INSTANTONS ON CONIC 4-MANIFOLDS: FREDHOLM THEORY

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ABSTRACT. We study the self-duality operator on conic 4-manifolds. The self-duality operator can be identified as a regular singular operator in the sense of Brüning and Seeley, based on which we construct its parametrizations and closed extensions. We also compute the indexes.

1. Introduction

The main purpose of the paper is to initiate the investigation of gauge theory on a class of manifolds that have the simplest kind of singularity.

Let X_0 be a compact smooth manifold with boundary Y. The associated topological conical space X is by definition the gluing $X_0 \cup_Y N$ along Y, where $N = [0, 1] \times Y / \{0\} \times Y$, i.e., one boundary $\{0\} \times Y$ collapsing into a single point c. Clearly X is a compact topological space but the manifold structure at c is singular. To best suit this type of singularity, we place the (close) conic metric of the form $dt^2 + t^2 g_Y$ on N away from c, where g_Y is any smooth Riemannian metric on Y.

Conical spaces and conic metrics arise naturally in many different contexts. In Algebraic Geometry, a conic singular complex surface is obtained by contracting a curve of negative self intersection in a smooth Kähler surface. The smooth Kähler metric descends to a conic metric on the complex quotient. The point of such a construction is that away from the singular point, the complex quotient is often topologically much simpler than the original Kähler surface. For example, contracting the sixteen (-2)-curves on a Kummer surface yields simply the quotient of a torus under the natural involution. It is well-known that negative curves are difficult to study since they are isolated and do not form any moduli. In Symplectic Geometry, for symplectic 4-manifolds with boundary a *concave* contact 3-manifold, the contact structure can be extended naturally into a symplectic form which is compatible with a conic metric on the end. A recent result of J. Etnyre and K. Honda [8] shows that any concave

contact structure is fillable by infinitely many non-isomorphic symplectic manifolds, thus quite different from convex contact structures. In String Theory, there is a 6-dimensional conifold X with base $S^3 \times S^2$ introduced in Gopakumar-Vafa [9]. The cone singularity can be resolved in two ways: either blow up the S^2 direction resulting in the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over S^2, or deform along the S^3 direction resulting in the cotangent bundle T^*S^3. The conjectured duality in [9] says that the Gromov-Witten invariant of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ is related to a Chern-Simons type knot invariant of S^3. A possible approach might be to study directly the conifold X and relate the invariants to those on the resolutions.

The (close) conic metric on X defined above is conformally equivalent to the open conic metric $ds^2 + s^2 g_Y$ and product metric $ds^2 + g_Y$, where $1 < s < +\infty$. However the geometry in the first case is considerably different and harder than the latter two cases. The main reason for such a disparity is due to the fact that the open conic and product metrics are both complete whereas the close conic metric is not. Thus the traditional analytic approach for complete metrics is no longer suitable and new phenomena do emerge in close conic metric geometry (see [1, 3, 4, 11]).

In the aforementioned examples, it is necessary to work with close conic metrics, since open or product metrics can not be Kähler or symplectic. (Of course being Kähler or symplectic metrics is not conformally invariant.) The conic singularities and manifolds with boundary are closely related, and topological and geometric properties on manifolds with conic singularities may be viewed as a ramification of special boundary value problems with non-local boundary conditions. The conic singularity is interesting in its own right and was initially studied by Cheeger [3, 4].

In the current paper, we introduce Yang-Mills theory on the conic 4-manifolds. This leads to two major changes from the standard theory in smooth 4-manifolds (e.g. [5]). (1) Since the metric is not differentiable at the cone singular point, the slice condition and d_A^* operator require a modification similar to [6]. (2) The functional-analytic framework and the Fredholm theory need different setups than [5, 6]. In particular, the self-duality operator on conic manifolds may have many different closed extensions which are all Fredholm but with different domains and indices. Our main results can be summarized as follows.

Theorem. (1) *The self-duality operator $d_A^+ \oplus d_A^*$ on a conic 4-manifold X can be identified with a first order regular singular operator $P_A = \frac{\partial}{\partial t} + t^{-1}P_0 + P_1$ in the neighborhood of the cone point. In particular it is of (weak) Fuchs type.*

(2) *Let P_A^{iD} be the restriction of $P_{A,\max}$ to the ideal Dirichlet boundary condition. The operator P_A^{iD} is closed and Fredholm.*

(3) *The index of P_A^{iD} is given by*

$$\int_X \tilde{c}_2(\mathrm{ad}P) - \frac{1}{2}(\eta_{P_0} + m_1) + \sum_{k \geq 1} \frac{2^{-2k}}{2k(2k+1)} \mathrm{Res}_1 \eta_{P_0}(2k),$$

where $\eta_{P_0}(z)$ is the eta function of the operator P_0 and m_1 is the multiplicity of eigenvalue $1/4$ of the Laplacian operator Δ^1 on coclosed $\Omega^1(Y)$ with the metric g_Y.

The paper is organized as follows. Section 2 gives a basic set-up and shows part (1) of the Theorem in two lemmas. The parametric construction, following from the method in [1, 2] (or the more general treatment in [11]), is given in section 3. Then we prove part (2) in Theorem 3.5. Unlike the smooth manifold case, there are different closed extensions of the operator P_A. By applying the results in [1, 2], we obtain the index formula part (3) in the last section.

2. A basic set-up

Let X be a closed, oriented conical 4-manifold with a cone point c. (The analysis can be naturally extended to the case of a finite number of cone points.) Let N be a neighborhood of c in X, $N = (0, \varepsilon] \times Y$, where the 3-manifold Y has a metric g_Y. Note that the 4-dimensional Yang-Mills theory is conformally unchanged. Thus up to conformal changes, N can be equipped with the metric

$$dt^2 + t^2 g_Y.$$

The metric satisfies the conformally conic manifold definition 1.1.1 of [11] (with $h = 1, c = 0 > -1$ and any δ). The manifold is a product near the singularity. Our point of view is to transfer the geometric singularity into a singularity of a geometric differential operator on a manifold with boundary. The natural differential operators involving the conic metric are differential operators of regularly singular operators [1, 2] or Fuchs type operators [11].

Let P be an $SU(2)$-principal bundle over X. Let $\mathcal{A}(P)$ be the space of $SU(2)$-connections on the principal bundle P, and $\mathcal{G}(P)$ be the automorphism group of P. Note that $\mathcal{A}(P)|_N$ is an affine space modeled on $\Omega^1(N, \mathrm{ad}P|_N)$. We can naturally identify $\Omega^1(N, \mathrm{ad}P|_N)$ with $\Omega^1(Y, \mathrm{ad}P_Y) \oplus \Omega^0(Y, \mathrm{ad}P|_Y)$. More generally and in order to perform further calculations, we give an identification between p-forms on N and p-forms, $(p-1)$-forms on Y:

(1) $$f_p : \Omega^p(Y, \mathrm{ad}P_Y) \oplus \Omega^{p-1}(Y, \mathrm{ad}P|_Y) \to \Omega^p(N, \mathrm{ad}P|_N)$$

$$(w_p, w_{p-1}) \mapsto t^{p-3/2}(\pi^* w_p + t^{-1}\pi^* w_{p-1} \wedge dt),$$

where $\pi : N = (0, \varepsilon) \times Y \to Y$ is the projection on the second factor. Note that one has

$$\|f_p(w_p, w_{p-1})\|^2_{L^2(\Omega^p(N, \mathrm{ad}P|_N))}$$
$$= \int_0^\varepsilon \{\|w_p\|^2_{L^2(\Omega^p(Y, \mathrm{ad}P_Y))} + \|w_{p-1}\|^2_{L^2(\Omega^{p-1}(Y, \mathrm{ad}P|_Y))}\}dt.$$

With respect to the metric $g_N = dt^2 + t^2 g_Y$,

$$
\begin{aligned}
&*_{g_N} f_p(w_p, w_{p-1}) \\
&= *_{g_N} (t^{p-3/2}(\pi^* w_p + t^{-1}\pi^* w_{p-1} \wedge dt)) \\
&= t^{3/2-p}\pi^*(*_{g_Y} w_p) \wedge dt + t^{5/2-p}\pi^*((-1)^{4-p} *_{g_Y} w_{p-1}) \\
&= t^{(4-p)-3/2}(\pi^*((-1)^{4-p} *_{g_Y} w_{p-1}) + t^{-1}\pi^*(*_{g_Y} w_p) \wedge dt) \\
&= f_{4-p}((-1)^{4-p} *_{g_Y} w_{p-1}, *_{g_Y} w_p).
\end{aligned}
$$

Hence we obtain the Hodge-star operator $*_{g_N}$ acting on the identification (1):

$$
(2) \qquad *_{g_N} f_p(w_p, w_{p-1}) = f_{4-p}((-1)^{4-p} *_{g_Y} w_{p-1}, *_{g_Y} w_p).
$$

The restricted gauge group over N is denoted by $\mathcal{G}(P)|_N$. For any connection $A \in \mathcal{A}(P)|_N$, there is a (temporal) gauge transformation in $\mathcal{G}(P)|_N$ under which A does not contain the covariant derivative in t-direction, i.e., $A = a(t)$, where each $a(t)$ is a connection on Y and $t \in (0, \varepsilon)$. Note that the connection $a(t)$ may not be defined at $t = 0$ corresponding the cone point in X. The covariant derivative $d_A : \Omega^0(N, \mathrm{ad}P|N) \to \Omega^1(N, \mathrm{ad}P|_N)$ can be transformed over to

$$
d_A : \Omega^0(Y, \mathrm{ad}P|_Y) \to \Omega^1(Y, \mathrm{ad}P|_Y) \oplus \Omega^0(Y, \mathrm{ad}P|_Y).
$$

We need to calculate $d_A^* : \Omega^1(N, \mathrm{ad}P|_N) \to \Omega^0(N, \mathrm{ad}P|N)$ through the identification (1).

