J. Korean Math. Soc. **44** (2007), No. 2, pp. 261–273

GENERALIZED FRÉCHET-URYSOHN SPACES

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ABSTRACT. In this paper, we introduce some new properties of a topological space which are respectively generalizations of Fréchet-Urysohn property. We show that countably AP property is a sufficient condition for a space being countable tightness, sequential, weakly first countable and symmetrizable to be ACP, Fréchet-Urysohn, first countable and semi-metrizable, respectively. We also prove that countable compactness is a sufficient condition for a countably AP space to be countably Fréchet-Urysohn. We then show that a countably compact space satisfying one of the properties mentioned here is sequentially compact. And we show that a countably compact and countably AP space is maximal countably compact if and only if it is Fréchet-Urysohn. We finally obtain a sufficient condition for the ACP closure operator $[\cdot]_{ACP}$ to be a Kuratowski topological closure operator and related results.

1. Introduction and preliminaries

All spaces are assumed to be Hausdorff. Our terminology is standard and follows [2] and [5]. Let X be a topological space and let c denote the closure operator on the space X. Let $\mathbb{N}$ denote the set of all natural numbers and $(x_n|n \in \mathbb{N})$(briefly (x_n)) a sequence of points of a set. The following functions $[\cdot]_{seq}$, $[\cdot]_{AP}$, and $[\cdot]_{ACP}$ of the power set $\mathcal{P}(X)$ of X to $\mathcal{P}(X)$ itself defined by for each subset A of X,

$[A]_{seq} = \{x \in X : (x_n)$ converges to x in X for some sequence (x_n) of points of $A\}$,

$[A]_{AP} = A \cup \{x \in c(A) - A : c(F) = F \cup \{x\}$ for some subset F of $A\}$, and

$[A]_{ACP} = A \cup \{x \in c(A) - A : c(F) = F \cup \{x\}$ for some countable subset F of $A\}$ are called the sequential closure operator on X [2], the AP closure operator on X [15], and the ACP closure operator on X, respectively. It is well known that the sequential closure operator $[\cdot]_{seq}$ satisfies the Kuratowski topological closure axioms except for idempotency in general (see [2]). We see easily that

Received July 22, 2005.

2000 *Mathematics Subject Classification.* 54A20, 54D20, 54D55, 54E25.

Key words and phrases. Fréchet-Urysohn, sequential, countably Fréchet-Urysohn, countable tightness, AP, countably AP, WAP, ACP, WACP, countably compact, and sequentially compact.

This paper was supported by Pusan National University Research Grant.

for each subset A of X,

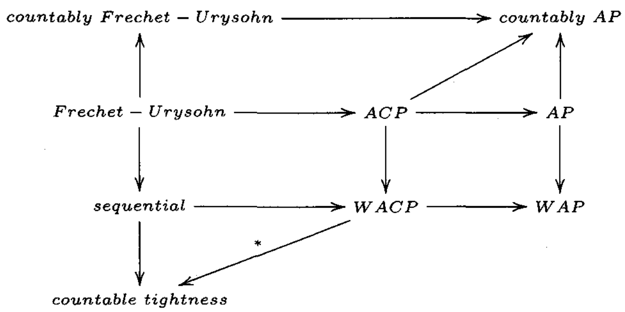
$$A \subset [A]_{seq} \subset [A]_{ACP} \subset [A]_{AP} \subset c(A),$$

for each countable subset A of X, $[A]_{AP} = [A]_{ACP}$, and $[\cdot]_{AP}$ and $[\cdot]_{ACP}$ do not satisfy the Kuratowski topological closure axioms in general.

Let us recall some properties and introduce new three properties of a topological space X.

(1) *Fréchet-Urysohn* [2] (also called Fréchet [6]) : for each subset A of X, $[A]_{seq} = c(A)$.
(2) *sequential* [6] : for each subset A of X which is not closed in X, $[A]_{seq} - A \neq \emptyset$.
(3) *countable tightness* [1] (also called determined by countable subsets [10], [12]) : for each subset $A \subset X$ and each $x \in c(A)$, there exists a countable subset B of A such that $x \in c(B)$.
(4) *countably Fréchet-Urysohn* [9] : for each countable subset A of X, $[A]_{seq} = c(A)$.
(5) *AP* (standing for Approximation by Points) [15] (also called Whyburn [11]) : for each subset A of X, $[A]_{AP}= c(A)$.
(6) *WAP* (standing for Weak Approximation by Points) [15] (also called weakly Whyburn [11]) : for each subset A of X which is not closed in X, $[A]_{AP} - A\neq \emptyset$.
(7) *countably AP* : for each countable subset A of X, $[A]_{AP} = c(A)$.
(8) *ACP* (standing for Approximation by Countable Points) : for each subset A of X, $[A]_{ACP}= c(A)$.
(9) *WACP* (standing for Weak Approximation by Countable Points) : for each subset A of X which is not closed in X, $[A]_{ACP} - A\neq \emptyset$.

