J. Korean Math. Soc. **44** (2007), No. 2, pp. 297–306

INEQUALITIES FOR STAR DUALS OF INTERSECTION BODIES

YUAN JUN, ZHU HUAWEI, AND LENG GANGSONG

ABSTRACT. In this paper, we present a new kind of duality between intersection bodies and projection bodies. Furthermore, we establish some counterparts of dual Brunn-Minkowski inequalities for intersection bodies.

1. Introduction

Intersection body and projection body are two basic concepts in geometric tomography. The projection bodies have been the objects of intense investigation during the past three decades. Lutwak [6] introduced the mixed projection bodies and studied them and their polar bodies systematically. The polar body of a convex body is an important object in the context of convex geometry. For example, two of the most important affine isoperimetric inequalities, the Blaschke-Santaló inequality and the Petty projection inequality, are closely related to the polar bodies. Hence, after we studied the mixed intersection bodies [11], it is natural to consider the inequalities for their polar bodies. But the intersection body of even a convex body generally is not convex, so is the mixed intersection body. Thus the inequalities for the polar body of the intersection body can not be given in the general cases.

In [7], Moszyńska introduced the notion of the star dual of a star body. Generally, star dual of a convex body is different from its polar dual. For every convex body K, let K^* and K° denote the polar body and the star dual of K, respectively. It is easy to verify that (see [7] for the equality case)

$$K^* \subset K^\circ,$$

and $K^* = K^\circ$ if and only if K is a centered ball.

In this paper, by applying the concept of star dual, we establish some inequalities for star duals of intersection bodies.

Received July 29, 2005.

2000 *Mathematics Subject Classification.* 52A40, 52A20.

Key words and phrases. star body, star dual, intersection body, radial Blaschke sum, harmonic Blaschke sum.

Supported in part by the National Natural Science Foundation of China. (Grant NO.10671117).

As Lutwak [5] shows (and as is further elaborated in Gardner's book [1]), there is a kind of duality between intersection and projection bodies. For a star body L, let IL denote the intersection body of L. For the star dual of the mixed intersection body $I(L_1, \ldots, L_{n-1})$, we will write $I^\circ(L_1, \ldots, L_{n-1})$, rather than $(I(L_1, \ldots, L_{n-1}))^\circ$.

The first aim of this paper is to establish the following star dual of the Busemann intersection inequality:

Theorem 1. *Let $L_1, \ldots, L_{n-1}$ be star bodies in $\mathbb{R}^n$. Then*

$$V(L_1) \cdots V(L_{n-1}) V(I^\circ(L_1, \ldots, L_{n-1})) \geq \left(\frac{k_n}{k_{n-1}}\right)^n, \tag{1.1}$$

with equality if and only if all L_i are dilatates.

This is just a dual form of the general Petty projection inequality which was given by Lutwak [6]:

Theorem 1*. *Let $K_1, \ldots, K_{n-1}$ be convex bodies in $\mathbb{R}^n$. Then*

$$V(K_1) \cdots V(K_{n-1}) V(\Pi^*(K_1, \ldots, K_{n-1})) \leq \left(\frac{k_n}{k_{n-1}}\right)^n,$$

with equality if and only if K_i are homothetic ellipsoids.

For two star bodies K and L, let $K \check{+} L$, $K \hat{+} L$ denote the radial Blaschke sum and the harmonic Blaschke sum of K and L, respectively. The other aim of this paper is to establish the dual Brunn-Minkowski inequalities for the star duals of the intersection bodies for the radial Blaschke sum and the harmonic Blaschke sum.

Theorem 2. *Let K, L be star bodies in $\mathbb{R}^n$. Then*

$$V(I^o(K \check{+} L))^{-\frac{1}{n}} \geq V(I^o K)^{-\frac{1}{n}} + V(I^o L)^{-\frac{1}{n}}, \tag{1.2}$$

with equality if and only if L is a dilatate of K.

Theorem 3. *Let K, L be star bodies in $\mathbb{R}^n$. Then*

$$\frac{V(I^o(K \hat{+} L))^{-\frac{n+1}{n(n-1)}}}{V(K \hat{+} L)} \geq \frac{V(I^o K)^{-\frac{n+1}{n(n-1)}}}{V(K)} + \frac{V(I^o L)^{-\frac{n+1}{n(n-1)}}}{V(L)}, \tag{1.3}$$

with equality if and only if L is a dilatate of K.

