 0,
\tag{2.12}
$$

which together with (2.1), (2.4), (2.8) and (2.9) implies $U = U_1, V = V_1, W = W_1$ and

$$
\begin{aligned}
&u(U) = v(V) = w(W) = 1,\\
&u(V) = u(W) = 0, \quad v(U) = v(W) = 0, \quad w(U) = w(V) = 0.
\end{aligned}
\tag{2.13}
$$

Here we notice that $\dim M = n + 3$ must be $4l + 2$ for some integer l since $\dim \mathscr{D}_x^\perp = 6$ at any point $x \in M$.

3. Fundamental equations for contact three CR-submanifold

Let M be as in section 2. We denote by ∇ and $\nabla^{\perp}$ the Levi-Civita connection on M and the normal connection on $TM^{\perp}$ induced from $\bar{\nabla}$, respectively. Then the Gauss and Weingarten equations are of the form

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{3.1}$$

$$\bar{\nabla}_X N_\alpha = -A_\alpha X + \nabla^{\perp}_X N_\alpha, \quad \alpha = 1,\dots,p \tag{3.2}$$

for any vector fields X, Y tangent to M. Here h denotes the second fundamental form and A_α is the shape operator in direction of N_α. They are related by

$$h(X,Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y) N_\alpha.$$

Furthermore we may put

$$\nabla^{\perp}_X N_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X) N_\beta, \tag{3.3}$$

where $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of $\nabla^{\perp}$. Moreover, since S^{4m+3} is of constant sectional curvature 1, the equations of Gauss, Codazzi and Ricci imply

$$\begin{aligned} g(R(X,Y)Z,W) &= g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &\quad + \sum_\alpha \{g(A_\alpha Y, Z) g(A_\alpha X, W)\} - g(A_\alpha X, Z) g(A_\alpha Y, W), \end{aligned} \tag{3.4}$$

$$\begin{aligned} &g((\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X, Z) \\ &= \sum_\beta \{g(A_\beta X, Z) s_{\beta\alpha}(Y) - g(A_\beta Y, Z) s_{\beta\alpha}(X)\}, \end{aligned} \tag{3.5}$$

$$g(R^{\perp}(X,Y)N_\beta, N_\alpha) = g([A_\beta, A_\alpha]X, Y) \tag{3.6}$$

for any vector fields X, Y, Z tangent to M, where R denotes the Riemannian curvature tensor of M and $R^{\perp}$ the curvature tensor of the normal connection $\nabla^{\perp}$ (cf. [2]).

Differentiating (2.3) covariantly and using (1.1), (1.2), (2.8), (2.11), (3.1) and (3.2), we have

$$\begin{aligned} (\nabla_Y F)X &= g(X,\xi)Y - g(X,Y)\xi - g(A_1 X, Y)U + u^1(X) A_1 Y, \\ (\nabla_Y u^1)X &= -\, g(A_1 F X, Y), \end{aligned} \tag{3.7}$$

$$\begin{aligned} (\nabla_Y G)X &= g(X,\eta)Y - g(X,Y)\eta - g(A_1 X, Y)V + v^1(X) A_1 Y, \\ (\nabla_Y v^1)X &= -\, g(A_1 G X, Y), \end{aligned} \tag{3.8}$$

$$(3.9)\qquad \begin{aligned}&(\nabla_Y H)X = g(X,\zeta)Y - g(X,Y)\zeta - g(A_1X,Y)W + w^1(X)A_1Y,\\ &(\nabla_Y w^1)X = -\, g(A_1HX,Y).\end{aligned}$$

Differentiating (2.1) covariantly and using (1.1), (1.2), (2.8) and (3.1)-(3.3), we have

$$(3.10)\qquad \begin{cases}\nabla_X U = FA_1X,\\ g(A_\alpha U,X) = -\sum_{\beta=2}^{p} s_{1\beta}(X)P^{\phi}_{\beta\alpha}, \quad \alpha = 2,\dots,p,\end{cases}$$

$$(3.11)\qquad \begin{cases}\nabla_X V = GA_1X,\\ g(A_\alpha V,X) = -\sum_{\beta=2}^{p} s_{1\beta}(X)P^{\psi}_{\beta\alpha}, \quad \alpha = 2,\dots,p,\end{cases}$$

$$(3.12)\qquad \begin{cases}\nabla_X W = HA_1X,\\ g(A_\alpha W,X) = -\sum_{\beta=2}^{p} s_{1\beta}(X)P^{\theta}_{\beta\alpha}, \quad \alpha = 2,\dots,p,\end{cases}$$

where we have put

$$P_\phi N_\alpha = \sum_{\beta=2}^{p} P^{\phi}_{\alpha\beta}N_\beta, \quad P_\psi N_\alpha = \sum_{\beta=2}^{p} P^{\psi}_{\alpha\beta}N_\beta, \quad P_\theta N_\alpha = \sum_{\beta=2}^{p} P^{\theta}_{\alpha\beta}N_\beta.$$

On the other hand, since ξ, η and ζ are tangent to M, it follows from (1.2) that

$$(3.13)\qquad \begin{cases}\nabla_X\xi = FX,\\ g(A_1\xi,X) = u^1(X), \quad \text{that is,} \quad A_1\xi = U,\\ A_\alpha\xi = 0, \quad \alpha = 2,\dots,p,\end{cases}$$

$$(3.14)\qquad \begin{cases}\nabla_X\eta = GX,\\ g(A_1\eta,X) = v^1(X), \quad \text{that is,} \quad A_1\eta = V,\\ A_\alpha\eta = 0, \quad \alpha = 2,\dots,p,\end{cases}$$

$$(3.15)\qquad \begin{cases}\nabla_X\zeta = HX,\\ g(A_1\zeta,X) = w^1(X), \quad \text{that is,} \quad A_1\zeta = W,\\ A_\alpha\zeta = 0, \quad \alpha = 2,\dots,p.\end{cases}$$