From definitions and Hausdorffness of X, one easily know that the following diagram except for $*$ exhibits the general relationships among the properties mentioned above. No arrows may be reversed as shown by Example below (see [1, 2, 3, 4, 6, 8, 9, 10, 11, 15]).



We begin by showing some examples related to the new three properties.

Example 1.1. (1) Let $X = \{(0,0)\} \cup (\mathbb{N} \times \mathbb{N})$. We define a topology τ on X by for each $(m,n) \in X - \{(0,0)\}$, $\{(m,n)\} \in \tau$ and $(0,0) \in U \in \tau$ if and only if for all but a finite number of integers m, the sets $\{n \in \mathbb{N} : (m,n) \notin U\}$ are each finite. Thus each point $(m,n) \in X - \{(0,0)\}$ is isolated and each open neighborhood of $(0,0)$ contains all but a finite number of points in each of all but a finite number of columns(see Arens-Fort space in [14]). Then it is clear that the space X is Hausdorff and there is a unique non-isolated point $(0,0)$ in X. In fact, X is normal. Note that any space with a unique non-isolated point is AP ([15, Proposition 2.1(10)]). It follows that since X countable, X is ACP and hence countable tightness, WACP, and countably AP. Clearly, $\{(0,0)\} = c(\mathbb{N} \times \mathbb{N}) - (\mathbb{N} \times \mathbb{N})$. But, there does not exist a convergent sequence of points of $\mathbb{N} \times \mathbb{N}$ ([14, p.54, 26(3)]). Thus, X is neither sequential, countably Fréchet-Urysohn, nor Fréchet-Urysohn.

(2) Let $X = \{z\} \cup \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers. We define a topology τ on X by for each $x \in \mathbb{R}$, $\{x\} \in \tau$ and $z \in U \in \tau$ if and only if $\mathbb{R} - U$ is countable. Clearly, X is Hausdorff and z is a unique non-isolated point in X. Thus X is AP and hence WAP and countably AP. But, it is neither countable tightness, sequential, WACP, nor ACP. For, $z \in c(\mathbb{R})$, but there does not exist a countable subset C of $\mathbb{R}$ such that $z \in c(C)$ since every countable subset of $\mathbb{R}$ is closed in X.

(3) Let $\mathbb{R}$ be the set of real numbers, τ_1 the usual topology on $\mathbb{R}$ and τ_2 the topology of countable complements on $\mathbb{R}$. We define τ to be the smallest topology on $\mathbb{R}$ generated by $\tau_1 \cup \tau_2$. Then a set U is open in the space $(\mathbb{R}, \tau)$ if and only if $U = O - K$ where $O \in \tau_1$ and K is a countable subset of $\mathbb{R}$(see Countable Complement Extension Topology in [14]). Clearly, the space $(\mathbb{R}, \tau)$ is Hausdorff and every countable subset of $\mathbb{R}$ is closed in $(\mathbb{R}, \tau)$. It is easy to check that if every countable subset of a topological space X is closed, then X is countably AP. Thus $(\mathbb{R}, \tau)$ is a countably AP space.

We now show that the space $\mathbb{R}$ is not AP. Suppose $\mathbb{R}$ is AP and let $A = [0,1] - Q$, where $[0,1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ and $\mathbb{Q}$ is the set of all rational numbers. Then since $\mathbb{R}$ is AP and $0 \in c(A) - A$, there exists a subset B of A such that $c(B) = B \cup \{0\}$. Since every countable subset of $\mathbb{R}$ is closed, clearly the set B must be uncountable. It follows that for each $x \in (0,1] \cap \mathbb{Q}$, $x \notin c(B)$ and hence there are $\epsilon_x > 0$ and a countable subset K_x of $\mathbb{R}$ such that $((x - \epsilon_x, x + \epsilon_x) - K_x) \cap B = \emptyset$. We then have that

$$(\cup\{(x - \epsilon_x, x + \epsilon_x) - K_x : x \in (0,1] \cap Q\}) \cap B = \emptyset.$$

Clearly,

$$\cup\{(x - \epsilon_x, x + \epsilon_x) - K_x : x \in (0,1] \cap Q\} \supset A - \cup\{K_x : x \in (0,1] \cap Q\}.$$

Thus,

$$(A - \cup\{K_x : x \in (0,1] \cap Q\}) \cap B = \emptyset,$$

and so $B \subset \cup\{K_x : x \in (0,1] \cap Q\}$; that is, an uncountable set B is a subset of a countable set $\cup\{K_x : x \in (0,1] \cap Q\}$, which is a contradiction.