Lemma 2.1. *The operator d_A^* can be identified through (1) with the operator:*

$$
\Omega^1(Y, \mathrm{ad}P|_Y) \oplus \Omega^0(Y, \mathrm{ad}P|_Y) \to \Omega^0(Y, \mathrm{ad}P|_Y)
$$

$$
(\alpha, \phi) \mapsto \frac{\partial \phi}{\partial t} + t^{-1} d_a^{*g_Y} \alpha + \frac{3}{2} t^{-1} \phi.
$$

Proof. The proof is given by a direct calculation as follows.

$$
\begin{aligned}
d_A^* f_1(\alpha, \phi) &= (-1)^{(p-1)n+1} *_{g_N} d_A(*_{g_N} f_1(\alpha, \phi)) \\
&= (-1) *_{g_N} d_A(f_3(- *_{g_Y} \phi, *_{g_Y} \alpha)) \\
&= (-1) *_{g_N} d_A(t^{3/2}(-\pi^*(*_{g_Y} \phi) + t^{1/2}\pi^*(*_{g_Y} \alpha) \wedge dt)) \\
&= *_{g_N}(\frac{3}{2} t^{1/2}\pi^*(*_{g_Y} \phi) \wedge dt \\
&\qquad + t^{3/2}\pi^*(*_{g_Y} \frac{\partial \phi}{\partial t} \wedge dt - t^{1/2}\pi^*(d_a *_{g_Y} \alpha) \wedge dt) \\
&= *_{g_N} f_4(0, *_{g_Y} \frac{\partial \phi}{\partial t} - t^{-1} d_a *_{g_Y} \alpha + t^{-1}\frac{3}{2} *_{g_Y} \phi) \\
&= f_0(\frac{\partial \phi}{\partial t} + t^{-1} d_a^{*g_Y} \alpha + \frac{3}{2} t^{-1}\phi, 0).
\end{aligned}
$$

The identity follows from (1) and (2), since the connection $A = \{a_t\}_{t \in (0, \varepsilon)}$ has no dt-component. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

With respect to the conic metric g_N on N, one has

$$d_A^+ : \Omega^1(N, \mathrm{ad}P|_N) \to \Omega^2_+(N, \mathrm{ad}P|_N).$$

Through the identifications (1) and (2), we have the following result.

Lemma 2.2. *The operator d_A^+ can be identified with the operator*

$$\Omega^1(Y, \mathrm{ad}P|_Y) \oplus \Omega^0(Y, \mathrm{ad}P|_Y) \to \Omega^1(Y, \mathrm{ad}P|_Y)$$

$$(\alpha, \phi) \mapsto \frac{\partial \alpha}{\partial t} + t^{-1}(*_{g_Y} d_a \alpha + d_a \phi - \frac{1}{2}\alpha).$$

Proof. We calculate the corresponding d_A first.

$$d_A f_1(\alpha, \phi) = d_A(t^{-1/2}(\pi^* \alpha + t^{-1} \pi^* \phi \wedge dt))$$

$$= t^{-1/2} \pi^* d_a \alpha - \frac{1}{2} t^{-3/2} \pi^* \alpha \wedge dt$$

$$+ t^{-1/2} \pi^*(\frac{\partial \alpha}{\partial t}) \wedge dt + t^{-2/3} \pi^*(d_a \phi) \wedge dt$$

$$= t^{2-3/2}(t^{-1} \pi^*(d_a \alpha) + t^{-1} \pi^*(\frac{\partial \alpha}{\partial t} + t^{-1} d_a \phi - \frac{1}{2} t^{-1} \alpha) \wedge dt)$$

$$= f_2(t^{-1} d_a \alpha, \frac{\partial \alpha}{\partial t} + t^{-1} d_a \phi - \frac{1}{2} t^{-1} \alpha).$$

Now by the definition of $d_A^+ = \frac{1 + *_{g_N}}{2} d_A$ and the above equality, we have $d_A^+ f_1(\alpha, \phi) = \frac{1}{2} f_2(w_2, w_1)$ by (2), where

$$w_2 = t^{-1} d_a \alpha + *_{g_Y} \frac{\partial \alpha}{\partial t} + t^{-1} *_{g_Y} d_a \phi - \frac{1}{2} t^{-1} *_{g_Y} \alpha,$$
$$w_1 = \frac{\partial \alpha}{\partial t} + t^{-1} d_a \phi - \frac{1}{2} t^{-1} \alpha + t^{-1} *_{g_Y} d_a \alpha.$$

The components w_2 and w_1 are identified by the Hodge operator $*_{g_Y}$ on Y. Therefore d_A^+ can be identified with w_1 through (1) and (2). $\square$

Remark. As in the usual instanton theory on smooth 4-manifolds, we need to use d_A^* to build a slice of the configuration space and to get the elliptic operator $d_A^+ \oplus d_A^*$. We will follow the method used in [1, 2] to build a parametrix for the self-duality operator with conic singularity. For the singular operator $d_A^+ \oplus d_A^*$, the construction of parameterization can be constructed from the general method in [11].

Thus the self-duality operator $d_A^+ \oplus d_A^*$ restricted on N can be identified with

$$P_A : \Omega^1(Y, \mathrm{ad}P|_Y) \oplus \Omega^0(Y, \mathrm{ad}P|_Y) \to \Omega^1(Y, \mathrm{ad}P|_Y) \oplus \Omega^0(Y, \mathrm{ad}P|_Y)$$

by Lemma 2.1 and Lemma 2.2 through the L^2-isometry. So we have:

$$(3) \qquad P_A(\alpha, \phi) = \begin{pmatrix} \frac{\partial}{\partial t} + \frac{1}{t} *_Y d_a - \frac{1}{2} & t^{-1} d_a \\ \frac{1}{t} d_a^{*Y} & \frac{\partial}{\partial t} + \frac{3}{2} t^{-1} \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \phi \end{pmatrix}.$$

As an operator,

$$(4) \qquad P_A = \frac{\partial}{\partial t} + \frac{1}{t} \begin{pmatrix} *_Y d_a - \frac{1}{2} & d_a \\ d_a^{*Y} & \frac{3}{2} \end{pmatrix}.$$

Let us recall Definition 1.1.2 of [11] about operators of Fuchs type.

Definition 2.3. 1. Let E, F be Hermitian vector bundles over X. We denote by $\mathrm{Diff}^{\mu}(E, F)$ the set of differential operators of order $\leq \mu$ acting from sections in E to sections in F.

2. Denote by t the operator of multiplication by t and put $D = -t\frac{\partial}{\partial t}$. A differential operator of Fuchs type of order $\mu \in \mathbf{Z}_{+}$ and weight $\nu \geq 0$ is an operator $P \in \mathrm{Diff}^{\mu}(E, F)$ such that

$$P|_N = t^{-\nu} \sum_{k=0}^{\mu} B_k D^k,$$

with $B_k \in C^{\infty}((0, \varepsilon), \mathrm{Diff}^{\mu-k}(E|_N, F|_N)) \cap C([0, \varepsilon), \mathrm{Diff}^{\mu-k}(E|_N, F|_N))$.

3. P is of weak Fuchs type if $B_k \in C^{\infty}((0, \varepsilon), \mathrm{Diff}^{\mu-k}(E|_N, F|_N))$ only.

Lemma 2.4. *The operator* $d_A^+ \oplus d_A^* : \Omega^1(X, \mathrm{ad}P) \to \Omega^0 \oplus \Omega_+^2(X, \mathrm{ad}P)$ *is a differential operator of weak Fuchs type of order 1 and weight 1.*

Proof. Through the identification with the operator P_A on the conic neighborhood N, it is sufficient to note that P_A can be written as

$$P_A = t^{-1}\left(-D + \begin{pmatrix} *_Y d_a - \frac{1}{2} & d_a \\ d_a^{*_Y} & \frac{3}{2} \end{pmatrix}\right);$$

so in the notations of Definition 2.3,

$$B_1 = Id \quad \text{and} \quad B_0(t) = \begin{pmatrix} *_Y d_a - \frac{1}{2} & d_a \\ d_a^{*_Y} & \frac{3}{2} \end{pmatrix}$$

are both smooth for $t \in (0, \varepsilon)$. Hence $d_A^+ \oplus d_A^*$ is of weak Fuchs type. $\square$

We obtain weak Fuchs type only since $a(t)$ and therefore $B_0(t)$ are undefined at $t = 0$ in general.

Note that the operator d_A^+ alone cannot be a differential operator of Fuchs type and the techniques and estimates in [1, 11] cannot be applied. To remedy this, we need to include d_A^*, i.e., consider the self-duality operator as a whole.

3. A parametrix construction

In this section, we construct parametrices for our first order self-duality operator of Fuchs type near the boundary (the neighborhood of the cone singularity). The important feature of the construction is that the space of closed extensions is finite dimensional and closed extensions are Fredholm operators for the operator of Fuchs type on a compact manifold with conic singularities.

To get the parametrices, we need some kind of ellipticity for the self-duality operator. The operator is certainly elliptic in the interior of X. In addition some condition must be imposed near the tip $t \to 0$. We follow Lesch's approach [11] here. So we need now to strengthen our requirement for the connection $A = a(t)$ near $t = 0$:

Assumption. There is a smooth connection a_0 on P_Y such that $\sup_Y |a(t) - a_0| \le Ct^{1+\nu}$ for some number $\nu > 0$.

This assumption will be used throughout the rest of our paper. With this assumption, $B_0(0)$ exists and $P_A = t^{-1}(B_0 + D)$ is a first order differential operator of Fuchs type as in Definition 2.3. Moreover we can express our operator as

$$P_A = \frac{\partial}{\partial t} + t^{-1}P_0 + P_1,$$

where

$$P_0 = \begin{pmatrix} *_Y d_{a_0} - \frac{1}{2} & d_{a_0} \\ d_{a_0}^{*_Y} & \frac{3}{2} \end{pmatrix}, P_1 = \frac{1}{t}\begin{pmatrix} *_{g_Y}(a(t) - a_0) & a(t) - a_0 \\ *_{g_Y}(a_0 - a(t))*_{g_Y} & 0 \end{pmatrix}$$

are smooth operators away from $t = 0$ and continuous up to $t = 0$. Also note that P_1 is a compact zero-th order operator.