From now on, we assume that the distinguished normal vector field N is parallel with respect to the normal connection $\nabla^\perp$. Then it follows from (3.3) that

$$(3.16)\qquad s_{1\beta} = 0, \quad \beta = 2,\dots,p,$$

which together with (3.10)-(3.12) implies

$$(3.17)\qquad A_\alpha U = 0, \quad A_\alpha V = 0, \quad A_\alpha W = 0, \quad \alpha = 2,\dots,p.$$

Moreover, (3.3) reduces to

$$\bar{\nabla}_X N_\alpha = \sum_{\beta=2}^{p} s_{\alpha\beta}(X) N_\beta, \quad \alpha = 2, \ldots, p. \tag{3.18}$$

for any vector field X tangent to M.

Finally we provide some lemmas for later use.

Lemma 3.1. *Let M be an $(n+3)$-dimensional contact three CR-submanifold of $(p-1)$ contact three CR-dimension in a $(4m+3)$-unit sphere S^{4m+3}, where $p = 4m - n$ denotes the codimension. Assume that the distinguished normal vector field N is parallel with respect to the normal connection. Then the commutativity conditions*

$$A_1 F = F A_1, \quad A_1 G = G A_1, \quad A_1 H = H A_1$$

hold on M if and only if

$$\nabla A_1 = 0.$$

Moreover, in this case

$$\begin{aligned} &{A_1}^2 = \lambda A_1 + I, \\ &A_1 U = \xi + \lambda U, \quad A_1 V = \eta + \lambda V, \quad A_1 W = \zeta + \lambda W, \end{aligned} \tag{3.19}$$

where $\lambda = u^1(A_1 U) = v^1(A_1 V) = w^1(A_1 W)$, which is locally constant.

Proof. We first assume that $\nabla A_1 = 0$. Differentiating the second equation of (3.13) covariantly along M and using the first equations of (3.10) and (3.13) and $\nabla A_1 = 0$, we can easily see that $A_1 F = F A_1$ holds on M. Similarly. from those of (3.11), (3.12), (3.14) and (3.15), we can verify that $A_1 G = G A_1$, $A_1 H = H A_1$ also hold on M.

The proofs of the converse and (3.19) have been given in [7, Lemma 4.1, p. 570]. □

Lemma 3.2. *Let M be as in Lemma 3.1. If the distinguished normal vector field N is parallel with respect to the normal connection, then*

$$A_\alpha F + F A_\alpha = 0, \quad A_\alpha G + G A_\alpha = 0, \quad A_\alpha H + H A_\alpha = 0, \tag{3.20}$$

$$\mathrm{tr} A_\alpha = 0, \quad \alpha = 2, \ldots, p. \tag{3.21}$$

Proof. Differentiating the third equation of (3.13) covariantly and using the first equation of (3.13), we have

$$(\nabla_X A_\alpha)\xi + A_\alpha F X = 0,$$

or equivalently

$$g((\nabla_X A_\alpha)Y, \xi) + g(A_\alpha F X, Y) = 0 \tag{3.22}$$

for any vector fields X, Y tangent to M. By means of (3.5), the third equation of (3.13) and (3.22), we can be easily obtain the first equation of (3.20). Similarly, from (3.14) and (3.15), we can get the other equations of (3.20).

Applying FX to both side of the first equation of (3.20) and using (2.5), (3.13)-(3.15) and (3.17), we have

$$A_\alpha X = FA_\alpha FX,$$

and consequently

$$g(A_\alpha GX, GY) = -g(A_\alpha HX, HY)$$

for any vector fields X, Y tangent to M. It is clear that those equations imply (3.21). □

4. Codimension reduction for contact three CR-submanifolds

In this section we let M be as in Lemma 3.1 and denote by A the shape operator A_1 in direction of the distinguished normal vector field N. We first prepare a lemma for later use.

Lemma 4.1. *Let M be as in Lemma 3.1. If, for any vector fields X, Y tangent to M, the equalities*

$$h(FX,Y) = -h(X,FY), \quad h(GX,Y) = -h(X,GY), \quad h(HX,Y) = -h(X,HY) \tag{4.1}$$

hold on M, then

$$AF = FA, \quad AG = GA, \quad AH = HA, \tag{4.2}$$

and $A_\alpha = 0$ for $\alpha = 2, \ldots, p$. Moreover, in this case, the distinguished normal vector field N is parallel with respect to the normal connection.