(4) Let X be the set containing of pairwise distinct objects of the following three types : points x_{mn} where $m, n \in \mathbb{N}$, points y_n where $n \in \mathbb{N}$ and a point

z. We set $V_k(y_n) = \{y_n\} \cup \{x_{mn} : m \geqq k\}$ for each $k \in \mathbb{N}$ and let γ denote the set of subsets W of X such that $z \in W$ and there exists a positive integer p such that $V_1(y_n) - W$ is finite and $y_n \in W$ for all $n \geqq p$. The collection

$$\{\{x_{mn}\} : m, n \in \mathbb{N}\} \cup \gamma \cup \{V_k(y_n) : k, n \in \mathbb{N}\}$$

is a base for a topology on X. It is clear that the space X with the topology generated by the base is Hausdorff sequential [2, p.13, Example 13] and hence WACP and WAP. Let $Y = \{x_{mn} : m, n \in \mathbb{N}\}$. Then, $z \in c(Y)$. We know that for each subset F of Y with $z \in c(F)$, $\{y_n : n \in \mathbb{N}\} \cap F$ is infinite. Hence, there does not exist a subset F of Y such that $c(F) = F \cup \{z\}$, and thus X is neither AP nor ACP.

(5) The space of ordinals $X = [0, \omega_1]$, where ω_1 is the first uncountable ordinal, is compact Hausdorff, WAP(see [14, p.70, 43(14)] and [15, Theorem 2.7]), and countably Fréchet-Urysohn ([9, Example (2)]) and hence countably AP. But, it is neither sequential ([6]) nor AP ([15, Corollary 2.10]).

Here we observe the implication $*$ in the diagram above.

Theorem 1.2. *Every WACP space is countable tightness.*

Proof. It is well-known that a topological space X is countable tightness if and only if for each non-closed subset A of X, there are $x \in c(A) - A$ and a sequence (x_n) of points of A such that (x_n) accumulates at x (see [12, Proposition 2.2]). Let X be a WACP space and A a non-closed subset of X. Then since X is WACP, $[A]_{ACP} - A \neq \emptyset$, and hence there are $x \in [A]_{ACP} - A$ and a countable subset C of A such that $c(C) = C \cup \{x\}$. By Hausdorffness of X, we see that C is countably infinite. Let $C = \{x_n : n \in \mathbb{N}\}$. Then since $x \in c(C) - C$, for each open set U_x in X containing x, $C \cap U_x$ is infinite. It follows that the sequence (x_n) accumulates at x. Thus we have that there are $x \in [A]_{ACP} - A \subset c(A) - A$ and a sequence (x_n) of points of A such that (x_n) accumulates at x. Therefore X is countable tightness. □

In [4, Theorem 2.1], A Bella and I. V. Yaschenko showed that there is a countable non WAP space. Since every countable space is countable tightness and every WACP space is WAP, we have that there is a countable tightness non WACP space and hence the reverse of $*$ is not true in general.

In [3, Proposition 3], A. Bella showed that a countably compact and WAP space is sequentially compact and in [15, Proposition 2.1(12) and Theorem 2.2], V. V. Tkachuk and I. V. Yaschenko showed that the space $\beta\mathbb{N} - \mathbb{N}$ is not WAP(and hence $\beta\mathbb{N}$ is neither WAP) and a countably compact and AP space is Fréchet-Urysohn.

In particular, in [7, Theorem 9.6](also in [13, Theorem 1.10] and [8, Theorem 2.2]), the authors showed a well-known and useful theorem that the following statements are equivalent :

(1) X is semi-metrizable.

(2) X is symmetrizable and first countable.

(3) X is symmetrizable and Fréchet-Urysohn.

Also, J. E. Vaughan in [16, p.590, 5.3] and S. P. Franklin in [6, Proposition 1.10] proved that a countably compact and sequential space is sequentially compact. In [2, p.58, Proposition 3], A. V. Arhangel'skii and L. S. Pontryagin showed that a compact and Fréchet-Urysohn space is sequentially compact.

Recently, in [9, Lemma 2.1], the author showed that in a countably Fréchet-Urysohn space X, the sequential closure operator $[\cdot]_{seq}$ on X satisfies the Kuratowski topological closure axioms and the space X endowed with the topology induced by $[\cdot]_{seq}$ is Fréchet-Urysohn. And in [9, Theorem 2.4(1)], he also showed that a sequentially compact and countably Fréchet-Urysohn space is Fréchet-Urysohn if and only if it is maximal sequentially compact.

In this paper, we introduce some new properties of a topological space which are respectively generalizations of Fréchet-Urysohn property. We then give some examples and investigate the relationships among the properties. We prove that countably AP property is a sufficient condition for a space being countable tightness, sequential, weakly first countable and symmetrizable to be ACP, Fréchet-Urysohn, first countable and semi-metrizable, respectively. We also prove that countable compactness is a sufficient condition for a countably AP space to be countably Fréchet-Urysohn. We then show that a countably compact space satisfying one of the properties mentioned above except for countable tightness is sequentially compact. And we show that a countably compact and countably AP space is maximal countably compact if and only if it is Fréchet-Urysohn. Finally, we show that if a topological space X is countably AP, then the ACP closure operator $[\cdot]_{ACP}$ on X satisfies the Kuratowski topological closure axioms and the space X endowed with the topology induced by $[\cdot]_{ACP}$ is ACP. Moreover, if X is a countably compact and countably AP space, then the ACP expansion of X obtained above is Fréchet-Urysohn.