As for ellipticity near $t = 0$, we observe that

$$\sigma_X^{1,1}(P_A)(z) := \sum_{k=0}^{\mu} B_k(0)z^k = B_0(0) + z$$

is a parameter dependent elliptic family of differential operators with parameter $z \in i\mathbf{R}$. Hence P_A is an elliptic operator of Fuchs type in the sense of [11]. There is a general method in [11] to get the parametrix construction. We would like to work out explicitly for the self-duality operator with cone singularity in order to understand better on the behaviors from analytic point of view.

Digression. Existence of a limiting connection at the conic singularity.

The assumption above is satisfied in our intended applications that will include orbifold singularities, blowing down negative curves etc. To illustrate this and also more specifically the difference between a close conic metric and an open conic metric/cube metric that was alluded in the Introduction, we pause to digest the simple case of $\mathbf{R}^4$ with the standard Euclidean metric. It is well known that a charge 1, $SU(2)$-instanton, with center at 0, is

$$A = \operatorname{Im}\left\{\frac{\bar{x}dx}{1 + x^2}\right\},$$

where $x \in \mathbf{R}^4 = \mathbf{H}$ is viewed as quaternion. Its curvature is $F = \frac{dx \wedge d\bar{x}}{(1+|x|^2)^2}$ with density $|F| = (1 + |x|^2)^{-2}$. We view the origin as the trivial cone point with base S^3, and we can rewrite the connection in terms of the generalized polar coordinates (r, θ) as

$$A = \frac{r^2}{1 + r^2}\operatorname{Im}(\theta^{-1}d\theta),$$

where $\theta \in S^3$ is viewed as a unit quaternion. This means that $a(r) \to 0$ here as we approach the cone tip $r = 0$, i.e., the assumption above is satisfied. Note that A is not flat near 0: $|F|$ approaches 1 as $r \to 0$. This contrasts with the open cone at ∞ of $\mathbf{R}^4$, where A is asymptotically flat. (Incidentally this

suggests that some kind of gluing procedure is possible between close and open conical instantons). Now transform $\mathbf{R}^4\backslash\{0\}$ into the cylinder $(-\infty,\infty)\times S^3$ via the map $(r,\theta)\mapsto(\ln r,\theta)$. Of course, the model connection A is still not asymptotically flat near the end $t=\ln r\to-\infty$ under the equivalent metric $dr^2+r^2d\theta^2=e^{2t}(dt^2+d\theta^2)$. However under the tube metric $dt^2+d\theta^2$, which is conformally equivalent to the Euclidean metric, A is asymptotically flat near $-\infty$, since the curvature density under the tube metric is $|F|=\frac{4}{\cosh^2 t}$, which goes to 0 as $t\to-\infty$. This simple picture explains why we can have exponential decays for curvature of instantons under tube metrics but we can not do so under close conic metrics. Moreover it indicates that conformal metrics can lead to quite different geometry.

Let us return to P_A. The operator P_A, acting on $L^2((0,\varepsilon),H)$ with domain $C_0^\infty((0,\varepsilon),H_P)$, amounts to integration of a first order ordinary differential equation, where $H=L_k^2(\Omega^1(Y,\mathrm{ad}P|_Y)\oplus\Omega^0(Y,\mathrm{ad}P|_Y))$ and H_P is the common domain of the family of self-adjoint operators P_0+tP_1, $t\in(0,\varepsilon]$, in $L_{k+1}^2(\Omega^1(Y,\mathrm{ad}P|_Y)\oplus\Omega^0(Y,\mathrm{ad}P|_Y))$. Hence we have the following operator norm inequality:

$$(5) \qquad \|P_1(|P_0|+1)^{-1}\|+\|(P_0+1)^{-1}P_1\|\leq C,$$

where the estimate is uniformly in $(0,\varepsilon]$ from our assumption on a_t. This is, in fact, a *first order regular singular operator* P_A from [1, 2, 3, 4].

For $f\in L^2(0,\infty)$, define two integral operators

$$P_{0,s}f(t)=\int_0^t(\frac{x}{t})^s f(x)dx, \quad s>-\frac{1}{2},$$

$$P_{1,s}f(t)=\int_1^t(\frac{x}{t})^s f(x)dx, \quad s<\frac{1}{2}.$$

On $L^2(0,\infty)$, these two integral operators satisfy the identity

$$(6) \qquad (\frac{\partial}{\partial t}+t^{-1}s)P_{0,s}=(\frac{\partial}{\partial t}+t^{-1}s)P_{1,s}=Id.$$

Our parameterizations are constructed from the combinations of $P_{0,s}$ and $P_{1,s}$ near $t=0$.

Lemma 3.1. *For $f\in L^2(0,\infty)$ and t sufficiently small, the following estimates hold.*

(1) $|P_{0,s}f(t)|\leq\sqrt{\frac{t}{|2s+1|}}\cdot\|f\|_{L^2(0,t)}$ *for $s>-1/2$;*

$$|P_{1,s}f(t)|\leq\begin{cases}\sqrt{\frac{t}{|2s+1|}}\cdot\|f\|_{L^2} & s<-1/2\\ \sqrt{t|\ln t|}\|f\|_{L^2(0,\delta)}+\sqrt{t|\ln\delta|}\|f\|_{L^2} & s=-1/2,\ 0<\delta<1.\end{cases}$$

(2) *for $|s|<1/2$,*

$$|P_{1,s}f(t)+t^{-s}\int_0^1 x^s f(x)dx|\leq\sqrt{\frac{t}{2s+1}}\|f\|_{L^2}.$$

(3) *for $0 < \varepsilon \le 1$,*

$$\|P_{0,s}\| + \|P_{1,-s}\| \le C_1(s,\varepsilon)(|s|+1)^{-1}, \quad s > -\frac{1}{2},$$

$$\|P_{1,s}\| + \|P_{0,-s}\| \le C_2(s,\varepsilon)(|s|+1)^{-1}, \quad s < 1/2,$$

where $\lim_{\varepsilon \to 0} C_i(s,\varepsilon) = 0$, $i = 1,2$, *and uniformly for* $|s| \ge 2$.

Proof. (1) and (2) follow from direct calculations. (3) follows from $(P_{0,s}t^r)^* = -t^r P_{1,s}$, where $P_{0,s}t^r$ is bounded in $L^2(0,\varepsilon)$, and from the standard estimates for integral operators (see [1] Lemma 2.1 and Lemma 2.2 and [7, 12]). $\qquad\square$

The maximum extension $P_{A,\max}$ of P_A has domain

$$\mathcal{D}(P_{A,\max}) = \{u \in L^2(\Omega^1 \oplus \Omega^0(Y, \mathrm{ad}P|_Y))| P_A u \in L^2(\Omega^1 \oplus \Omega^0(Y, \mathrm{ad}P|_Y))\}.$$

A kind of *ideal Dirichlet boundary condition* for P_A is defined to be

$$\mathcal{D}^{iD}(P_A) = \{u \in \mathcal{D}(P_{A,\max})|\|u(t)\|_H = o(1) \quad \text{as } t \to 0\},$$

Denote by P_A^{iD} the restriction of $P_{A,\max}$ to the domain $\mathcal{D}^{iD}(P_A)$. The boundary parametrix Q^{iD} is given by

$$Q^{iD} = \oplus_{s \in \mathrm{Spec}P_0, s \ge 0} P_{0,s} \oplus \oplus_{s \in \mathrm{Spec}P_0, s < 0} P_{1,s},$$

with each term $P_{i,s}$ ($i = 0,1$) in the appropriate eigenspace of P_0.

Lemma 3.2. *If* $\psi \in C_0^\infty(-1,1)$, *then* ψQ^{iD} *maps* $L^2((0,1), H)$ *into* $\mathcal{D}^{iD}(P_A)$.

Proof. By Lemma 3.1 (1) and (2), $f(t) = \oplus_{s \in \mathrm{Spec}P_0} f_s(t)$ has an unique expression with $f_s(t)$ projection on the eigenspace of P_0 for the eigenvalue s.

$$\| \oplus_{s \ge 0} P_{0,s} f_s(t)\|_H^2 \le \|t \cdot \sum_{s \ge 0} \frac{s}{2s+1} \int_0^t |f_s(x)|^2 dx\|_H \le Ct \int_0^t \|f(t)\|_H^2 dt.$$