Proof. Sine $n = 4l + 3$ and $4m - n = 4q + 1$ for some integers $l > 1$ and $q > 1$, and since the subspace ν is $\{\phi, \psi, \theta\}$-invariant (see also (2.12)), we can take a local orthonormal basis $\{N, N_a, N_{a^*}, N_{a^{**}}, N_{a^{***}}\}_{a=1,\ldots,q}$ of normal vectors to M such that

$$N_{a^*} := \phi N_a, \quad N_{a^{**}} := \psi N_a, \quad N_{a^{***}} := \theta N_a.$$

Then we can express the second fundamental form h as

$$h(X,Y) = g(AX,Y)N + \sum_{a=1}^{q}\{g(A_aX,Y)N_a + g(A_{a^*}X,Y)N_{a^*} + g(A_{a^{**}}X,Y)N_{a^{**}} + g(A_{a^{***}}X,Y)N_{a^{***}}\},$$

where $A_a, A_{a^*}, A_{a^{**}}, A_{a^{***}}$ the shape operators corresponding to the normals $N_a, N_{a^*}, N_{a^{**}}, N_{a^{***}}$, respectively. Hence the assumption (4.1) implies

$$\begin{aligned} &AF = FA, \quad AG = GA, \quad AH = HA, \\ &A_{a^*}F = FA_{a^*}, \quad A_{a^*}G = GA_{a^*}, \quad A_{a^*}H = HA_{a^*}, \\ &A_{a^{**}}F = FA_{a^{**}}, \quad A_{a^{**}}G = GA_{a^{**}}, \quad A_{a^{**}}H = HA_{a^{**}} \\ &A_{a^{***}}F = FA_{a^{***}}, \quad A_{a^{***}}G = GA_{a^{***}}, \quad A_{a^{***}}H = HA_{a^{***}}. \end{aligned} \tag{4.3}$$

On the other hand, the Weingarten equations (3.2) are rewritten in the form

$$(3.2_1)\quad \bar{\nabla}_X N = -AX + \sum_{a=1}^{q}\{s_a(X)N_a + s_{a^*}(X)N_{a^*} + s_{a^{**}}(X)N_{a^{**}} + s_{a^{***}}(X)N_{a^{***}}\},$$

$$(3.2_2)\quad \bar{\nabla}_X N_a = -A_aX - s_a(X)N + \sum_{b=1}^{q}\{s_{ab}(X)N_b + s_{ab^*}(X)N_{b^*} + s_{ab^{**}}(X)N_{b^{**}} + s_{ab^{***}}(X)N_{b^{***}}\},$$

$$(3.2_3)\quad \bar{\nabla}_X N_{a^*} = -A_{a^*}X - s_{a^*}(X)N + \sum_{b=1}^{q}\{s_{a^*b}(X)N_b + s_{a^*b^*}(X)N_{b^*} + s_{a^*b^{**}}N_{b^{**}} + s_{a^*b^{***}}N_{b^{***}}\},$$

$$(3.2_4)\quad \bar{\nabla}_X N_{a^{**}} = -A_{a^{**}}X - s_{a^{**}}(X)N + \sum_{b=1}^{q}\{s_{a^{**}b}(X)N_b + s_{a^{**}b^*}(X)N_{b^*} + s_{a^{**}b^{**}}N_{b^{**}} + s_{a^{**}b^{***}}N_{b^{***}}\},$$

$$(3.2_5)\quad \bar{\nabla}_X N_{a^{***}} = -A_{a^{***}}X - s_{a^{***}}(X)N + \sum_{b=1}^{q}\{s_{a^{***}b}(X)N_b + s_{a^{***}b^*}(X)N_{b^*} + s_{a^{***}b^{**}}N_{b^{**}} + s_{a^{***}b^{***}}N_{b^{***}}\}.$$

Since the structure vectors ξ, η, ζ are tangent to M, applying ϕ to the both side of (3.2_2) and using (1.6), (2.1) and (2.3), we have

$$\bar{\nabla}_X N_{a^*} = -FA_aX - u(A_aX)N + s_a(X)U + \sum_{b=1}^{q}\{s_{ab}(X)N_{b^*} - s_{ab^*}(X)N_b + s_{ab^{**}}(X)N_{b^{***}} - s_{ab^{***}}(X)N_{b^{**}}\},$$

which and (3.2_3) imply

$$A_{a^*}X = FA_aX - s_a(X)U, \quad s_{a^*}(X) = u(A_aX).$$

Similarly, from $(3.2_2) - (3.2_5)$, we can easily obtain that

$$(4.4_1)\quad \begin{aligned} A_{a^{**}}X &= GA_aX - s_a(X)V \\ &= HA_{a^*}X - s_{a^*}(X)W = -FA_{a^{***}}X + s_{a^{***}}(X)U, \end{aligned}$$

$$(4.4_2)\quad \begin{aligned} A_{a^{***}}X &= HA_aX - s_a(X)W \\ &= -GA_{a^*}X + s_{a^*}(X)V = FA_{a^{**}}X - s_{a^{**}}(X)U, \end{aligned}$$

$$\begin{aligned} A_aX &= -FA_{a^*}X + s_{a^*}(X)U \\ &= -GA_{a^{**}}X + s_{a^{**}}(X)V = -HA_{a^{***}}X + s_{a^{***}}(X)W, \end{aligned} \tag{4.4$_3$}$$

$$\begin{aligned} A_{a^*}X &= FA_aX - s_a(X)U \\ &= -HA_{a^{**}}X + s_{a^{**}}(X)W = GA_{a^{***}}X - s_{a^{***}}(X)V, \end{aligned} \tag{4.4$_4$}$$

$$\begin{aligned} &s_{a^{**}}(X) = v(A_aX) = w(A_{a^*}X) = -u(A_{a^{***}}X), \\ &s_{a^{***}}(X) = w(A_aX) = -v(A_{a^*}X) = u(A_{a^{**}}X), \\ &s_a(X) = -u(A_{a^*}X) = -v(A_{a^{**}}X) = -w(A_{a^{***}}X), \\ &s_{a^*}(X) = u(A_aX) = -w(A_{a^{**}}X) = v(A_{a^{***}}X), \end{aligned} \tag{4.4$_5$}$$