2. Results

We now show the relationships among the properties.

Theorem 2.1. (1) *Every countably Fréchet-Urysohn and countable tightness space is Fréchet-Urysohn.*
(2) *Every countably AP and sequential space is Fréchet-Urysohn.*
(3) *Every countably AP and countable tightness space is ACP.*
(4) *Every WAP and countable tightness space is WACP.*

Proof. (1) See [10, Proposition 8.7].

(2) Let X be a countably AP and sequential space, A a subset of X and $x \in c(A)$. Then since X is sequential and hence countable tightness, there is a countable subset C of A such that $x \in c(C)$. Since X is countably AP and C is countable, there is a subset F of C such that $c(F) = F \cup \{x\}$; i.e., $\{x\} = c(F) - F$. Since X is sequential, there exists a sequence (x_n) of points

of F such that (x_n) converges to x. Thus $x \in [F]_{seq} \subset [C]_{seq} \subset [A]_{seq}$, and so X is Fréchet-Urysohn.

(3) Let X be a countably AP and countable tightness space, A a subset of X and $x \in c(A)$. Then since X is countable tightness, there exists a countable subset C of A such that $x \in c(C)$. Since X is countably AP and C is countable, there exists a subset F of C such that $c(F) = F \cup \{x\}$, and hence $x \in [A]_{ACP}$. Thus $c(A) = [A]_{ACP}$, and so X is ACP.

(4) Let X be a WAP and countable tightness space and A a non-closed subset of X. Then since X is countable tightness, there exist $x \in c(A) - A$ and a sequence (x_n) of points of A such that (x_n) accumulates at x. Clearly, $x \in c(\{x_n : n \in \mathbb{N}\})$. Since X is WAP, there exists a subset F of $\{x_n : n \in \mathbb{N}\}$ such that $c(F) = F \cup \{x\}$. Hence we have that there exist $x \in c(A) - A$ and a countable subset F of A such that $c(F) = F \cup \{x\}$, and thus X is WACP. □

By Theorems 1.2 and 2.1 above, we have immediately the following corollary.

Corollary 2.2. (1) *Every WACP and countably AP space is ACP.*

(2) *Every AP and countable tightness space is ACP.*
(3) *Every countably AP and countable tightness space is AP.*
(4) *Every countable tightness and countably AP space is WACP.*
(5) *Every countably Fréchet-Urysohn and WACP space is Fréchet-Urysohn.*
(6) *Every sequential and AP space is Fréchet-Urysohn* (see [15, Proposition 2.1(6)]).

Note that by Example 1.1(1), we see that every countably AP and countable tightness space need not be countably Fréchet-Urysohn in general.

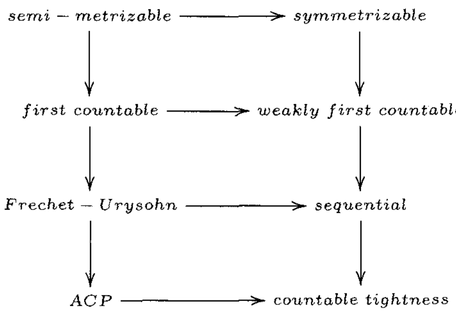
Recall that a topological space X is called *weakly first countable* [7] (also called g-first countable [13]) if for each $x \in X$, there exists a family $\{B(x,n) : n \in \mathbb{N}\}$ of subsets of X such that the following conditions are satisfied :

(i) $x \in B(x,n+1) \subset B(x,n)$ for all $n \in \mathbb{N}$,
(ii) a subset U of X is open if and only if for every $x \in U$ there exists an $n \in \mathbb{N}$ such that $B(x,n) \subset U$.

Such a family $\{B(x,n) : n \in \mathbb{N}\}$ is called a weak base at x.

A topological space X is called *symmetrizable* [7] if there exists a symmetric (= a metric except for the triangle inequality) d on X satisfying the following condition : a subset U of X is open if and only if for every $x \in U$ there is a positive real number r such that $B(x,r) \subset U$, where $B(x,r)$ denotes the set $\{y \in X : d(x,y) < r\}$. A space X is *semi-metrizable* [7] if and only if there exists a symmetric d on X such that for each $x \in X$, the family $\{B(x,r) : r > 0\}$ forms a (not necessarily open) neighborhood base at x.

By definitions and the first diagram above, we have that the following second diagram indicates the general nature of these properties above (see [7, 8, 13]).