Similarly, we have

$$\| \oplus_{s < 0} P_{1,s} f_s(t)\|_H^2 \le C\Big(\sum_{-1/2 < s < 0} t^{-2s} \|f_s\|_{L^2}^2 + t|\ln t| \|f_{-1/2}\|_{L^2}^2$$

$$+ t \sum_{s < -1/2} \frac{|s|}{|2s+1|} \|f_s\|_{L^2}^2 \Big) = o(1)\|f\|_{L^2}^2,$$

as $t \to 0$. So $Q^{iD}f(t) = o(1)$ as $t \to 0$. One has the following:

$$P_A(\psi Q^{iD}f) = \psi P_A Q^{iD}f + \psi' Q^{iD}f$$

$$= \psi(\frac{\partial}{\partial t} + t^{-1}P_0 + P_1)(\oplus_{s \in \mathrm{Spec}P_0} f_s(t)) + \psi' Q^{iD}f$$

$$= \psi(\sum_{s \in \mathrm{Spec}P_0} (\frac{\partial}{\partial t} + t^{-1}P_0)f_s) + \psi P_1 Q^{iD}f + \psi' Q^{iD}f$$

$$= \psi f + \psi P_1 Q^{iD}f + \psi' Q^{iD}f,$$

where $f_s(t) = P_{0,s}f(t)$ if $s \geq 0$, and $P_{1,s}f(t)$ if $s < 0$. The last equality follows from (6). It now suffices to estimate

$$\|(|P_0| + 1)Q^{iD}f\|_H^2 = \sum_{s \geq 0}(|s| + 1)^2\|P_{0,s}f_s\|_H^2 + \sum_{s < 0}(|s| + 1)^2\|P_{1,s}f_s\|_H^2$$

$$\leq \sum_{s \geq 0}C_1(s, \varepsilon)^2\|f_s\|_{L^2}^2 + \sum_{s < 0}C_2(s, \varepsilon)^2\|f_s\|_{L^2}^2$$

$$\leq \max\{C_1(s, \varepsilon)^2, C_2(s, \varepsilon)^2\}\|f\|^2,$$

by Lemma 3.1 (3). Thus ψQ^{iD} maps into the correct domain. $\square$

Lemma 3.3. *If $u \in \mathcal{D}^{iD}(P_A)$ and $u(t) \equiv 0$ for $t \geq 1/2$, then $Q^{iD}P_A u = u + (Q^{iD}P_1)u$.*

Proof. Let $\{e_s\}_{s \in \operatorname{Spec}P_0}$ be an orthonormal basis in H with $P_0 e_s = s e_s$. Define $v(t) = (\frac{\partial}{\partial t} + t^{-1}P_0)u(t)$. So one has $v(t) = P_A u(t) - P_1 u(t)$. Thus $v_s(t) = \langle v(t), e_s \rangle$ can be calculated by

$$v_s(t) = \langle (\frac{\partial}{\partial t} + t^{-1}P_0)u(t), e_s \rangle$$

$$= \langle u'(t) + t^{-1}P_0 u(t), e_s \rangle$$

$$= u_s'(t) + t^{-1}s u_s(t), \quad s \in \operatorname{Spec}P_0.$$

So $v_s \in L^2(0, 1)$. Since $u_s(1) = 0$ for all $s \in \operatorname{Spec}P_0$, for $s < 1/2$,

$$P_{1,s}v_s(t) = P_{1,s}(u_s'(t) + t^{-1}s u_s(t))$$

$$= \int_1^t (\frac{x}{t})^s (u_s'(x) + x^{-1}s u_s(x))dx$$

$$= u_s(t),$$

from the integration by parts; for $s \geq 0$, we use $P_{0,s}$ on $v_s(t)$:

$$P_{0,s}v_s(t) = \int_0^t (\frac{x}{t})^s (u_s'(x) + x^{-1}s u_s(x))dx$$

$$= t^{-s}\lim_{\delta \to 0}\int_\delta^t x^s u_s'(x)dx + t^{-s}\int_0^t s x^{s-1}u_s(x)dx$$

$$= u_s(t) - t^{-s}\lim_{\delta \to 0}\delta^s u_s(\delta).$$

For $s \geq 0$ and $v \in L^1$, we have $|P_{0,s}v(t)| \leq \int_0^t |v(x)|dx = o(1)$, and $u_s(t) = o(1)$ as $u \in \mathcal{D}^{iD}(P_A)$ for $t \to 0$ sufficiently small. Hence $P_{0,s}v(t) - u_s(t) = o(1)(= -t^{-s}\lim_{\delta \to 0}\delta^s u_s(\delta))$ as $t \to 0$. Therefore $\lim_{\delta \to 0}\delta^s u_s(\delta) = 0$ for $s > 0$, i.e., $u_s(t) = P_{0,s}v_s(t)$ for $s \geq 0$. This completes the proof with the right domain. $\square$

Lemma 3.4. *There is a positive $\varepsilon \leq 1$ such that for $\phi, \psi \in C_0^\infty(-\varepsilon, \varepsilon)$ with $\psi\phi = \phi$ and $u \in \mathcal{D}^{iD}(P_A)$,*

$$\phi u = V\psi Q^{iD}P_A \phi u,$$

for some bounded operator $V \in L^2((0,1), H)$. *Moreover* $\|\phi(|P_0| + 1)u\|_H \leq C\|P_A\phi u\|_H$.

Proof. Choose $\chi \in C_0^\infty(-\varepsilon, \varepsilon)$ with $\chi\psi = \psi$. Since $\phi u \in \mathcal{D}^{iD}(P_A)$, we have $f = P_A(\phi u) \in \mathcal{D}^{iD}(P_A)$ with $f(t) \equiv 0$ for $t \geq 1/2$. By Lemma 3.3,

$$f = \phi u = Q^{iD} P_A f - Q^{iD} P_1 f = Q^{iD} P_A \phi u - Q^{iD} P_1 \phi u.$$

By the choices of smooth functions $\psi\phi = \phi, \chi\psi = \psi$ and $\chi\phi = \phi$, we get

$$\phi u = \psi Q^{iD}(\chi P_A f) - \psi Q^{iD} P_1 \chi \phi u$$

$$= \psi Q^{iD}(\chi P_A f) + (-\psi Q^{iD} P_1 \chi)(\psi Q^{iD}(\chi P_A f) - \psi Q^{iD} P_1 \chi \phi u)$$

$$= \sum_{j=0}^{n} (-\psi Q^{iD} P_1 \chi)^j (\psi Q^{iD}(\chi P_A f) + (-\psi Q^{iD} P_1 \chi \phi u)^{n+1}.$$

The operator norm has $\|\psi Q^{iD} P_1 \chi\|_{L^2((0,\varepsilon), H)} < 1/2$ for ε sufficiently small, and $V = \sum_{j \geq 0}(-\psi Q^{iD} P_1 \chi)^j$ is a bounded operator with $\phi u = V \psi Q^{iD} P_A \phi u$, i.e., $V \psi Q^{iD} P_A = Id$. $\square$

Theorem 3.5. (1) P_A^{iD} *is a closed operator, and*
(2) $P_A^{iD} : \mathcal{D}^{iD}(P_A) \to L^2(\Omega^1 \oplus \Omega^0(Y, \mathrm{ad}P|_Y))$ *is a Fredholm operator.*

Proof. (1) If $u_n \in \mathcal{D}^{iD}(P_A)$ with $u_n \to u$ and $P_A^{iD} u_n \to v$ in $L^2(\Omega^1 \oplus \Omega^0(Y, \mathrm{ad}P|_Y))$, then $u \in \mathcal{D}(P_{A,\max})$ and $v = P_A u$ in $L^2(\Omega^1 \oplus \Omega^0(Y, \mathrm{ad}P|_Y))$. For ε in Lemma 3.4, $\phi \in C_0^\infty(-\varepsilon, \varepsilon)$ with $\phi = 1$ near 0, we get

$$\phi u_n = V \psi Q^{iD}(P_A \phi u_n)$$

$$= V \psi Q^{iD}(\phi' u_n + \phi P_{A,\max} u_n)$$

$$\to V \psi Q^{iD}(\phi' u + \phi v)$$

$$= \phi u,$$

and $\|u(t)\|_H = o(1)$ as $t \to 0$. Thus P_A^{iD} is a closed operator.

(2) By (1), $\mathcal{D}^{iD}(P_A)$ is a Hilbert space under the graph norm. It suffices to show that P_A^{iD} has finite dimensional kernel and cokernel for the Fredholm property. This is amount to constructing right and left parametrices for P_A^{iD}.

Let $\phi, \tilde{\phi} \in C_0^\infty(-\varepsilon, \varepsilon)$ be such that $\phi = 1$ near 0 and $\tilde{\phi} = 1$ near $\mathrm{supp}\phi$, and $\psi, \tilde{\psi} \in C_0^\infty(M)$ such that $\phi + \psi = 1$ and $\tilde{\psi} = 1$ in a neighborhood of $\mathrm{supp}\psi$. Let $P_i : L^2(\Omega^1 \oplus \Omega^0(Y, \mathrm{ad}P|_Y)) \to L_{1,loc}^2(\Omega^1 \oplus \Omega^0(Y, \mathrm{ad}P|_Y))$ be an interior parametrix for P_A (see [5, 6]) with

(7) $$P_A \tilde{\psi} P_i \psi = \psi + R_i; \quad \tilde{\psi} P_i \psi P_A = \psi + L_i,$$

where R_i and L_i are compact in $L^2(\Omega^1 \oplus \Omega^0(Y, \mathrm{ad}P|_Y))$. Define $Q_A = \tilde{\phi} Q_A^{iD} \phi + \tilde{\psi} P_i \psi$. By Lemma 3.2, Q_A^{iD} maps into $\mathcal{D}^{iD}(P_A)$ and

$$P_A^{iD} Q_A = P_A^{iD}(\tilde{\phi} Q_A^{iD} \phi + \tilde{\psi} P_i \psi)$$

$$= Id + \tilde{\phi}' Q_A^{iD} \phi + \tilde{\phi} P_1 Q_A^{iD} \phi + R_i,$$

where $\tilde{\phi}'$ stands for $(\frac{\partial}{\partial t} + t^{-1}P_0)\tilde{\phi}$. If the supp$\phi$ is sufficiently small, then $\|\tilde{\phi}'Q_A^{iD}\phi + \tilde{\phi}P_1Q_A^{iD}\phi\| \le 1/4$ and $P_A^{iD}Q_A = Id + R + R_i$ with R_i compact operator and $\|R\| \le 1/4$. This implies that

$$P_A^{iD}Q_A(Id + R)^{-1} = Id + R_i(Id + R)^{-1},$$

and P_A^{iD} has finite-dimensional cokernel.