On the other hand, (2.11) and (4.3) imply $FA_aU = 0$, from which together with (1.5), (2.5), (3.13) and (4.4$_5$), it follows that

$$A_aU = u(A_aU)U = s_{a^*}(U)U.$$

Similarly, from the other equations of (4.3), we can easily verify that

$$\begin{aligned} &A_aU = s_{a^*}(U)U, \quad A_aV = s_{a^{**}}(V)V, \quad A_aW = s_{a^{***}}(W)W, \\ &A_{a^*}U = -s_a(U)U, \quad A_{a^*}V = -s_{a^{***}}(V)V, \quad A_{a^*}W = s_{a^{**}}(W)W, \\ &A_{a^{**}}U = s_{a^{***}}(U)U, \quad A_{a^{**}}V = -s_a(V)V, \quad A_{a^{**}}W = -s_{a^*}(W)W, \\ &A_{a^{***}}U = -s_{a^{**}}(U)U, \quad A_{a^{***}}V = s_{a^*}(V)V, \quad A_{a^{***}}W = -s_a(W)W. \end{aligned} \tag{4.5}$$

Hence, from (4.4$_5$) and (4.5), we have

$$\begin{aligned} &s_a(X) = s_a(U)u(X) = s_a(V)v(X) = s_a(W)w(X), \\ &s_{a^*}(X) = s_{a^*}(U)u(X) = s_{a^*}(V)v(X) = s_{a^*}(W)w(X), \\ &s_{a^{**}}(X) = s_{a^{**}}(U)u(X) = s_{a^{**}}(V)v(X) = s_{a^{**}}(W)w(X), \\ &s_{a^{***}}(X) = s_{a^{***}}(U)u(X) = s_{a^{***}}(V)v(X) = s_{a^{***}}(W)w(X), \end{aligned}$$

from which together with (2.13), it is clear that

$$s_a = s_{a^*} = s_{a^{**}} = s_{a^{***}} = 0, \tag{4.6}$$

namely, the distinguished normal vector field N is parallel with respect to the normal connection.

Next, we combine (4.4$_1$) – (4.4$_4$) and (4.6), Then we have

$$FA_a = A_{a^*}, \quad GA_a = A_{a^{**}}, \quad HA_a = A_{a^{***}}, \quad a = 1, \ldots, q. \tag{4.7}$$

Therefore, for any tangent vectors X, Y to M, we have from the first equation of (4.7)

$$g(A_{a^*}FX, Y) = -g(A_aFX, FY)$$

and consequently

$$g(A_{a^*}FX, Y) = g(A_{a^*}FY, X) = -g(FA_{a^*}X, Y),$$

that is,

$$A_{a^*}F = -FA_{a^*},$$

which and (4.3) imply $FA_{a^*} = 0$. Thus it is clear from (4.7) that $F^2A_a = 0$, which together with (2.5), (3.13) and (3.17) yields $A_a = 0$. Hence it follows from (4.7) that

$$A_a = 0, \quad A_{a^*} = 0, \quad A_{a^{**}} = 0, \quad A_{a^{***}} = 0, \quad a = 1, \ldots, q.$$

□

For the submanifold M given in Lemma 4.1, we can easily see that its first normal space is contained in Span$\{N\}$ which is invariant under parallel translation with respect to the normal connection from our assumption. Thus we may apply Erbacher's reduction theorem ([3, p.339]) and this yields

Theorem 4.2. *Let M be as in Lemma 3.1. If, for any vector fields X, Y tangent to M, the equalities* (4.1) *hold on M, then there is an $(n+4)$-dimensional totally geodesic unit sphere S^{n+4} such that $M \subset S^{n+4}$.*

Finally, using Theorem 4.2, we prove

Theorem 4.3. *Let M be as in Lemma 3.1. If, for any vector fields X, Y tangent to M, the equalities* (4.1) *hold on M, then M is locally isometric to*

$$S^{4n_1+3}(r_1) \times S^{4n_2+3}(r_2)$$

for some portion (n_1, n_2) of $(n-3)/4$ and some r_1, r_2 with $r_1^2 + r_2^2 = 1$.

Proof. By means of Theorem 4.2, there exists a real $(n+4)$-dimensional totally geodesic unit sphere S^{n+4} such that $M \subset S^{n+4}$. We notice that $n+4$ is of type $4r+3$ for some integer r. Moreover, since the tangent space T_xS^{n+4} of the totally geodesic submanifold S^{n+4} at x in M is $T_xM \oplus \text{Span}\{N\}$, S^{n+4} is an invariant submanifold of S^{4m+3} with respect to the Sasakian 3-structure $\{\xi, \eta, \zeta\}$ (that is, ξ, η and ζ are all tangent to S^{n+4} and

$$\phi(T_xS^{n+4}) \subset T_xS^{n+4}, \quad \psi(T_xS^{n+4}) \subset T_xS^{n+4}, \quad \theta(T_xS^{n+4}) \subset T_xS^{n+4}$$

for any x in S^{n+4}), because of (2.1) and (2.3) (for definition, cf. [7, 12]). Hence the submanifold M can be regarded as a real hypersurface of S^{n+4} which is totally geodesic invariant submanifold of S^{4m+3}.