2. Basic definitions and notation

As usual, let B^n denote the unit ball in Euclidean n-space, $\mathbb{R}^n$. While its boundary is S^{n-1} and the origin is denoted by o. If u is a unit vector, that is, an element of S^{n-1}, we denote by $u^\perp$ the $(n-1)$-dimensional linear subspace orthogonal to u.

For a compact subset L of $\mathbb{R}^n$, with $o \in L$, star-shaped with respect to o, the radial function $\rho(L, \cdot) : S^{n-1} \to \mathbb{R}$, is defined by

$$\rho(L, u) = \rho_L(u) = \max\{\lambda : \lambda u \in L\}. \tag{2.1}$$

If $\rho(L,\cdot)$ is continuous and positive, L will be called a star body.

Let φ_o^n denote the set of star bodies in $\mathbb{R}^n$. Two star bodies $K, L \in \varphi_o^n$ are said to be dilatate (of each other) if $\rho(K,u)/\rho(L,u)$ is independent of $u \in S^{n-1}$.

Also associated with a star body $L \in \varphi_o^n$ is its star dual L^o, which was introduced by Moszyńska [7] (and was improved in [8]). Let i be the inversion of $\mathbb{R}^n\backslash\{0\}$, with respect to S^{n-1}:

$$i(x) := \frac{x}{||x||^2}.$$

Then the star dual L^o of a star body $L \in \varphi_o^n$ is defined by

$$L^o = \mathrm{cl}(\mathbb{R}^n\backslash i(L)).$$

It is easy to verify that for every $u \in S^{n-1}$ [7],

$$\rho(L^o,u) = \frac{1}{\rho(L,u)}. \tag{2.2}$$

Let $L \in \varphi_o^n$, $i \in \mathbb{R}$. The dual volume $\widetilde{V}_i(L)$ and dual quermassintegral $\widetilde{W}_{n-i}(L)$ of L are defined by

$$\widetilde{V}_i(L) = \widetilde{W}_{n-i}(L) = \frac{1}{n}\int_{S^{n-1}} \rho_L(u)^i du. \tag{2.3}$$

Let $K \in \varphi_o^n$. The intersection body IK of K is a star body such that

$$\rho_{IK}(u) = \lambda_{n-1}(K \cap u^{\perp}) = \frac{1}{n-1}\int_{S^{n-1}\cap u^{\perp}} \rho_K(v)^{n-1} d\lambda_{n-2}(v), \tag{2.4}$$

where λ_i denote the i-dimensional volume.

Let $K_1, \dots, K_{n-1} \in \varphi_o^n$. The mixed intersection body $I(K_1, \dots, K_{n-1})$ of star bodies $K_1, \dots, K_{n-1}$ is defined by

$$\begin{aligned}\rho_{I(K_1,\dots,K_{n-1})}(u) &= \widetilde{v}(K_1 \cap u^{\perp}, \dots, K_{n-1} \cap u^{\perp}) \\ &= \frac{1}{n-1}\int_{S^{n-1}\cap u^{\perp}} \rho_{K_1}(v)\cdots\rho_{K_{n-1}}(v) d\lambda_{n-2}(v),\end{aligned} \tag{2.5}$$

where $\widetilde{v}$ is (n-1)-dimensional dual mixed volume.

If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = L$, then $I(K_1, \dots, K_{n-1})$ will be denoted as $I_i(K,L)$. If $K = B^n$, then $I_i(B^n, L)$ is called the intersection body of order i of L; it will often be written as I_iK. Specially, $I_{n-1}L{=}IL$. This term was introduced by Zhang [13].

Let $K_1, \dots, K_{n-1} \in \varphi_o^n$. In accordance with (2.5), we define the ith mixed intersection body $I_i(K_1, \dots, K_{n-1})$, by:

$$\rho_{I_i(K_1,\dots,K_{n-1})}(u) = \frac{1}{n-1}\int_{S^{n-1}\cap u^{\perp}} \rho_{K_1}^{\frac{i}{n-1}}(v)\cdots\rho_{K_{n-1}}^{\frac{i}{n-1}}(v) d\lambda_{n-2}(v). \tag{2.6}$$

It follows that $I_{n-1}(K_1, \dots, K_{n-1}){=}I(K_1, \dots, K_{n-1})$ and $I_i\,(K, \dots, K){=}I_iK$.