Theorem 2.3. *If a space X satisfies one of the properties in the second column of the second diagram above and is also countably AP, then it satisfies the corresponding property in the first column.*

Proof. In Theorem 2.1(2) and (3), we have shown that every countable tightness and countably AP space is ACP and every sequential and countably AP space is Fréchet-Urysohn.

Let X be a weakly first countable and countably AP space. Then since X is weakly first countable, for each $x \in X$, there is a weak base $\{B(x,n) : n \in \mathbb{N}\}$ at x. To prove this, it is sufficient to show that for each $n \in \mathbb{N}$, $B(x,n)$ is a neighborhood of x in the space X. Suppose on the contrary that there are $x \in X$ and $n \in \mathbb{N}$ such that $x \notin \text{int}(B(x,n))$, where $\text{int}(B(x,n)$ is the interior of $B(x,n)$ in X. Then, clearly, $x \in c(X - B(x,n))$. Since X is weakly first countable and countably AP, it is sequential and countably AP. By Theorem 2.1(2), X is Fréchet-Urysohn and hence AP. It follows that there exists a subset Y of $X - B(x,n)$ such that $c(Y) = Y \cup \{x\}$. Since $X - c(Y)$ is open and $B(x,n) \subset ((X - c(Y)) \cup \{x\})$, by (ii) of the definition of weak first countability, we have that the set $(X - c(Y)) \cup \{x\}$ is open in X containing x and $Y \cap ((X - c(Y)) \cup \{x\}) = \emptyset$, which is a contradiction. Thus we have that a weakly first countable and countably AP space is first countable.

Finally, by the above result, it follows that every symmetrizable and countably AP space is first countable and hence semi-metrizable. □

From Theorem 2.3, we have immediately the following corollaries and hence we omit the proofs.

Corollary 2.4. *Let X be a sequential space. Then the following statements are equivalent :*

(1) *X is Fréchet-Urysohn.*
(2) *X is ACP.*
(3) *X is countably AP.*

Corollary 2.5. *Let X be a weakly first countable space. Then the following statements are equivalent :*

(1) *X is first countable.*
(2) *X is Fréchet-Urysohn.*
(3) *X is ACP.*
(4) *X is countably AP.*

Corollary 2.6. *Let X be a symmetrizable space. Then the following statements are equivalent :*
(1) *X is semi-metrizable.*
(2) *X is first countable.*
(3) *X is Fréchet-Urysohn.*
(4) *X is ACP.*
(5) *X is countably AP.*

We also obtain the result of [7, Theorem 9.6] (also [13, Theorem 1.10] and [8, Theorem 2.2]) as a corollary.

By Example 1.1(5), we know that a WAP and countably AP space need not be an AP space. Hence, in Theorem 2.3 above, we cannot replace "ACP$\longrightarrow$ countable tightness" by "AP$\longrightarrow$ WAP".

We recall that a topological space X is *countably compact* if and only if every countable open cover of X has a finite subcover ; equivalently, every sequence of points of X has an accumulation point.

We are going to prove that countably compactness is a sufficient condition for a countably AP to be countably Fréchet-Urysohn.

Theorem 2.7. *Every countably compact and countably AP space is countably Fréchet-Urysohn.*

Proof. In [15, Theorem 2.2], V. V. Tkachuk and I. V. Yaschenko showed that a countably compact and AP space is Fréchet-Urysohn. It can be proved using the very similar arguments of the proof of [15, Theorem 2.2]. Hence we omit the proof. □

Remarks 2.8. (1) From Example 1.1(5), we know that a compact and WAP space need not be sequential in general.

(2) Still there is a very natural question left open: Is every countably compact(or compact) and WACP space sequential?

Recall that a topological space X is *sequentially compact* if and only if every sequence of points of X has a convergent subsequence. It is obvious that every sequentially compact space is countably compact, but the reverse is not true in general.

Next, we show that a countably compact space satisfying one of the properties mentioned above except for countable tightness is sequentially compact.

Theorem 2.9. *Every countably compact and countably AP space is sequentially compact.*