On the other hand, by Lemma 3.2, we have, similarly,

$$Q_A P_A^{iD} = (\tilde{\phi}Q_A^{iD}\phi + \tilde{\psi}P_i\psi)P_A^{iD}$$
$$= Id + (\tilde{\phi}Q_A^{iD}P_1\phi - \tilde{\phi}Q_A^{iD}\phi') + L_i,$$

for sufficiently small ε,

$$\|L\| = \|(\tilde{\phi}Q_A^{iD}P_1\phi - \tilde{\phi}Q_A^{iD}\phi')\| \le 1/4,$$

and L_i is a compact operator. Hence

$$(Id + L)^{-1}Q_A P_A^{iD} = Id + (Id + L)^{-1}L_i,$$

and P_A^{iD} has finite-dimensional kernel. Thus the desired result follows. □

The proof of Theorem 3.5 is standard in [1, 11]. The result can also follow from the general theory in [11] for the differential operators of Fuchs type.

Corollary 3.6. *Let $P(t) \in \mathcal{L}(H_s, H)$ be a smooth function of $t \in (0, 1]$. Then for $\chi \in C_0^\infty(-\varepsilon, \varepsilon)$ with ε sufficiently small and $\tilde{P}_A = P_A^{iD} + \chi tP(t)$ is a Fredholm operator on $\mathcal{D}^{iD}(P_A)$ with $\mathrm{Ind}\,(P_A^{iD}) = \mathrm{Ind}\,(\tilde{P}_A)$.*

Proof. Note that $\chi tP(t) = P(t)(|P_0|+1)^{-1}\chi t(|P_0|+1)$ is bounded from $\mathcal{D}^{iD}(P_A)$ to $L^2(\Omega^1 \oplus \Omega^0(Y, adP|_Y))$. The family $P_A^{iD}(s) = P_A^{iD} + s\chi tP(t)$ is a continuous function of $s \in [0, 1]$ with values in $\mathcal{L}(\mathcal{D}^{iD}(P_A), L^2(\Omega^1 \oplus \Omega^0(Y, adP|_Y)))$. By Theorem 3.5, with $\tilde{\psi}$ such that $\chi\tilde{\psi} = 0$, $P_A^{iD}(s)$ is a Fredholm operator for each $s \in [0, 1]$ and the index is constant. □

Therefore the index of P_A^{iD} can be conveniently computed by the index of the operator $\frac{\partial}{\partial t} + t^{-1}P_0$ without P_1 term. We will apply Corollary 3.6 in section 5 for the index calculation.

4. Closed Extensions for P_A

For the operator $P_A = \frac{\partial}{\partial t} + t^{-1}P_0 + P_1$, we have the formal adjoint

$$P_A^* = -\frac{\partial}{\partial t} + t^{-1}P_0^* + P_1^*,$$

where P_0 and P_1 are self-adjoint due to the self-adjointness of $P_0 + tP_1$.

Lemma 4.1. *If $u \in \mathcal{D}^{iD}(P_A)$ with $u \equiv 0$ for $t \ge 1$, then there exists a operator $Q_{A,\max} = \oplus_{s<1/2}P_{1,s} \oplus \oplus_{s\ge 1/2}P_{0,s}$ such that*

$$Q_{A,\max}P_A u = u + (Q_{A,\max}P_1)u,$$

with $Q_{A,\max}P_1$ is bounded in $L^2((0, 1), H)$ and $s \in \mathrm{Spec}P_0$.

Proof. Similar to Lemma 3.2, we take an orthonormal basis of eigensections $\{e_s\}_{s\in\text{Spec}P_0}$. Let $u, P_A u \in L^2$. Define

$$h(t) = u^{'}(t) + t^{-1}P_0 u(t) = P_A u(t) - P_1(u(t)).$$

By the uniform estimates, one gets

$$\begin{aligned}
\|P_1 e_s\|_H &= \|P_1(|P_0|+1)^{-1}(|P_0|+1)e_s\|_H \\
&\leq C\|(|P_0|+1)e_s\|_H \\
&\leq C(|s|+1).
\end{aligned}$$

Hence for each $s \in \text{Spec}P_0$, $h_s(t) = \langle h(t), e_s \rangle_H$ is given by

$$h_s(t) = \langle P_A u, e_s \rangle - \langle u, P_1^* e_s \rangle \in L^2$$

since both u and $P_A u$ are in L^2. By definition $u(t) \equiv 0$ for $t \geq 1$, $u_s(t) \equiv 0$ for $t \geq 1$, and for $s < 1/2$,

$$P_{1,s}h_s = P_{1,s}(u_s^{'} + t^{-1}su_s) = u_s \in L^2.$$

For $s \geq 1/2$, using integration by parts we have

$$t^{-s}\int_0^1 x^s h_s(x)dx = P_{0,s}h_s(t) - P_{1,s}h_s(t) \in L^2.$$

By the same argument in Lemma 3.3, $\int_0^1 x^s h_s(x)dx = 0$. Hence $P_{0,s}h_s = P_{1,s}h_s = u_s$ for $s \geq 1/2$. For all $s \in \text{Spec}P_0$, $Q_{A,\max}h = u$ gives the desired equality

$$Q_{A,\max}P_A u = u + (Q_{A,\max}P_1)u.$$

$\square$

Theorem 4.2. (1) $P_{A,\max}$ and $P_{A,\min}$ are Fredholm operators;

(2) *The extensions of $P_{A,\min}$ are all Fredholm operators and correspond to the subspaces of the finite-dimensional space $\mathcal{D}^{iD}(P_A)/\mathcal{D}^{iD}(P_A^*)$.*

Proof. Choose $\phi, \tilde{p}, \psi, \tilde{\psi}$ as in Theorem 3.5, and define the parametrix $Q = \tilde{\phi}Q_{A,\max}\phi + \tilde{\psi}P_i\psi$. Then, by Lemma 4.1, we have

$$\begin{aligned}
QP_{A,\max}u &= (\tilde{\phi}Q_{A,\max}\phi + \tilde{\psi}P_i\psi)P_{A,\max}u \\
&= u + (\tilde{\phi}Q_{A,\max}P_1\phi - \tilde{\phi}Q_{A,\max}\phi^{'})u + L_i u.
\end{aligned}$$

By the same estimates in Theorem 3.5, the operator $(\tilde{\phi}Q_{A,\max}P_1\phi - \tilde{\phi}Q_{A,\max}\phi^{'})$ has norm $\leq 1/4$ and L_i is compact. Therefore $QP_{A,\max}$ is Fredholm with finite nullity. So the operator $P_{A,\max}$ has finite nullity. Since it is an extension of the Fredholm operator P_A^{iD}, it also has closed range with finite codimension. Thus $P_{A,\max}$ is Fredholm.

The same argument applies to $P_{A,\max}^*$ and its adjoint $P_{A,\min}$ is Fredholm too. Note that $\mathcal{D}(P_{A,\min}) \subset \mathcal{D}(P_{A,\max})$ with finite codimension since both operators $P_{A,\max}$ and $P_{A,\min}$ are Fredholm. Thus

$$\text{Ind}\,(P_{A,\max}) = \text{Ind}\,(P_{A,\min}) + \dim\mathcal{D}^{iD}(P_{A,\max})/\mathcal{D}^{iD}(P_{A,\min}),$$

where $\mathcal{D}^{iD}(P_{A,\min}) = \mathcal{D}^{iD}(P_A^*)$ and $\mathcal{D}^{iD}(P_{A,\max}) = \mathcal{D}^{iD}(P_A)$.   □

The finite-dimensional space $\mathcal{D}^{iD}(P_{A,\max})/\mathcal{D}^{iD}(P_{A,\min})$ is isomorphic to $\oplus_{|s|<1/2} \ker(P_0 - s)$ by results in [1, 11]. Closed extensions of $P_{A,\min}$ are determined by the asymptotic behavior of the elements in their domains.

Lemma 4.3. *For $s \in \mathrm{Spec}P_0$ with $|s| < 1/2$, there are C^0-linear functionals c_s on $\mathcal{D}(P_{A,\max})$ such that for $t \in (0,1)$, $0 < \varepsilon < 1$ and $u \in \mathcal{D}(P_{A,\max})$,*

$$\|u(t) - \sum_{|s|<1/2} c_s(u)t^{-s}e_s\|_H \leq \varepsilon\sqrt{t|\ln t|} + C_{\varepsilon,u}\sqrt{t},$$

The same estimate holds for P_A^*, mutatis mutandis.

Proof. By the same argument of applying Lemma 3.4 as in Theorem 3.5, Lemma 3.2 implies that u is an element of $\mathcal{D}(P_{A,\max})$. Thus $\|(|P_0|+1)u(t)\|_H \in L^2((0,1))$ and

$$\int_0^1 \|(|P_0| + 1)u(t)\|_H^2 dt \leq C(\|P_A u\|^2 + \|u\|^2).$$

For $s > -1/2$, $h = u' + t^{-1}P_0 u$, there is a $c_s(u)$ with

$$u_s(t) = t^{-s}[c_s(u) + \int_0^t x^s h_s(x)dx]$$

from integration by parts. Since $\int_0^t (\frac{x}{t})^s h_s(x)dx \in L^2$ by Lemma 3.2, for $s \geq 1/2$, $c_s(u) = 0$. For $|s| < 1/2$ and let $t = 1$,

$$c_s(u) = u_s(1) - \int_0^1 x^s h_s(x)dx,$$

and $u_s(t) = t^{-s}u_s(1) + P_{1,s}h_s(t)$ for $s < 1/2$. By interior regularity, $u' \in L^2((1/2,1), H)$ and

$$|u_s(1)|^2 \leq t^s|u_s(t)|^2 + |P_{1,s}h_s(t)|^2,$$

$$\|u(1)\|_H^2 \leq C\int_{1/2}^1 (\|u(t)\|_H^2 + \|u'(t)\|_H^2)dt.$$

Hence $c_s(u)$ is continuous on $\mathcal{D}^{iD}(P_A)$, and

$$\sum_{s \leq -1/2} t^{-2s}|u_s(1)|^2 \leq C_u t.$$

Combining Lemma 3.1, for $0 < \delta < 1$,

$$\sum_{s \leq -1/2} |P_{1,s}h_s(t)|^2 \leq t[\|h\|_H^2 + 2|\ln \delta|\|h_{-1/2}\|_H^2 + 2|\ln t| \int_0^\delta |h_{-1/2}|^2 dx].$$

By Lemma 3.1 (1)

$$\sum_{s > -1/2} |u_s(t) - c_s(u)t^{-s}|^2 = \sum_{s > -1/2} |P_{0,s}h_s(t)|^2 \leq Ct\|h\|_H^2.$$

Thus the result follows. □

With Lemma 4.3, we can define various extensions of $P_{A,\min}$ and compute their indexes in terms of the extension determined by the subspaces of $\oplus_{|s|<1/2}\ker(P_0 - s)$. Let $P_{A,W}$ be an extension of $P_{A,\min}$ by restricting $P_{A,\max}$ to the domain $\mathcal{D}(P_{A,W}) = \{u \in \mathcal{D}(P_{A,\max})| \sum_{|s|<1/2} c_s(u)e_s \in W\}$, where W is a subspace of $\oplus_{|s|<1/2}\ker(P_0 - s)$. So $P_{A,W}$ is automatically closed since the functionals c_s are continuous on the domain $\mathcal{D}(P_{A,\max})$. Our main result of this section is the following theorem.