The above elementary results (and definitions) are from the theory of convex bodies. The reader may consult the standard works on the subject [1, 3, 10] for reference.

3. Counterparts of Busemann intersection inequality for star duals of intersection bodies

In fact, we will establish the following inequality more general than Theorem 1:

Theorem 3.1. *Let $L_1, \dots, L_{n-1} \in \varphi_o^n$ and $1 \le i \le n-1$. Then*

$$\prod_{j=1}^{n-1} V(L_j)^{\frac{i}{n-1}} V(I_i^\circ(L_1, \dots, L_{n-1})) \ge \frac{k_n^{i+1}}{k_{n-1}^n}, \tag{3.1}$$

with equality if and only if all L_i are dilatates.

Remark 1. Taking $i = n-1$ in Theorem 3.1, we obtain Theorem 1.

To prove Theorem 3.1, the following preliminary result will be needed:

Lemma 3.2. ([2]) *Let $L \in \varphi_o^n$ and $1 \le i < j \le n-1$. Then*

$$\left(\frac{\widetilde{V}_i(L)}{\widetilde{V}_i(B^n)}\right)^j \le \left(\frac{\widetilde{V}_j(L)}{\widetilde{V}_j(B^n)}\right)^i, \tag{3.2}$$

with equality if and only if L is a ball.

Lemma 3.3. *Let $L \in \varphi_o^n$ and $1 \le i \le n-1$. Then*

$$V(I_i L) \le \frac{k_{n-1}^n}{k_n^{i-1}} V(L)^i, \tag{3.3}$$

with equality if and only if L is a ball.

Proof. From the define of $I_i L$, we have[13]

$$\rho_{I_i L}(u) = \widetilde{v}_i(L \cap u^\perp). \tag{3.4}$$

Then

$$\begin{aligned} V(I_i L) &= \frac{1}{n} \int_{S^{n-1}} \rho_{I_i L}(u)^n du \\ &\le k_{n-1}^{\frac{n(n-1-i)}{n-1}} \frac{1}{n} \int_{S^{n-1}} \rho_{IL}(u)^{\frac{in}{n-1}} du \\ &\le (k_{n-1}^n k_n)^{\frac{(n-1-i)}{n-1}} V(IL)^{\frac{i}{n-1}} \\ &\le \frac{k_{n-1}^n}{k_n^{i-1}} V(L)^i. \end{aligned}$$

The first inequality uses (3.4) and (3.2). The second inequality uses the Hölder integral inequality. By the equality conditions in Hölder integral inequality and Lemma 3.2, the equality in (3.3) holds if and only if L is a ball. □

Remark 2. The particular case of inequality (3.3) for $i = n-1$ is the well-known Busemann intersection inequality.

Lemma 3.4. ([12]) *Let $L \in \varphi_o^n$. Then*

$$V(L)V(L^\circ) \geq k_n^2, \tag{3.5}$$

with equality if and only if L is a ball.

Proof of Theorem 3.1. According to (2.6), (3.3) and the Hölder integral inequality, we have

$$\begin{aligned}
&V(I_i(L_1,\dots,L_n))\\
&= \frac{1}{n}\int_{S^{n-1}} \rho^n_{I_i(L_1,\dots,L_{n-1})}(u)du\\
&= \frac{1}{n}\int_{S^{n-1}} \left(\frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho_{L_1}^{\frac{i}{n-1}}(v)\cdots\rho_{L_{n-1}}^{\frac{i}{n-1}}(v)d\lambda_{n-2}(v)\right)^n du\\
&\leq \frac{1}{n}\int_{S^{n-1}} \prod_{j=1}^{n-1}\left(\frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho_{L_j}^{i}(v)d\lambda_{n-2}(v)\right)^{\frac{n}{n-1}} du\\
&= \frac{1}{n}\int_{S^{n-1}} \prod_{j=1}^{n-1} \rho_{I_iL_j}(u)^{\frac{n}{n-1}}du\\
&\leq \prod_{j=1}^{n-1} V(I_iL_j)^{\frac{1}{n-1}}\\
&\leq \frac{k_{n-1}^n}{k_n^{i-1}}\prod_{j=1}^{n-1} V(L_j)^{\frac{i}{n-1}}.
\end{aligned}$$

To complete the proof of the inequality, it suffices to apply Lemma 3.4.