Tentatively we denote S^{n+4} by M' and by i_1 the immersion of M into M' and i_2 the totally geodesic immersion of M' into S^{4m+3}. Then, from the Gauss equation (3.1), it follows that

$$\nabla'_{i_1X} i_1Y = i_1\nabla_XY + h'(X,Y) = i_1\nabla_XY + g(A'X,Y)N',$$

where h' is the second fundamental form of M in M', N' a unit normal vector field to M in M' and A' the corresponding shape operator. Since $i = i_2 \circ i_1$, we have

$$\begin{aligned} \bar\nabla_{i_2\circ i_1X} i_2 \circ i_1Y &= i_2\nabla'_{i_1X} i_1Y + \bar h(i_1X, i_1Y) \\ &= i_2(i_1\nabla_XY + g(A'X,Y)N'), \end{aligned} \tag{4.8}$$

because M' is totally geodesic in S^{4m+3}. Comparing (4.8) with (3.1), we easily see that

$$N = i_2 N', \quad A = A'.$$

Since M' is an invariant submanifold of S^{4m+3}, for any $X' \in TM'$,

$$\phi i_2 X' = i_2 \phi' X', \quad \psi i_2 X' = i_2 \psi' X', \quad \theta i_2 X' = i_2 \theta' X'$$

is valid, where $\{\phi', \psi', \theta'\}$ is the induced Sasakian 3-structure on $M' = S^{n+4}$. Thus it follows from (2.3) that

$$\begin{aligned}
\phi iX &= \phi i_2 \circ i_1 X = i_2 \phi' i_1 X = i_2(i_1 F'X + u'(X)N') \\
&= iF'X + u'(X) i_2 N' = iF'X + u'(X)N, \\
\psi iX &= \psi i_2 \circ i_1 X = i_2 \psi' i_1 X = i_2(i_1 G'X + v'(X)N') \\
&= iG'X + v'(X) i_2 N' = iG'X + v'(X)N, \\
\theta iX &= \theta i_2 \circ i_1 X = i_2 \theta' i_1 X = i_2(i_1 H'X + w'(X)N') \\
&= iH'X + w'(X) i_2 N' = iH'X + w'(X)N.
\end{aligned}$$

Comparing those equations with (2.3), we have

$$F = F', \ u' = u^1 \ ; \ G = G', \ v' = v^1 \ ; \ H = H', \ w' = w^1.$$

Hence M is a real hypersurface of S^{n+4} which satisfies

$$F'A' = A'F', \quad G'A' = A'G', \quad H'A' = A'H'.$$

By means of Lemma 3.1 $\nabla A' = 0$ and also

$$\begin{aligned}
A'(\rho_i U + \xi) = \rho_i(\rho_i U + \xi), \quad A'(\rho_i V + \eta) = \rho_i(\rho_i V + \eta), \\
A'(\rho_i W + \zeta) = \rho_i(\rho_i W + \zeta), \quad i = 1, 2,
\end{aligned}$$

where $\rho_i (i = 1, 2)$ are the solutions of equation $\rho^2 - \lambda\rho - 1 = 0$. Hence we can easily verify that M is locally isometric to

$$S^{4n_1+3}(r_1) \times S^{4n_2+3}(r_2) \quad (r_1^2 + r_2^2 = 1).$$

for some integers n_1, n_2 with $4n_1 + 4n_2 = n - 3$ (for more details, see [7, 8]). □

5. An integral formula for compact contact three CR-submanifolds

Let M be an $(n+3)$-dimensional compact contact three CR-submanifold of $(p-1)$ contact three CR-dimension in S^{4m+3}, where $p = 4m - n$. Assume that the distinguished normal vector field N is parallel with respect to the normal connection $\nabla^\perp$.

The equation (3.4) of Gauss implies

$$\text{Ric}(X, Y) = (n+2)g(X, Y) + \sum_\alpha \{(\text{tr}A_\alpha) g(A_\alpha X, Y) - g(A_\alpha^2 X, Y)\}, \tag{5.1}$$

$$\rho = (n+2)(n+3) + (n+3)^2 \|\mu\|^2 - \sum_\alpha \text{tr}{A_\alpha}^2, \tag{5.2}$$

where Ric and ρ denote the Ricci tensor and the scalar curvature, respectively, and

$$\mu = \frac{1}{n+3}\sum_{\alpha}(\mathrm{tr}A_\alpha)N_\alpha \tag{5.3}$$

is the mean curvature vector (cf [1, 2, 12]).

Now we prove

Lemma 5.1. *Let M be an $(n+3)$-dimensional compact contact three CR-submanifold of $(p-1)$ contact three CR-dimension in a $(4m+3)$-unit sphere S^{4m+3}, where $p = 4m-n$ denotes the codimension. If the distinguished normal vector field N is parallel with respect to the normal connection and if the inequality*

$$\begin{aligned}&\frac{1}{3}\{\mathrm{Ric}(U,U)+\mathrm{Ric}(V,V)+\mathrm{Ric}(W,W)\\&\quad+\|A_1U\|^2+\|A_1V\|^2+\|A_1W\|^2\}+\rho-(n+3)^2\|\mu\|^2 \;\geq\; n^2+5n+5\end{aligned}$$

holds on M, then we have

$$A_1F = FA_1, \quad A_1G = GA_1, \quad A_1H = HA_1 \tag{5.4}$$

and $A_\alpha = 0, \alpha = 2,\dots,p$.