Proof. Let X be a countably compact and countably AP space and let (x_n) be a sequence of points of X. Then since X is countably compact, (x_n) has an accumulation point. Let x be an accumulation point of (x_n) in X. Clearly, $x \in c(\{x_n : n \in \mathbb{N}\})$, where $\{x_n : n \in \mathbb{N}\}$ is the range of (x_n). By Theorem 2.7, X is countably Fréchet-Urysohn and hence $c(\{x_n : n \in \mathbb{N}\}) = [\{x_n : n \in \mathbb{N}\}]_{seq}$. It follows that there exists a sequence (y_n) of points of $\{x_n : n \in \mathbb{N}\}$ such that (y_n) converges to x. Set $y_n = x_{\mu(n)}$ for each $n \in \mathbb{N}$. Note that (y_n) need not be a subsequence of (x_n) in general. We now construct a sequence $(x_{\phi(n)})$ as follows : Let $\phi(1) = \mu(1)$, $\phi(2) =$ the first(least) element of $\{\mu(k)|\phi(1) < \mu(k), k \in \mathbb{N}\}$ and $\phi(3) =$ the first element of $\{\mu(k)|\phi(2) < \mu(k)$ and $p < k\}$, where p is the number satisfying $x_{\phi(2)} = x_{\mu(p)} = y_p$. Assume that for $k \in \mathbb{N}$, $\phi(1) < \phi(2) < \phi(3) < \cdots < \phi(k)$ have been defined, let $\phi(k+1) =$ the first element of $\{\mu(k)|\phi(k) < \mu(k)$ and $p < k\}$, where p is the number satisfying $x_{\phi(k)} = x_{\mu(p)} = y_p$. Then we obtain, by Induction, a sequence $(x_{\phi(n)})$. It is obvious that the sequence $(x_{\phi(n)})$ is a subsequence of (x_n) and also a subsequence of (y_n). Since (y_n) converges to x, $(x_{\phi(n)})$ converges to x. Hence we have shown that there exists a subsequence $(x_{\phi(n)})$ of (x_n) which converges to x. Thus X is sequentially compact. □

Corollary 2.10. *A countably compact space which satisfies one of the properties mentioned above except for countable tightness is sequentially compact.*

Proof. This follows directly from [3, Proposition 3] and Theorem 2.9. □

We obtain immediately the results of [16, p.590, 5.3], [6, Proposition 1.10], and [2, p.53, Proposition 3] from Corollary 2.10.

Lemma 2.11. ([9, Lemma 2.1]) *If $(X, \mathcal{T})$ is a countably Fréchet-Urysohn space, then the sequential closure operator $[\cdot]_{seq}$ on X satisfies the Kuratowski topological closure operator axioms and $(X, \mathcal{T}_{[\cdot]_{seq}})$ is a Fréchet-Urysohn space, where $\mathcal{T}_{[\cdot]_{seq}}$ is the topology for X induced by the closure operator $[\cdot]_{seq}$ on X.*

Note that $(X, \mathcal{T}_{[\cdot]_{seq}})$ is a Fréchet-Urysohn expansion of a countably Fréchet-Urysohn space $(X, \mathcal{T})$.

Lemma 2.12. (1) *If $(X, \mathcal{T})$ and $(X, \mathcal{T}^*)$ are sequentially compact spaces with $\mathcal{T} \subset \mathcal{T}^*$, then for each sequence (x_n) of points of X, (x_n) converges to x in $(X, \mathcal{T})$ if and only if (x_n) converges to x in $(X, \mathcal{T}^*)$.*

(2) *Let $(X, \mathcal{T})$ be a countably Fréchet-Urysohn space. Then, $(X, \mathcal{T})$ is countably compact (sequentially compact) if and only if the Fréchet-Urysohn space $(X, \mathcal{T}_{[\cdot]_{seq}})$ obtained in Lemma 2.11 is countably compact (resp. sequentially compact).*

Proof. (1) See [9, Lemma 2.3].

(2) Note that a topological space X is countably compact if and only if every countably infinite subset of X has at least one cluster point. Let F be a countably infinite subset of X. Since $(X, \mathcal{T})$ is countably Fréchet-Urysohn and

F is countable, we have $c(F) = [F]_{seq} = c_{\mathcal{T}_{[\cdot]_{seq}}}(F)$, where $c_{\mathcal{T}_{[\cdot]_{seq}}}(F)$ denotes the closure of F in $(X, \mathcal{T}_{[\cdot]_{seq}})$. By countable compactness of X, $F \subsetneqq c(F)$, and hence $F \subsetneqq c_{\mathcal{T}_{[\cdot]_{seq}}}(F)$. Thus $(X, \mathcal{T}_{[\cdot]_{seq}})$ is countably compact.

Conversely, let F be a countably infinite subset of X. Since $(X, \mathcal{T}_{[\cdot]_{seq}})$ is countably compact and Fréchet-Urysohn, we have that $F \subsetneqq c_{\mathcal{T}_{[\cdot]_{seq}}}(F)$ and $c(F) = [F]_{seq} = c_{\mathcal{T}_{[\cdot]_{seq}}}(F)$, and hence $F \subsetneqq c(F)$. Thus $(X, \mathcal{T})$ is countably compact.

By Theorem 2.9, for sequential compactness, it is trivial. □

Theorem 2.13. *Let $(X, \mathcal{T})$ be a countably compact and countably AP space. Then, X is maximal countably compact if and only if X is Fréchet-Urysohn.*

Proof. Suppose $(X, \mathcal{T})$ is not Fréchet-Urysohn. Since $(X, \mathcal{T})$ is countably compact and countably AP, $(X, \mathcal{T})$ is countably Fréchet-Urysohn by Theorem 2.7. Hence, by Lemma 2.11, there is the Fréchet-Urysohn expansion $(X, \mathcal{T}_{[\cdot]_{seq}})$ of $(X, \mathcal{T})$. Since $(X, \mathcal{T})$ is not Fréchet-Urysohn, it follows clearly $\mathcal{T} \subsetneqq \mathcal{T}_{[\cdot]_{seq}}$. But, by maximal countable compactness and Lemma 2.12(2), $\mathcal{T} = \mathcal{T}_{[\cdot]_{seq}}$. This is a contradiction.