Theorem 4.4. *The operators $P_{A,W}$ give all closed extensions of $P_{A,\min}$ for all subspace $W \subset \oplus_{|s|<1/2}\ker(P_0 - s)$. $P^*_{A,W} = P_{A,W^\perp}$, where $W^\perp$ is the complement subspace of W in $\oplus_{|s|<1/2}\ker(P_0 - s)$. Furthermore,*

$$\text{Ind}\,(P_{A,W}) = \text{Ind}\,(P_{A,\min}) + \dim W.$$

Proof. For $u \in \mathcal{D}(P_{A,\max})$ and $v \in \mathcal{D}(P^*_{A,\max})$,

$$(8) \qquad \langle P_A u, v\rangle = \langle u, P^*_A v\rangle - \sum_{|s|<1/2} c_s(u)c^*_{-s}(v),$$

where c^*_s are functionals for $P^*_{A,\max}$ on the domain $\mathcal{D}(P^*_{A,\max})$. By Lemma 4.3, with $\phi \in C^\infty_0(-1,1)$ and $\phi \equiv 0$ near $t = 0$,

$$u(t) = \phi(t) \sum_{|s|<1/2} c_s(u)t^{-s}e_s + \tilde{u}(t),$$

$$v(t) = \phi(t) \sum_{|s|<1/2} c^*_{-s}(v)t^s e_s + \tilde{v}(t),$$

with $\|\tilde{u}(t)\| + \|\tilde{v}(t)\| \le C\sqrt{t|\ln t|}$ as $t \to 0$. Thus we obtain

$$\langle P_A u, v\rangle = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \langle P_A u, \phi v\rangle_H + \langle u, P^*_A(1-\phi)v\rangle$$

$$= \lim_{\varepsilon \to 0} -\langle u(\varepsilon), v(\varepsilon)\rangle_H + \langle u, P^*_A v\rangle$$

$$= \langle u, P^*_A v\rangle - \sum_{|s|<1/2} c_s(u)c^*_{-s}(v).$$

For $|s| < 1/2$ with any constant c_s, set $u(t) = \phi \sum_{|s|<1/2} c_s t^{-s}e_s$. Then one can compute the term

$$P_A u(t) = \phi'(t) \sum_{|s|<1/2} c_s t^{-s}e_s + \phi \sum_{|s|<1/2} c_s t^{-s}[P_1(|P_0|+1)^{-1}](|s|+1)e_s \in L^2.$$

Note that $P^*_{A,\max} = (P_{A,\min})^*$ and $(P^*_{A,\max})^* = P_{A,\min}$. $c_s(u) = 0$ for all s if and only if $u \in \mathcal{D}(P_{A,\min})$. Thus the equation (8) implies $u \in \mathcal{D}((P^*_{A,\max})^*)$, the converse follows from the continuous functional c_s.

For any extension P of $P_{A,\min}$, define $W = \{\sum_{|s|<1/2} c_s(u)e_s|u \in \mathcal{D}(P)\}$. Then it is clear that $P \subset P_{A,W}$. For $v \in \mathcal{D}(P_{A,W})$, there is an element $u \in \mathcal{D}(P)$ such that $c_s(v - u) = 0$ for all s by definition. So $u - v \in \mathcal{D}(P_{A,\min}) \subset \mathcal{D}(P)$ by

the above necessary and sufficient condition, therefore $P = P_{A,W}$. The index formula for $P_{A,W}$ is clear from Theorem 4.2 and $\mathcal{D}(P_{A,W})/\mathcal{D}(P_{A,\min}) \cong W$. $\square$

Note that for $u = (\alpha, \phi)$, $P_0 u = su$ is equivalent to the system of equations:

$$*_Y d_{a_0}\alpha + d_{a_0}\phi = (s + 1/2)\alpha$$
$$d_{a_0}^{*_Y}\alpha = (s - 3/2)\phi$$

Thus, by applying $d_{a_0}^{*_Y}$ to the first equation, one gets

$$\Delta^0 \phi = d_{a_0}^{*_Y} d_{a_0}\phi = (s + 1/2)d_{a_0}^{*_Y}\alpha = (s + 1/2)(s - 3/2)\phi.$$

Let $H_{\lambda(s)}^0 = \{\phi \in \Omega^0(Y, \mathrm{ad}P|_Y)|\Delta^0\phi = \lambda(s)\phi\}$. Hence for the nonnegative operator Δ^0 the eigenvalues $\lambda(s) = (s+1/2)(s-3/2) \geq 0$. So $s \in (-\infty, -1/2] \cup [3/2, \infty)$ if and only if $\lambda(s) \geq 0$. For $-1/2 < s < 3/2$, $H_{\lambda(s)}^0 = \{0\}$.

Corollary 4.5. *Let $H_{\lambda,cc}^1 = \{\alpha \in \Omega^1(Y, \mathrm{ad}P|_Y)|d_{a_0}^{*_Y}\alpha = 0, \Delta^1 = \lambda\alpha\}$. Then we have*

$$\oplus_{|s|<1/2} \ker(P_0 - s) \cong \oplus_{|s|<1/2} H_{(s+1/2)^2,cc}^1.$$

In particular, if $\oplus_{|s|<1/2} H_{(s+1/2)^2,cc}^1 = \{0\}$, then there is a unique closed extension $\overline{P}_A$ of P_A, where Δ^i is the Laplacian operator on $\Omega^i(Y, \mathrm{ad}P|_Y)$ with the twisted connection a_0 $(i = 0, 1)$.

Proof. It suffices to see $\oplus_{|s|<1/2} \ker(P_0 - s) = \{0\}$. For $|s| < 1/2$, one must have $\phi \equiv 0$ since $\lambda(s) < 0$ from the above discussion. Thus the equations reduce to

$$*_Y d_{a_0}\alpha = (s + 1/2)\alpha, \quad d_{a_0}^{*_Y}\alpha = 0.$$

Thus the space $\oplus_{|s|<1/2} \ker(P_0 - s)$ can be reduced to $\{(\alpha, \phi) : \phi = 0, \alpha \in H_{(s+1/2)^2,cc}^1\}$. The results follow. $\square$

5. The index calculation for P_A

By Corollary 3.6, the index of P_A with respect to a closed extension in §4 can be reduced to the index of $\frac{\partial}{\partial t} + t^{-1}P_0$ with respect to the closed extension. By Theorem 4.4, we can just calculate the index of P_A^{iD} (the restriction of $P_{A,\max}$ to the ideal boundary Dirichlet domain $\mathcal{D}^{iD}(P_A)$) with $P_1 \equiv 0$.

Note that P_A^{iD} is closed and $\Delta_A^+ = (P_A^{iD})^* P_A^{iD}$, $\Delta_A^- = P_A^{iD}(P_A^{iD})^*$ are non-negative, self-adjoint. By a straightforward calculation, one gets

$$\Delta_A^+ = -\frac{\partial^2}{\partial t^2} + t^{-2}(P_0^2 + P_0),$$

$$\Delta_A^- = -\frac{\partial^2}{\partial t^2} + t^{-2}(P_0^2 - P_0).$$

By the completing square, we have $P_0^2 \pm P_0 \geq 1/4$. Any nonzero eigenvalue of Δ_A^+ and Δ_A^- coincide with same multiplicity, i.e., $u \mapsto P_A^{iD}u$ and $v \mapsto (P_A^{iD})^*v$ are injective between the corresponding eigenspaces. The idea is to show that $(\Delta_A^\pm + \lambda)^{-m}$ are trace class and then

$$\mathrm{tr}\,(\Delta_A^+ + \lambda)^{-m} - \mathrm{tr}\,(\Delta_A^- + \lambda)^{-m} = \lambda^{-m}Ind(P_A^{iD})$$

for appropriate m, since all terms approach to zero as $\lambda \to 0$ except the term λ^{-m}.