By the equality conditions of the Hölder integral inequality, Lemma 3.3 and Lemma 3.4, the equality in (3.1) holds if and only if L is a ball. □

Let $L_1 = \cdots = L_{n-1} = L$ and $i = n-1$ in Theorem 3.1. We get

Corollary 3.5. *Let $L \in \varphi_o^n$. Then*

$$V(L)^{n-1}V(I^\circ L) \geq \left(\frac{k_n}{k_{n-1}}\right)^n, \tag{3.6}$$

with equality if and only if L is a ball.

This is just a dual form of the Petty projection inequality which was given by Petty [9]:

Theorem 3.5*. *Let K be a convex body in $\mathbb{R}^n$. Then*

$$V(K)^{n-1}V(\Pi^* K) \leq \left(\frac{k_n}{k_{n-1}}\right)^n, \tag{3.7}$$

with equality if and only if K is an ellipsoid.

Let $L_1 = \cdots = L_{n-1} = L$ in Theorem 3.1 we get

Corollary 3.6. *Let $L \in \varphi_o^n$ and $1 \le i \le n-1$. Then*

$$V(L)^i V(I_i^\circ L) \ge \frac{k_n^{i+1}}{k_{n-1}^n}, \tag{3.8}$$

with equality if and only if L is a ball.

This is just a dual form of the general Petty projection inequality for ith projection body which was given by Lutwak [6]:

Theorem 3.6*. *Let K be a convex body in $\mathbb{R}^n$ and $1 \le i \le n-1$. Then*

$$V(K)^i V(\Pi_i^* K) \le \frac{k_n^{i+1}}{k_{n-1}^n}, \tag{3.9}$$

with equality if and only if K is an ellipsoid.

4. The dual Brunn-Minkowski inequalities for star duals of intersection bodies

For $K, L \in \varphi_o^n$ and $\lambda, \mu \ge 0$, the radial Blaschke sum, $\lambda \cdot K \check{+} \mu \cdot L$, was defined by Lutwak [5]. Its radial function is given by:

$$\rho(\lambda \cdot K \check{+} \mu \cdot L, \cdot)^{n-1} = \lambda \rho(K, \cdot)^{n-1} + \mu \rho(L, \cdot)^{n-1}. \tag{4.1}$$

Obviously, radial Blaschke and Minkowski scalar multiplications are related by $\lambda \cdot K = \lambda^{\frac{1}{n-1}} K$.

For the radial Blaschke sum, we will prove a result more general than Theorem 2.

Theorem 4.1. *Let $K, L \in \varphi_o^n$, $0 \le i < n$ and $0 \le \alpha \le 1$. Then*

$$\begin{aligned} \widetilde{W}_i(I^o(K \check{+} L))^{-\frac{1}{n-i}} \ge & \widetilde{W}_i\left[I^o(\alpha \cdot K \check{+} (1-\alpha) \cdot L)\right]^{-\frac{1}{n-i}} \\ & + \widetilde{W}_i\left[I^o((1-\alpha) \cdot K \check{+} \alpha \cdot L)\right]^{-\frac{1}{n-i}}, \end{aligned} \tag{4.2}$$

with equality if and only if $\alpha \cdot K \check{+} (1-\alpha) \cdot L$ is a dilatate of $(1-\alpha) \cdot K \check{+} \alpha \cdot L$.