Conversely, suppose $(X, \mathcal{T})$ is not maximal countably compact. Then since $(X, \mathcal{T})$ is Fréchet-Urysohn, by Corollary 2.10, $(X, \mathcal{T})$ is not maximal sequentially compact and hence there exists a sequentially compact space $(X, \mathcal{T}^*)$ such that $\mathcal{T} \subsetneqq \mathcal{T}^*$. Let $U \in \mathcal{T}^* - \mathcal{T}$. Clearly, $X - U$ is not closed in $(X, \mathcal{T})$, and so $c(X - U) - (X - U) \neq \emptyset$, where c is the closure operator on $(X, \mathcal{T})$. Let $p \in c(X - U) - (X - U)$. Then since X is Fréchet-Urysohn, $p \in [X - U]_{seq} = c(X - U)$ and hence there exists a sequence (x_n) of points of $X - U$ such that (x_n) converges to p in $(X, \mathcal{T})$. By Lemma 2.12(1), (x_n) converges to p in $(X, \mathcal{T}^*)$. Thus,

$$p \in c_{\mathcal{T}^*}(\{x_n : n \in \mathbb{N}\}) \subset c_{\mathcal{T}^*}(X - U) = X - U,$$

which is a contradiction. □

Corollary 2.14. ([9, Theorem 2.4(1)]) *Let X be a sequentially compact and countably Fréchet-Urysohn space. Then, X is maximal sequentially compact if and only if X is Fréchet-Urysohn.*

Proof. This follows immediately from Theorems 2.9 and 2.13. □

Corollary 2.15. *Let X be a countably compact and countably AP space. Then, X is maximal countably compact if and only if X is countable tightness.*

Proof. This follows from Theorems 2.1(1), 2.7, and 2.13. □

It is easy to check that the ACP closure operator $[\cdot]_{ACP}$ on a topological space X is not a Kuratowski topological closure operator on X in general. We finally obtain a sufficient condition for $[\cdot]_{ACP}$ to be a Kuratowski topological closure operator and related results.

Theorem 2.16. *If $(X,\mathcal{T})$ is a countably AP space, then the ACP closure operator $[\cdot]_{ACP}$ on X satisfies the Kuratowski topological closure axioms and $(X,\mathcal{T}_{[\cdot]_{ACP}})$ is an ACP space, where $\mathcal{T}_{[\cdot]_{ACP}}$ is the topology for X induced by $[\cdot]_{ACP}$. Moreover, if $(X,\mathcal{T})$ is a countably compact and countably AP space, then $(X,\mathcal{T}_{[\cdot]_{ACP}})$ is a countably compact and ACP space and hence Fréchet-Urysohn.*

Proof. First, we show that the ACP closure operator $[\cdot]_{ACP}$ on X satisfies the Kuratowski topological closure axioms. By the definition of $[\cdot]_{ACP}$, it is obvious that $[X]_{ACP} = X$, $[\emptyset]_{ACP} = \emptyset$ and for each subsets A and B of X, $[A]_{ACP} \cup [B]_{ACP} \subset [A \cup B]_{ACP}$. Let $x \in [A \cup B]_{ACP}$. Then there exists a countable subset F of $A \cup B$ such that $c(F) = F \cup \{x\}$. Put $F_A = A \cap F$ and $F_B = B \cap F$. Then since

$$x \in c(F) = c(F_A \cup F_B) = c(F_A) \cup c(F_B),$$

$x \in c(F_A)$ or $x \in c(F_B)$. Without loss of generality, assume $x \in c(F_A)$. Since X is countably AP and F_A is countable, there exists a subset G of F_A such that $c(G) = G \cup \{x\}$ and hence $x \in [A]_{ACP}$. Thus, for each subsets A and B of X, $[A]_{ACP} \cup [B]_{ACP} = [A \cup B]_{ACP}$. Hence, it remains to prove that for each subset A of X, $[A]_{ACP} = [[A]_{ACP}]_{ACP}$. Clearly, $[A]_{ACP} \subset [[A]_{ACP}]_{ACP}$. Conversely, let $x \in [[A]_{ACP}]_{ACP}$. Then there exists a countable subset F of $[A]_{ACP}$ such that $c(F) = F \cup \{x\}$. Since F is countable, $F - A$ is at most countable. Let $F - A = \{y_n : n \in \mathbb{N}\}$. Then for each $n \in \mathbb{N}$, since $y_n \in [A]_{ACP} - A$, there exists a countable subset A_{y_n} of A such that $c(A_{y_n}) = A_{y_n} \cup \{y_n\}$. Clearly, $A \cap F = F - \{y_n : n \in \mathbb{N}\}$ and it is countable. So, $(A \cap F) \cup \bigcup_{n\in\mathbb{N}} A_{y_n}$ is a countable subset of A. Since for each $n \in \mathbb{N}$, $c(A_{y_n}) = A_{y_n} \cup \{y_n\}$ and $y_n \in c(A_{y_n}) \subset c(\bigcup_{n\in\mathbb{N}} A_{y_n})$, we have that