Let Δ_A^+ be the operator in $L^2(R_+, H)$ with the boundary condition $u(t) = o(1)$ and $P_A^{iD}u = \frac{\partial u}{\partial t} + t^{-1}P_0u = O(1)$. Similarly for Δ_A^- with $u(t) = o(1)$ and $(P_A^{iD})^*u = -\frac{\partial u}{\partial t} + t^{-1}P_0u = o(1)$. By the standard result in [12], the resolvent for $\Delta_A^\pm$ is obtained by

$$(\Delta_A^\pm + \lambda)^{-1} = \oplus_{s \in \mathrm{Spec}\,P_0}(\Delta_s^\pm + \lambda)^{-1} \otimes \pi_s,$$

where $\Delta_s^\pm = -\frac{\partial^2}{\partial t^2} + t^{-2}(s^2 \pm s)$ and π_s is the projection on the s-eigenspace of P_0. Note that

$$P_0^2 + P_0 = \begin{pmatrix} \Delta^1 & 0 \\ 0 & \Delta^0 \end{pmatrix} + 2\begin{pmatrix} 0 & d_{a_0} \\ d_{a_0}^{*Y} & 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{15}{4} \end{pmatrix},$$

$$P_0^2 - P_0 = \begin{pmatrix} \Delta^1 & 0 \\ 0 & \Delta^0 \end{pmatrix} - 2\begin{pmatrix} *_Y d_{a_0} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix},$$

where Δ^i is the Laplacian operator on $\Omega^i(Y, \mathrm{ad}P|_Y)$ with the twisted connection a_0.

Let us first compute the resolvent kernel of $\Delta_s^+ + z^2$. Let u_1 and u_2 be the independent solutions, $0 < t < \infty$,

$$\begin{cases} (-\frac{\partial^2}{\partial t^2} + t^{-2}(s^2 + s) + z^2)u & = 0 \\ u'(t) + t^{-1}P_0u(t) = O(1) & u(t) = o(1) \end{cases}$$

By Theorem 16 of Chapter XIII of [7] in page 1329 (or §8.4 of [12]), the resolvent is given by

$$(\Delta_s^+ + z^2)^{-1}f(t) = R(z, s)f = \int_0^\infty f(x)K(t, x; z)dx$$

for $f \in L_2(0, \infty)$, and the kernel is given by

$$K(t, x; z) = \begin{cases} \frac{u_1(t,z)u_2(x,z)}{W(u_1, u_2)} & x < t \\ \frac{u_1(x,z)u_2(t,z)}{W(u_1, u_2)} & x > t, \end{cases}$$

where $W(u_1, u_2) = u_1'u_2 - u_1u_2'$ is the Wronskian of u_1 and u_2. By substituting $u_i = t^{1/2}v_i$, so v_1, v_2 are fundamental solutions of

$$(9) \qquad\qquad -t^2v'' - tv' + ((s + 1/2)^2 + (tz)^2)v = 0.$$

Hence the general solution can be expressed $u = c_1t^{1/2}v_1 + c_2t^{1/2}v_2$. Theorem 16 of [7] shows that there is exactly one solution $\psi(t, z) = t^{1/2}v_2(tz)$ satisfying the condition at $t = \infty$. Unless $-1/2 < s < 0$, $\phi(t, z) = t^{1/2}v_1(tz)$ is the unique solution satisfying the condition as $t \to 0$. Since $W(\phi, \psi) = tW(v_1, v_2)$

and v_1, v_2 are solutions of (9), one gets

$$
\begin{aligned}
tv_1' v_2 &= (-t^2 v_1'' + ((s+1/2)^2 + (tz)^2)v_1)v_2 \\
&= -t^2 v_1'' v_2 + v_1((s+1/2)^2 + (tz)^2))v_2 \\
&= -t^2 v_1'' v_2 + v_1(t^2 v_2'' + tv_2').
\end{aligned}
$$

Hence $v_1' v_2 - v_1 v_2' = -t(v_1' v_2 - v_1 v_2')'$ and $v_1' v_2 - v_1 v_2' = 1/t$. So $W(\phi, \psi) = 1$ and the kernel is

$$
K(t, x; z) = \begin{cases} \frac{\phi(t,z)\psi(x,z)}{W(\phi,\psi)} \\ \frac{\phi(x,z)\psi(t,z)}{W(\phi,\psi)} \end{cases} = \begin{cases} (tx)^{1/2} v_1(tz) v_2(xz) & x < t \\ (tx)^{1/2} v_1(xz) v_2(tz) & x > t \end{cases}
$$

The solutions v_1 and v_2 are depending on s and z. Let $\nu_\pm = |s \pm 1/2|$. If $\operatorname{Im} z^2 \neq 0, t \leq x$,, the Kernel $K^+(t, x; z)$ of $(\Delta_s^+ + z^2)^{-1}$ can be expressed as

$$
\begin{cases} (tx)^{1/2} v_{1,\nu_+}(tz) v_{2,\nu_+}(xz) & \text{if } s \leq -1/2 \text{ or } s \geq 0 \\ (tx)^{1/2} v_{1,-\nu_+}(tz) v_{2,\nu_+}(xz) & \text{if } -1/2 < s < 0; \end{cases}
$$

Similarly, the kernel $K^-(t, x; z)$ of $(\Delta_s^- + z^2)^{-1}$ is

$$
\begin{cases} (tx)^{1/2} v_{1,\nu_-}(tz) v_{2,\nu_-}(xz) & \text{if } s < 0 \text{ or } s \geq 1/2 \\ (tx)^{1/2} v_{1,-\nu_-}(tz) v_{2,\nu_-}(xz) & \text{if } 0 \leq s < 1/2; \end{cases}
$$

Let $P_i^\pm$ be the standard pseudo-differential parametrix for $(\Delta_s^\pm + \lambda)^{-m}$ in the interior away from $t = 0$. If $\phi, \psi \in C^\infty$ vanishing near $t = 0$ and $\psi \equiv 1$ near $\operatorname{supp}\phi$, then

$$
(\Delta_s^\pm + \lambda)^m \psi P_i^\pm \phi = \phi - R_i^\pm,
$$

with $\|R_i\|_{\mathrm{tr}} = O(\lambda^{-k})$, where k can be arbitrarily large. Choose $\chi \in C_0^\infty(-1, x_0)$ with $x_0 < \varepsilon$ and $\chi \equiv 1$ near $t = 0$ and $\tilde{\chi} \in C_0^\infty(-1, \varepsilon)$ with $\tilde{\chi} \equiv 1$ for $0 \leq t \leq t_0$ with $t_0 > x_0$. Let $\chi_i = 1 - \chi$ and $\tilde{\chi}_i \in C_0^\infty(X)$ be vanishing near $t = 0$ with $\tilde{\chi}_i \equiv 1$ near $\operatorname{supp}\chi_i$. Define

$$
P^\pm = \tilde{\chi}(P_A + \lambda)^{-m}\chi + \tilde{\chi}_i P_i^\pm \chi_i.
$$

Then $(\Delta_A^\pm + \lambda)P^\pm = Id - R^\pm$, where

$$
R^\pm = R_i^\pm + \sum_{j>0} c_{\alpha\beta\gamma, jk} \chi^{(j)} t^\alpha \frac{\partial^\beta}{\partial t^\beta} P_0^\gamma (P_A + \lambda)^{-k}\chi.
$$

By Lemma 4.2 of [1], $\|R^\pm\|_{\mathrm{tr}} \leq C\lambda^{-k}$. Hence for large λ, $(\Delta_A^\pm + \lambda)^{-m} = P^\pm(\sum_{j\geq 0}(R^\pm)^j)$ with $\|P \sum_{j\geq 1}(R^\pm)^j\|_{\mathrm{tr}} \leq C\lambda^{-k}$. The asymptotic of $\operatorname{tr}(\Delta_A^\pm + \lambda)^{-m}$ from $P^\pm$ is required for the calculation. Note that

$$
(\Delta_A^\pm + \lambda)^{-m} = \frac{1}{(m-1)!}(-\frac{1}{2z}\frac{\partial}{\partial z})^{m-1}((\Delta_A^\pm + \lambda)^{-1}.
$$

Substitute $\xi = tz$ and define formally,

$$\sigma_+(t,\xi) = \frac{t^{2m-1}}{(m-1)!} (-\frac{1}{\xi}\frac{\partial}{\partial\xi})^{(m-1)} \Big[\sum_{s \le -1/2} + \sum_{s \ge 0} v_{1,\nu_+}(\xi) v_{2,\nu_+}(\xi)$$

$$+ \sum_{-1/2 < s < 0} v_{1,-\nu_+}(\xi) v_{2,\nu_+}(\xi) \Big],$$

$$\sigma_-(t,\xi) = \frac{t^{2m-1}}{(m-1)!} (-\frac{1}{\xi}\frac{\partial}{\partial\xi})^{(m-1)} \Big[\sum_{s \ge 1/2} + \sum_{s < 0} v_{1,\nu_-}(\xi) v_{2,\nu_-}(\xi)$$

$$+ \sum_{0 \le s < 1/2} v_{1,-\nu_-}(\xi) v_{2,\nu_-}(\xi) \Big]$$

following [1, 2]. Applying the same method in [1, 2], one obtains

$$(10) \quad \operatorname{Ind} P_A^{iD} = \int_X \chi_i \omega(P_A) + \int_0^\infty \frac{\xi^{2m-1}}{(2m-1)!} [\sigma_+^{(2m-1)}(0,\xi) - \sigma_-^{(2m-1)}(0,\xi)] d\xi,$$

where $\omega(P_A)$ is the local index form of the self-duality operator P_A on X and the second integral is given by

$$(11) \qquad -\frac{1}{2}(\eta_{P_0} + \dim \ker P_0) + \sum_{k \ge 1} \alpha_k \operatorname{Res}_1 \eta_{P_0}(2k),$$

where $\alpha_k = \frac{2^{-2k}}{2k(2k+1)}$ and the eta function of P_0 is defined by

$$\eta_{P_0}(z) = \sum_{s \in \operatorname{Spec} P_0 \setminus \{0\}} |s|^{-z} \operatorname{sgn} s,$$

in which $\eta_{P_0}(z)$ is meromorphic in the whole plane with possible simple poles but holomorphic at $z = 0$, and the η-invariant of P_0, $\eta_{P_0} = \eta_{P_0}(0)$, is well-defined.