Proof. In order to prove our lemma we use the following integral formula established by Yano([11]):

(5.5)

$$\begin{aligned}&\int_M \mathrm{div}\{\nabla_U U+\nabla_V V+\nabla_W W-(\mathrm{div}U)U-(\mathrm{div}V)V-(\mathrm{div}W)W\}*1\\&=\int_M\{\mathrm{Ric}(U,U)+\frac{1}{2}\|\mathcal{L}_U g\|^2-\|\nabla U\|^2-(\mathrm{div}U)^2\\&\qquad+\mathrm{Ric}(V,V)+\frac{1}{2}\|\mathcal{L}_U g\|^2-\|\nabla V\|^2-(\mathrm{div}V)^2\\&\qquad+\mathrm{Ric}(W,W)++\frac{1}{2}\|\mathcal{L}_W g\|^2-\|\nabla W\|^2-(\mathrm{div}W)^2\}*1=0.\end{aligned}$$

Now we take an orthonormal basis

$$\{U,\ V,\ W,\ \xi,\ \eta,\ \zeta,\ e_a,\ e_{a^*},\ e_{a^{**}}, e_{a^{***}}\}_{a=1,\dots,\ t=(n-3)/4}$$

of tangent vectors to M such that

$$e_{a^*} := Fe_a, \quad e_{a^{**}} := Ge_a, \quad e_{a^{***}} = He_a.$$

Then it is clear from (2.6), (2.10)-(2.11) and (3.10) that

$$\begin{aligned}\mathrm{div}\ U = \mathrm{tr}(FA_1) = \sum_{a=1}^{t}\{&g(FA_1e_a,e_a)+g(FA_1e_{a^*},e_{a^*})\\&+g(FA_1e_{a^{**}},e_{a^{**}})+g(FA_1e_{a^{***}},e_{a^{***}})\}=0.\end{aligned}$$

Similarly, from (3.11)-(3.12), we have

$$\operatorname{div} U = 0, \quad \operatorname{div} V = 0, \quad \operatorname{div} W = 0. \tag{5.6}$$

From (3.10)-(3.12), we also have

$$(\mathscr{L}_U g)(X,Y) = g((FA_1 - A_1F)X, Y), \ (\mathscr{L}_V g)(X,Y) = g((GA_1 - A_1G)X, Y),$$
$$(\mathscr{L}_W g)(X,Y) = g((HA_1 - A_1H)X, Y)$$

and consequently

$$\begin{aligned} \|\mathscr{L}_U g\|^2 = \|FA_1 - A_1F\|^2, \quad \|\mathscr{L}_V g\|^2 = \|GA_1 - A_1G\|^2, \\ \|\mathscr{L}_W g\|^2 = \|HA_1 - A_1H\|^2. \end{aligned} \tag{5.7}$$

Using (2.6), (2.10)-(2.11) and (3.10)-(3.15), we also have

$$\begin{aligned} \|\nabla U\|^2 = \operatorname{tr}A_1^2 - 1 - \|A_1U\|^2, \|\nabla V\|^2 = \operatorname{tr}A_1^2 - 1 - \|A_1V\|^2, \\ \|\nabla W\|^2 = \operatorname{tr}A_1^2 - 1 - \|A_1W\|^2. \end{aligned} \tag{5.8}$$

On the other hand (5.2) yields

$$\operatorname{tr}A_1^2 = -\rho + (n+2)(n+3) + (n+3)^2\|\mu\|^2 - \sum_{\alpha=2}^{p} \operatorname{tr}{A_\alpha}^2. \tag{5.9}$$

Substituting (5.6)-(5.9) into (5.5), we obtain

(5.10)

$$\begin{aligned} \int_M \{ & \frac{1}{2}(\|FA_1 - A_1F\|^2 + \|GA_1 - A_1G\|^2 + \|HA_1 - A_1H\|^2) \\ & + \operatorname{Ric}(U,U) + \operatorname{Ric}(V,V) + \operatorname{Ric}(W,W) + \|A_1U\|^2 + \|A_1V\|^2 + \|A_1W\|^2 \\ & + 3\rho - 3(n+3)^2\|\mu\|^2 - 3(n^2+5n+5) + 3\sum_{\alpha=2}^{p} \operatorname{tr}A_\alpha^2\} * 1 = 0. \end{aligned}$$

Thus, if the inequality

$$\begin{aligned} & \frac{1}{3}\{\operatorname{Ric}(U,U) + \operatorname{Ric}(V,V) + \operatorname{Ric}(W,W) \\ & + \|A_1U\|^2 + \|A_1V\|^2 + \|A_1W\|^2\} + \rho - (n+3)^2\|\mu\|^2 \ \geq \ n^2 + 5n + 5 \end{aligned}$$

holds on M, then we have

$$A_1F = FA_1, \quad A_1G = GA_1, \quad A_1H = HA_1$$

and $A_\alpha = 0, \alpha = 2, \ldots, p$. □

For the submanifold M given in Lemma 5.1, we can easily see that its first normal space is contained in Span$\{N\}$ which is invariant under parallel translation with respect to the normal connection from our assumption. Thus we may apply Erbacher's reduction theorem ([3]) and this yields.

Theorem 5.2. *Let M be as in Lemma 5.1. If the distinguished normal vector field N is parallel with respect to the normal connection and if the inequality*

$$\frac{1}{3}\{\mathrm{Ric}(U,U)+\mathrm{Ric}(V,V)+\mathrm{Ric}(W,W)$$
$$+\|A_1U\|^2+\|A_1V\|^2+\|A_1W\|^2\}+\rho-(n+3)^2\|\mu\|^2 \ \geq\ n^2+5n+5$$

holds on M, then there is an $(n+4)$-dimensional totally geodesic unit sphere S^{n+4} such that $M\subset S^{n+4}$.