Proof. For $u \in S^{n-1}$, from (2.4) and (4.1), we have

$$\begin{aligned} \rho(I(\lambda \cdot K \check{+} \mu \cdot L, u) = & \frac{1}{n-1} \int_{S^{n-1} \cap u^\perp} \rho(\lambda \cdot K \check{+} \mu \cdot L, v)^{n-1} d\lambda_{n-2}(v) \\ = & \lambda \rho(IK, u) + \mu \rho(IL, u). \end{aligned}$$

Then, using (2.2) and the Minkowski integral inequality($i - n < 0$), we have

$$\begin{aligned} & \widetilde{W}_i(I^o(K \check{+} L))^{-\frac{1}{n-i}} \\ = & \left[\frac{1}{n} \int_{S^{n-1}} \rho(I(K \check{+} L), u)^{i-n} du\right]^{-\frac{1}{n-i}} \\ = & \left[\frac{1}{n} \int_{S^{n-1}} (\rho(IK, u) + \rho(IL, u))^{i-n} du\right]^{-\frac{1}{n-i}} \end{aligned}$$

$$= \left[\frac{1}{n}\int_{S^{n-1}} (\alpha\rho(IK,u) + (1-\alpha)\rho(IL,u) + (1-\alpha)\rho(IK,u) + \alpha\rho(IL,u))^{i-n}\, du\right]^{-\frac{1}{n-i}}$$

$$= \left[\frac{1}{n}\int_{S^{n-1}} \left(\rho_{I[\alpha\cdot K\check{+}(1-\alpha)\cdot L]}(u) + \rho_{I[(1-\alpha)\cdot K\check{+}\alpha\cdot L]}(u)\right)^{i-n} du\right]^{-\frac{1}{n-i}}$$

$$\geq \left[\frac{1}{n}\int_{S^{n-1}} \rho_{I[\alpha\cdot K\check{+}(1-\alpha)\cdot L]}(u))^{i-n} du\right]^{-\frac{1}{n-i}} + \left[\frac{1}{n}\int_{S^{n-1}} \rho_{I[(1-\alpha)\cdot K\check{+}\alpha\cdot L]}(u))^{i-n} du\right]^{-\frac{1}{n-i}}$$

$$= \widetilde{W}_i\left[I^o(\alpha\cdot K\check{+}(1-\alpha)\cdot L)\right]^{-\frac{1}{n-i}} + \widetilde{W}_i\left[I^o((1-\alpha)\cdot K\check{+}\alpha\cdot L)\right]^{-\frac{1}{n-i}}.$$

From the equality conditions for the Minkowski integral inequality, the equality holds if and only if $\alpha\cdot K\check{+}(1-\alpha)\cdot L$ is a dilatate of $(1-\alpha)\cdot K\check{+}\alpha\cdot L$. □

Taking $\alpha = 1$ in Theorem 4.1, we get

Corollary 4.2. *Let $K, L \in \varphi_o^n$ and $0 \leq i < n$. Then*

$$\widetilde{W}_i(I^o(K\check{+}L))^{-\frac{1}{n-i}} \geq \widetilde{W}_i(I^oK)^{-\frac{1}{n-i}} + \widetilde{W}_i(I^oL)^{-\frac{1}{n-i}}, \tag{4.3}$$

with equality if and only if L is a dilatate of K.

Remark 3. As in the proof of Theorem 4.1, we have

$$\widetilde{W}_i\left[I^o(\alpha\cdot K\check{+}(1-\alpha)\cdot L)\right]^{-\frac{1}{n-i}} \geq \alpha\widetilde{W}_i(I^oK)^{-\frac{1}{n-i}} + (1-\alpha)\widetilde{W}_i(I^oL)^{-\frac{1}{n-i}}, \tag{4.4}$$

$$\widetilde{W}_i\left[I^o((1-\alpha)\cdot K\check{+}\alpha\cdot L)\right]^{-\frac{1}{n-i}} \geq (1-\alpha)\widetilde{W}_i(I^oK)^{-\frac{1}{n-i}} + \alpha\widetilde{W}_i(I^oL)^{-\frac{1}{n-i}}. \tag{4.5}$$

From (4.4) and (4.5), we get

$$\widetilde{W}_i\left[I^o(\alpha\cdot K\check{+}(1-\alpha)\cdot L)\right]^{-\frac{1}{n-i}} + \widetilde{W}_i\left[I^o((1-\alpha)\cdot K\check{+}\alpha\cdot L)\right]^{-\frac{1}{n-i}} \geq \widetilde{W}_i(I^oK)^{-\frac{1}{n-i}} + \widetilde{W}_i(I^oL)^{-\frac{1}{n-i}}.$$

Hence, (4.2) is a stronger form of (4.3).