With $P_1 \equiv 0$ for $t \le \varepsilon$, the index formula of P_A is given by

$$(12) \qquad \begin{aligned} \operatorname{Ind} P_A^{iD} = &-\frac{1}{2}(\eta_{P_0} + \dim \ker P_0) + \sum_{k \ge 1} \alpha_k \operatorname{Res}_1 \eta_{P_0}(2k) \\ &+ \int_{X \setminus \{t \le \varepsilon\}} \frac{1}{8\pi^2} \operatorname{Tr}(F_A \wedge F_A) \end{aligned}$$

following from (10) and (11). Define $\int_X \tilde{c}_2(\operatorname{ad} P)$ to be the limit

$$\lim_{\varepsilon \to 0} \int_{t > \varepsilon} c_2(\operatorname{ad} P) = \lim_{\varepsilon \to 0} \int_{t > \varepsilon} \frac{1}{8\pi^2} \operatorname{Tr}(F_A \wedge F_A).$$

Theorem 5.1. *With the conic metric $dt^2 + t^2 g_Y$ at a point on X, the self-duality operator P_A^{iD} is Fredholm with index*

$$\int_X \tilde{c}_2(\operatorname{ad} P) - \frac{1}{2}(\eta_{P_0} + m_1) + \sum_{k \ge 1} \frac{2^{-2k}}{2k(2k+1)} \operatorname{Res}_1 \eta_{P_0}(2k),$$

where $\eta_{P_0}(z)$ is the eta function of the operator P_0 and $m_1 = \dim H^1_{1/4,cc}$ is the multiplicity of eigenvalue $1/4$ of the Laplacian operator Δ^1 on the coclosed $\Omega^1(Y, \mathrm{ad}P|_Y)$ with the twisted connection a_0.

Proof. $\ker P_0$ can be identified with $H^1_{1/4,cc}$ by Corollary 4.5. The result follows from the above discussion and the method in [1, 2] as well as (12). $\quad\square$

Corollary 5.2. *The closed extension $P_{A,W}$ of P_A is Fredholm with index*

$$\int_X \tilde{c}_2(\mathrm{ad}P) - \frac{1}{2}(\eta_{P_0} + m_1) + \sum_{k \geq 1} \frac{2^{-2k}}{2k(2k+1)} \mathrm{Res}_1 \eta_{P_0}(2k)$$

$$+ \dim W - \sum_{-1/2 < s < 0} \dim \ker(P_0 - s).$$

Proof. Let $W \subset \oplus_{|s|<1/2} \ker(P_0 - s)$ be a subspace with

$$W^- = \oplus_{-1/2<s<0} \ker(P_0 - s), \quad W^+ = \oplus_{1/2>s\geq0} \ker(P_0 - s).$$

Note that $\mathcal{D}(P_A^{iD}) = \mathcal{D}(P_{A,\min}) \oplus W^-$. By Theorem 4.4, $P_A^{iD} = P_{A,W^-}$ and

$$\mathrm{Ind}P_{A,W} = \mathrm{Ind}P_{A,\min} + \dim W = \mathrm{Ind}P_A^{iD} + \dim W - \dim W^-.$$

$\quad\square$

Let $E_s(P)$ be the eigenspace of the operator P with eigenvalue s. Then we have the following identification ($s \neq -1/2$)

$$(13) \qquad E_s(P_0) = E^1_{s+1/2,cc}(*_Y d_{a_0}) \oplus E^0_{\lambda(s)}(\Delta^0),$$

where $E^1_{t,cc}(*_Y d_{a_0}) = \{\alpha_1 \in \Omega^1 : *_Y d_{a_0} \alpha_1 = t\alpha_1, d^{*Y}_{a_0} \alpha_1 = 0\}$. For $u = (\alpha, \phi) \in E_s(P_0)$, define a map

$$F(\alpha, \phi) = (\alpha - (s+1/2)^{-1} d_{a_0}\phi, \phi).$$

The map F is bijective since the equation $P_0 u = su$ is equivalent to $\Delta^0 \phi = \lambda(s)\phi$ and

$$*_Y d_{a_0}(\alpha - (s+1/2)^{-1} d_{a_0}\phi) = (s+1/2)(\alpha - (s+1/2)^{-1} d_{a_0}\phi),$$
$$d^{*Y}(\alpha - (s+1/2)^{-1} d\phi) = 0.$$

Note that for $s = -1/2$, $E_{-1/2}(P_0) = H^1(Y, \mathrm{ad}a_0) \oplus H^0(Y, \mathrm{ad}a_0)$ and $E_{3/2}(P_0) = E^1_{2,cc}(*_Y d_{a_0}) \oplus H^0(Y, \mathrm{ad}a_0)$. For $-1/2 < s < 3/2$, $E_s(P_0) \cong H^1_{s+1/2,cc}(*_Y d_{a_0})$. Let $m^1_t = \dim E^1_{t,cc}(*_Y d_{a_0})$ and $m^0_t = \dim E^0_t(\Delta^0)$. Using the above discussion,

one obtains

$$\eta_{P_0}(z) = \sum_{s \in \mathrm{Spec} P_0 \setminus \{0\}} \mathrm{sgn} s |s|^{-z}$$

$$= -\sum_{s<0} + \sum_{s>0} |s|^{-z}$$

$$= -(h^1 + h^0)2^z + h^0(\frac{3}{2})^{-z} - \sum_{s<0, s \neq -1/2} m^1_{s+1/2}|s|^{-z}$$

$$+ \sum_{s>0} m^1_{s+1/2}|s|^{-z} - \sum_{s<-1/2} m^0_{\lambda(s)}|s|^{-z} + \sum_{s>3/2} m^0_{\lambda(s)}|s|^{-z}$$

$$= -(h^1 + h^0)2^z + h^0(\frac{2}{3})^z - 2\sum_{-1/2<s<0} m^1_{s+1/2}|s|^{-z}$$

$$+ \overline{\eta}_{*_Y d_{a_0}}(z) + \overline{\eta}_{\Delta^0}(z),$$

where $h^i = \dim \ker \Delta^i$ and

$$\overline{\eta}_{*_Y d_{a_0}}(z) = \sum_{\lambda \in \mathrm{Spec}(*_Y d_{a_0}) \setminus \{0\}} \mathrm{sgn}\lambda |\lambda - \frac{1}{2}|^{-z},$$

$$\overline{\eta}_{\Delta^0}(z) = \sum_{\lambda \in \mathrm{Spec}\Delta^0 \setminus \{0\}} m^0_{\lambda}(|\frac{1}{2} + \sqrt{1 + \lambda}|^{-z} - |\frac{1}{2} - \sqrt{1 + \lambda}|^{-z})$$

from the equation $\lambda(s) = (s + 1/2)(s - 3/2)$. The more explicit calculation of the index will be discussed elsewhere.

Remark. Our eventual goal will be the study of gauge theory type of topological invariants for the conic 4-manifolds. Up to now, we focus on the analytic aspects of the linear theory. Deforming the metrics $dt^2 + (t^{1-r})^2 g_Y$ with $0 \leq r \leq 1$, the operator $P_A(r)$ changes from $P_A(0)$ which we have discussed in this paper to $P_A(1) = \frac{\partial}{\partial t} + \begin{pmatrix} *_Y d_a & d_a \\ d_a^{*Y} & 0 \end{pmatrix}$ under the product metric. The closed extension $P_{A,W^-}(0)$ of $P_A(0)$ changes to $P_{A,W^-}(1)$ with the corresponding domain $\{u = \sum_s u_s(t) : s \in \mathrm{Spec}(P_0(1)), s < 0\}$ for $P_0(1) = \begin{pmatrix} *_Y d_a & d_a \\ d_a^{*Y} & 0 \end{pmatrix}$. This is precisely the Atiyah-Patodi-Singer global boundary condition for 4-manifold X with a boundary Y. The index formula is given by $\int_X \tilde{c}_2(\mathrm{ad}P) - \frac{1}{2}(\eta_{P_0(1)} + \dim \ker P_0(1))$ from $\mathrm{tr}(P_A^*(1)P_A(1)) - tr(P_A(1)P_A^*(1))$. The last term $\sum_{k\geq 1} \frac{2^{-2k}}{2k(2k+1)} \mathrm{Res}_1 \eta_{P_0}(2k)$ in Theorem 5.1 vanishes since the $m = -1$ case has non-residue terms in the calculation.

References

[1] J. Brüning and R. Seeley, *An Index theorem for first order regular singular operators*, American J. Math. **110** (1988), no. 4, 650–711.

[2] ――――, *Regular singular asymptotics*, Adv. Math. **58** (1985), no. 2, 133–148.

[3] J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Proc. Symp. Pure. Math. **36** (1980), 91–146.

[4] ――――, *Spectral geometry of singular Riemannian spaces*, J. Differential Geom. **18** (1983), no. 4, 575–657.

[5] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-manifolds, Oxford Mathematical Monographs*, Oxford University Press, Oxford, 1990.

[6] S. Donaldson and D. Sullivan, *Quasiconformal 4-manifolds*, Acta Math. **163** (1989), no. 3-4, 181–252.

[7] N. Dunford and J. Schwartz, *Linear operators Part II : Sepctral theory, Sefl adjoint operators in Hilbert space*, Interscience, New York 1963.

[8] J. Etnyre and K. Honda, *On symplectic cobordisms*, Math. Ann. **323** (2002), no. 1, 31–39.

[9] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, Adv. Theor. Math. Phys. **3** (1999), no. 5, 1415–1443.

[10] P. B. Kronheimer and T. S. Mrowka, *Gauge theory for embedded surfaces, I, II*, Topology **32** (1993), no. 4, 773–826 and **34** (1995), no. 1, 37–97.

[11] M. Lesch, *Operators of Fuchs Type, Conical Singularities, and Asymptotic Methods*, Teubner-Texte zur Mathematik 136, Stuttgart, Leipzig 1997.

[12] J. Weidmann, *Linear Operators in Hilbert Spaces*, GTM **68**, Springer-Verlag 1980.

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