Moreover, since the tangent space T_xS^{n+4} of the totally geodesic submanifold S^{n+4} at x in M is $T_xM\oplus\mathrm{Span}\{N\}$, S^{n+4} is an invariant submanifold of S^{4m+3} with respect to $\{\phi,\psi,\theta\}$ because of (3.2) and (3.11). Hence the submanifold M satisfying the assumptions given in Lemma 5.1 can be regarded as a real hypersurface of S^{n+4} which is totally geodesic invariant submanifold of S^{4m+3} . By the same method as shown in the proof of Theorem 4.3, we can see that M satisfies the commutativity condition

$$A'F'=F'A',\quad A'G'=G'A',\quad A'H'=H'A'.$$

Thus we have

Theorem 5.3. *Let M be as in Lemma 5.1. If the distinguished normal vector field N is parallel with respect to the normal connection and if the inequality*

$$\frac{1}{3}\{\mathrm{Ric}(U,U)+\mathrm{Ric}(V,V)+\mathrm{Ric}(W,W)$$
$$+\|A_1U\|^2+\|A_1V\|^2+\|A_1W\|^2\}+\rho-(n+3)^2\|\mu\|^2 \ \geq\ n^2+5n+5$$

holds on M, then M is isometric to

$$S^{4n_1+3}(r_1)\times S^{4n_2+3}(r_2)\quad (r_1^2+r_2^2=1)$$

for some portion (n_1,n_2) of $(n-3)/4$. In particular, if $\lambda=0$, then M is isometric to

$$S^{4n_1+3}(1/\sqrt{2})\times S^{4n_2+3}(1/\sqrt{2}).$$

Proof. Our assumptions imply

$$A'F'=F'A',\quad A'G'=G'A',\quad A'H'=H'A'.$$

as mentioned above, and hence the former part of the theorem can be easily proved by the same method as shown in the proof of Theorem 4.3. In particular, if $\lambda=0$, (3.19) yields $r_1=r_2=1/\sqrt{2}$. □

Remark. We consider special generalized Clifford tori in an $(n+4)$-unit sphere S^{n+4} defined by

$$M_{n_1,n_2}:=S^{4n_1+3}(1/\sqrt{2})\times S^{4n_2+3}(1/\sqrt{2})$$
$$=\{(x_1,\dots,x_{n+5})\in R^{n+5}\mid \sum_{i=1}^{n+5}x_i^2=1,\ \sum_{i=1}^{4n_1+4}x_i^2=1/2,\ \sum_{i=4n_1+5}^{n+5}x_i^2=1/2\},$$

where $4n_1 + 4n_2 = n - 3$ and $n = 4s - 1$ for some integer s. Since M_{n_1,n_2} is a real hypersurface of S^{n+4}, its shape operator A_1 is of the form

$$A_1 = \mathrm{diag}(1, -1)$$

for suitable orthonormal basis. The multiplicities of 1 and -1 are $4n_1 + 3$ and $4n_2 + 3$, respectively (cf. [9]). Moreover, on the real hypersurface M_{n_1,n_2},

$$A_1U = \xi, \quad A_1V = \eta, \quad A_1W = \zeta$$

(for details, see [8]) and consequently

$$\|A_1U\|^2 = \|A_1V\|^2 = \|A_1W\|^2 = 1,$$
$$\lambda = g(A_1U, U) = g(A_1V, V) = g(A_1W, W) = 0.$$

Applying (5.1)-(5.3) to M_{n_1,n_2}, we also obtain

$$\mathrm{Ric}(U, U) = n + 1, \quad \mathrm{Ric}(V, V) = n + 1, \quad \mathrm{Ric}(W, W) = n + 1,$$
$$\mathrm{tr}A_1 = 4(n_1 - n_2), \quad \mathrm{tr}A_1^2 = n + 3, \quad \rho = (n + 1)(n + 3) + 16(n_1 - n_2)^2.$$

Hence, for M_{n_1,n_2}, we have

$$\frac{1}{3}\{\mathrm{Ric}(U, U) + \mathrm{Ric}(V, V) + \mathrm{Ric}(W, W) + \|A_1U\|^2 + \|A_1V\|^2 + \|A_1W\|^2\} + \rho - (n + 3)^2\|\mu\|^2 = n^2 + 5n + 5.$$

6. Some characterizations concerning sectional curvature

In this section we let M be as in Lemma 5.1. Suppose that the distinguished normal vector field N is parallel with respect to the normal connection and that the trace of the shape operator A_1 in direction of N vanishes, that is,

$$\mathrm{tr}A_1 = 0. \tag{6.1}$$

Then (3.5) with $\alpha = 1$, (3.16) and (6.1) yield

$$\sum(\nabla_i A_1)e_i = 0, \tag{6.2}$$

where $\{e_i\}_{i=1,\dots,n}$ is an orthonormal basis of tangent vectors to M and $\nabla_i := \nabla_{e_i}$. Hence it follows from (3.5) with $\alpha = 1$ and (6.2) that

$$\sum(\nabla_i\nabla_i A_1)X = \sum(R(e_i, X)A_1)e_i$$

for any vector X tangent to M, and consequently we have

$$g(\nabla^2 A_1, A_1) = \sum_{i,j} g((R(e_i, e_j)A_1)e_i, A_1e_j). \tag{6.3}$$

Thus we have

Theorem 6.1. *Let M be as in Lemma 5.1 and let the distinguished normal vector field N be parallel with respect to the normal connection. Suppose that the trace of the shape operator A_1 in direction of N vanishes and that the minimum of sectional curvatures of M is zero. Then M is minimal and $\nabla A_1 = 0$ on M.*

Proof. The minimality of M is easily followed by our assumptions and Lemma 3.2.