In [4], Lutwak introduced the harmonic Blaschke sum. Suppose $K, L \in \varphi_o^n$. To define the harmonic Blaschke sum, $K\hat{+}L$, first define $\xi > 0$ by

$$\xi^{1/(n+1)} = \frac{1}{n}\int_{S^{n-1}} [V(K)^{-1}\rho(K,u)^{n+1} + V(L)^{-1}\rho(L,u)^{n+1}]^{n/(n+1)} dS(u). \tag{4.6}$$

The body $K\hat{+}L \in \varphi_o^n$ is defined as the body whose radial function is given by

$$\xi^{-1}\rho(K\hat{+}L,\cdot)^{n+1} = V(K)^{-1}\rho(K,\cdot)^{n+1} + V(L)^{-1}\rho(L,\cdot)^{n+1}. \tag{4.7}$$

In fact, we will establish the following theorem more general than Theorem 3.

Theorem 4.3. *Let $K, L \in \varphi_o^n$ and $0 \le i \le n-1$. Then*

$$\frac{\widetilde{W}_i(I^o(K\hat{+}L))^{\frac{n+1}{(i-n)(n-1)}}}{V(K\hat{+}L)} \ge \frac{\widetilde{W}_i(I^oK)^{\frac{n+1}{(i-n)(n-1)}}}{V(K)} + \frac{\widetilde{W}_i(I^oL)^{\frac{n+1}{(i-n)(n-1)}}}{V(L)}, \tag{4.8}$$

with equality if and only if L is a dilatate of K.

Proof. By (4.6), (4.7) and the polar coordinate formula for volume, we get $\xi = V(K\hat{+}L)$. Hence from (4.7), we obtain

$$V(K\hat{+}L)^{-1}\rho(K\hat{+}L,\cdot)^{n+1} = V(K)^{-1}\rho(K,\cdot)^{n+1} + V(L)^{-1}\rho(L,\cdot)^{n+1}. \tag{4.9}$$

For $u \in S^{n-1}$. By (2.4), (4.9), we have

$$\begin{aligned} &\rho(I(K\hat{+}L),u) \\ =\ & \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho(K\hat{+}L,v)^{n-1}d\lambda_{n-2}(v) \\ =\ & \frac{1}{n-1}\cdot V(K\hat{+}L)^{\frac{n-1}{n+1}} \\ & \times\int_{S^{n-1}\cap u^\perp}\left(\frac{\rho(K,v)^{n+1}}{V(K)}+\frac{\rho(L,v)^{n+1}}{V(L)}\right)^{\frac{n-1}{n+1}} d\lambda_{n-2}(v). \end{aligned}$$

Applying the Minkowski integral inequality($0 < \frac{n-1}{n+1} < 1$) and (2.4), we have

$$\begin{aligned} &\rho(I(K\hat{+}L),u) \\ \ge\ & V(K\hat{+}L)^{\frac{n-1}{n+1}}\left(\frac{1}{V(K)}\cdot\rho(IK,u)^{\frac{n+1}{n-1}}+\frac{1}{V(L)}\cdot\rho(IL,u)^{\frac{n+1}{n-1}}\right)^{\frac{n-1}{n+1}}. \end{aligned}$$

That is,

$$\frac{\rho(I(K\hat{+}L),u)^{\frac{n+1}{n-1}}}{V(K\hat{+}L)} \ge \frac{\rho(IK,u)^{\frac{n+1}{n-1}}}{V(K)} + \frac{\rho(IL,u)^{\frac{n+1}{n-1}}}{V(L)}. \tag{4.10}$$

By the equality conditions of the Minkowski integral inequality, the equality of (4.10) holds if and only if $\frac{1}{V(K)}\cdot\rho(K,\cdot)^{n+1}$ and $\frac{1}{V(L)}\cdot\rho(L,\cdot)^{n+1}$ are proportional. That is, L is a dilatate of K.

Let $a = \dfrac{V(K\hat{+}L)}{V(K)}$ and $b = \dfrac{V(K\hat{+}L)}{V(L)}$. The Minkowski integral inequality $(-\frac{n(n-1)}{n+1} < 0)$ combined with (2.2) and (2.3), we have

$$\begin{aligned}
&\widetilde{W}_i(I^o(K\hat{+}L))^{\frac{n+1}{(i-n)(n-1)}}\\
&= \left(\frac{1}{n}\int_{S^{n-1}} \rho(I(K\hat{+}L),u)^{i-n}du\right)^{\frac{n+1}{(i-n)(n-1)}}\\
&\geq \left(\frac{1}{n}\int_{S^{n-1}} \left(a\cdot\rho(IK,u)^{\frac{n+1}{n-1}} + b\cdot\rho(IL,u)^{\frac{n+1}{n-1}}\right)^{\frac{(i-n)(n-1)}{n+1}} du\right)^{\frac{n+1}{(i-n)(n-1)}}\\
&\geq a\left(\frac{1}{n}\int_{S^{n-1}} \rho(IK,u)^{i-n}du\right)^{\frac{n+1}{(i-n)(n-1)}}\\
&\quad + b\left(\frac{1}{n}\int_{S^{n-1}} \rho(IL,u)^{i-n}du\right)^{\frac{n+1}{(i-n)(n-1)}}\\
&= a\widetilde{W}_i(I^oK)^{\frac{n+1}{(i-n)(n-1)}} + b\widetilde{W}_i(I^oL)^{\frac{n+1}{(i-n)(n-1)}}.
\end{aligned}$$

So the inequality (4.8) is proved. By the equality conditions for the Minkowski integral inequality and (4.10), the equality in (4.8) holds if and only if L is a dilatate of K. □

Acknowledgment. The authors are most grateful to the referee for his helpful suggestions.

References

[1] R. J. Gardner, *Geometric Tomography*, Encyclopedia of Mathematics and its Applications, 58, Cambridge University Press, New York, 1995.

[2] R. J. Gardner and S. Vassallo, *Inequalities for dual Isoperimetric Deficits*, Mathematika, **45** (1998), no. 2, 269–285.

[3] K. Leichtweiß, *Affine Geometry of Convex Bodies*, Johann Ambrosius Barth, Heidelberg, 1998.

[4] E. Lutwak, *Centroid bodies and dual mixed volumes*, Proc. London Math. Soc. (3) **60** (1990), no. 2, 365–391.

[5] ———, *Intersection bodies and dual mixed volumes*, Adv. in Math. **71** (1988), no. 2, 232–261.

[6] ———, *Mixed projection inequalities*, Trans. Amer. Math. Soc. **287** (1985), no. 1, 91–105.

[7] M. Moszyńska, *Quotient star bodies, intersection bodies and star duality,* J. Math. Anal. Appl. **232** (1999), no. 1, 45–60.

[8] ———, *Selected Topics in Convex Geometry*, Springer Verlag, 2005.

[9] C. M. Petty. *Isoperimetric problems,* Proc. Conf. Convexity and Combinatorial Geometry, (Univ. Oklahoma, 1971), University of Oklahoma, 1971, pp. 26–41.

[10] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia of Mathematics and its Applications, 44, Cambridge University Press, Cambridge, 1993.

[11] S. Yuan and G. Leng, *Inequality for mixed intersection bodies*, Chinese Ann. Math. Series B **26** (2005), no. 3, 423–436.

[12] J. Yuan, S. Yuan, and G. Leng, *Inequalities for dual harmonic quermassintegrals,* J. Korean. Math. Soc. **43** (2006), no. 3, 593–607.

[13] G. Zhang, *Centered bodies and dual mixed volumes*, Tran. Amer. Math. **345** (1994), no. 2, 777–801.

YUAN JUN
SCHOOL OF MATHEMATICS AND COMPUTER SCIENCE
NANJING NORMAL UNIVERSITY
NANJING 210097, P. R. CHINA
E-mail address: yuanjun@graduate.shu.edu.cn

ZHU HUAWEI
SOFTWARE ENGINEERING INSTITUTE
GUANGZHOU UNIVERSITY
GUANGZHOU 510405, P. R. CHINA
E-mail address: huawei000@263.net

LENG GANGSONG
DEPARTMENT OF MATHEMATICS
SHANGHAI UNIVERSITY
SHANGHAI 200436, P. R. CHINA
E-mail address: gleng@staff.shu.edu.cn