Taking account of the Laplacian of $\mathrm{tr}A_1^2$, we have

$$\int_M \|\nabla A_1\|^2 * 1 = -\int_M g(\nabla^2 A_1, A_1) * 1,$$

which together with (6.3) yields

$$0 \leq \int_M \|\nabla A_1\|^2 * 1 = -\int_M \sum_{i,j} g((R(e_i, e_j)A_1)e_i, A_1 e_j) * 1. \tag{6.4}$$

Now we choose an orthonormal frame $\{e_j\}$ of M such that

$$A_1 e_j = \lambda_j e_j \quad (j = 1, \ldots, n).$$

Then it is clear that

$$\begin{aligned}&\sum_{i,j} g((R(e_i, e_j)A_1)e_i, A_1 e_j)\\ &= \sum_{i,j}\{g((R(e_i, e_j)A_1 e_i, A_1 e_j) - g(A_1 R(e_i, e_j)e_i, A_1 e_j)\} = \frac{1}{2}\sum_{i,j}(\lambda_i - \lambda_j)^2 K_{ij},\end{aligned}$$

where K_{ij} denotes the sectional curvature of the plane section spanned by $\{e_i, e_j\}$. Hence, if the minimum of sectional curvatures of M is zero, the above equation and (6.4) imply $\nabla A_1 = 0$. □

By means of Theorem 6.1 we can obtain the following theorem under additional condition:

Theorem 6.2. *Let M be as in Lemma 5.1 and assume that there exists an orthonormal basis $\{N, N_\alpha\}_{\alpha=2,\ldots,p}$ of normal vectors to M each of which is parallel with respect to the normal connection. If the trace of the shape operator A_1 in direction of N vanishes and if the minimum of sectional curvatures of M is zero, then there is an $(n+4)$-dimensional totally geodesic unit sphere S^{n+4} of S^{4m+3} such that $M \subset S^{n+4}$.*

Proof. Under our assumptions it follows from Theorem 6.1 that

$$\mathrm{tr}A_\alpha = 0, \quad \alpha = 2, \ldots .p.$$

Moreover, it is clear from (3.5) that, for any vector fields X, Y tangent to M,

$$(\nabla_X A_\alpha)Y - (\nabla_Y A_\alpha)X = 0$$

because of $s_{\alpha\beta} = 0,\ 1 \leq \alpha, \beta \leq p$, and consequently

$$\sum(\nabla_i A_\alpha)e_i = 0,$$

where $\{e_i\}_{i=1,\ldots,n}$ is an orthonormal basis of tangent vectors to M. Taking account of the Laplacian of $\mathrm{tr}A_\alpha^2$ and using the quite similar method as shown in the proof of Theorem 6.1, we can easily see that

$$\nabla_X A_\alpha = 0, \quad \alpha = 2, \ldots, p \tag{6.5}$$

for any vector field X tangent to M.

Differentiating the third equation of (3.13) covariantly and using the first equation of (3.13) and (6.5), we have

$$A_\alpha FX = 0$$

for any vector fields X, Y tangent to M. Inserting FX instead of X in this equation and using (2.5), (3.13) and (3.17), we have

$$A_\alpha = 0, \quad \alpha = 2, \ldots, p.$$

Hence the first normal space of M is contained in Span$\{N\}$, which is invariant under parallel translation with respect to the normal connection from our assumption. Thus we may apply Erbacher's reduction theorem ([3]), which gives the proof of our theorem. □

Combining Theorem 6.2 and a theorem provided in [7, Theorem 5.2, p. 436], we have

Theorem 6.3. *Let M be as in Lemma 5.1 and assume that there exists an orthonormal basis $\{N, N_\alpha\}_{\alpha=2,\ldots,p}$ of normal vectors to M each of which is parallel with respect to the normal connection. If the trace of the shape operator A_1 in direction of N vanishes and if the minimum of sectional curvatures of M is zero, then M is isometric to a generalized Clifford surface:*

$$S^{4n_1+3}(((4n_1+3)/(n+3))^{\frac{1}{2}}) \times S^{4n_2+3}(((4n_2+3)/(n+3))^{\frac{1}{2}})$$

for some portion (n_1, n_2) of $(n-3)/4$.

Proof. By means of Theorem 6.2 M can be regarded as a real minimal hypersurface of S^{n+2} which is a totally geodesic invariant submanifold of S^{4m+3}. Moreover, since $\nabla A_1 = 0$ and $A_1 F = FA_1, A_1 G = GA_1, A_1 H = HA_1$, we can easily see that M is isometric to

$$S^{4n_1+3}(r_1) \times S^{4n_2+3}(r_2)$$

for some portion (n_1, n_2) of $(n-3)/4$ and some r_1, r_2 with $r_1^2 + r_2^2 = 1$ as shown in the proof of Theorem 4.3. On the other hand, M is minimal and consequently $r_1 = ((4n_1+3)/(n+3))^{\frac{1}{2}}, r_2 = ((4n_2+3)/(n+3))^{\frac{1}{2}}$. Moreover, using (3.19), we can easily see that the minimum of sectional curvatures of those hypersurfaces is zero. □

Corollary 6.4. *Let M be a compact, minimal real hypersurface tangent to the structure vector fields ξ, η, ζ of a (4m+3)-dimensional unit sphere S^{4m+3}. If the minimum of sectional curvatures of M is zero, then M is isometric to*

$$S^{4n_1+3}(((4n_1+3)/(4m+2))^{\frac{1}{2}}) \times S^{4n_2+3}(((4n_2+3)/(4m+2))^{\frac{1}{2}})$$

for some portion (n_1, n_2) of $m-1$.

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