$$\begin{aligned} F &= (A \cap F) \cup (F - A) = (A \cap F) \cup \{y_n : n \in \mathbb{N}\} \\ &\subset (A \cap F) \cup c(\bigcup_{n\in\mathbb{N}} A_{y_n}) \subset c((A \cap F) \cup \bigcup_{n\in\mathbb{N}} A_{y_n}), \end{aligned}$$

and hence $x \in c(F) \subset c((A \cap F) \cup \bigcup_{n\in\mathbb{N}} A_{y_n})$. Since X is countably AP and $(A \cap F) \cup \bigcup_{n\in\mathbb{N}} A_{y_n}$ is countable, there exists a subset G of $(A \cap F) \cup \bigcup_{n\in\mathbb{N}} A_{y_n}$ such that $c(G) = G \cup \{x\}$, and thus $x \in [A]_{ACP}$. Therefore, $[\cdot]_{ACP}$ satisfies the Kuratowski topological closure axioms.

Second, we show that $(X,\mathcal{T}_{[\cdot]_{ACP}})$ is an ACP space. Let A be a subset of X. Then by the definitions, it is obvious $[A]_{ACP-\mathcal{T}_{[\cdot]_{ACP}}} \subset c_{\mathcal{T}_{[\cdot]_{ACP}}}(A)$, where $[A]_{ACP-\mathcal{T}_{[\cdot]_{ACP}}}$ and $c_{\mathcal{T}_{[\cdot]_{ACP}}}(A)$ are the ACP closure of A and the closure of A in $(X,\mathcal{T}_{[\cdot]_{ACP}})$, respectively. Hence it is sufficient to show that $c_{\mathcal{T}_{[\cdot]_{ACP}}}(A) \subset [A]_{ACP-\mathcal{T}_{[\cdot]_{ACP}}}$. Let $x \in c_{\mathcal{T}_{[\cdot]_{ACP}}}(A)$. Since $c_{\mathcal{T}_{[\cdot]_{ACP}}}(A) = [A]_{ACP}$, $x \in [A]_{ACP}$ and hence there exists a countable subset F of A such that $c(F) = F \cup \{x\}$. Note that in a countably AP space X, for each countable subset F of X, $c(F) = [F]_{ACP} = [F]_{AP}$. Thus $[F]_{ACP} = F \cup \{x\}$, and so $x \in [A]_{ACP-\mathcal{T}_{[\cdot]_{ACP}}}$. Therefore, $(X,\mathcal{T}_{[\cdot]_{ACP}})$ is an ACP space.

Finally we show that $(X, \mathcal{T}_{[\cdot]_{ACP}})$ is a Fréchet-Urysohn space. It is trivial that $(X, \mathcal{T}_{[\cdot]_{ACP}})$ is Hausdorff since $(X, \mathcal{T})$ is Hausdorff and $\mathcal{T} \subset \mathcal{T}_{[\cdot]_{ACP}}$. By [15, Theorem 2.2], we have known that every countably compact and AP (ACP implies AP) space is Fréchet-Urysohn. Hence, it is sufficient to prove that $(X, \mathcal{T}_{[\cdot]_{ACP}})$ is countably compact. Let C be a countably infinite subset of X. We now assert $C \subsetneqq c_{\mathcal{T}_{[\cdot]_{ACP}}}(C)$. Since $(X, \mathcal{T})$ is countably compact, clearly $C \subsetneqq c(C)$. Since $(X, \mathcal{T})$ is countably AP and C is countable, we have that

$$c(C) = [C]_{AP} = [C]_{ACP} = [C]_{ACP-\mathcal{T}_{[\cdot]_{ACP}}} = c_{\mathcal{T}_{[\cdot]_{ACP}}}(C).$$

Thus $C \subsetneqq c_{\mathcal{T}_{[\cdot]_{ACP}}}(C)$, and so $(X, \mathcal{T}_{[\cdot]_{ACP}})$ is countably compact. Therefore, $(X, \mathcal{T}_{[\cdot]_{ACP}})$ is a Fréchet-Urysohn space. □

Note that if $(X, \mathcal{T})$ is an ACP (Fréchet-Urysohn) space, then $\mathcal{T} = \mathcal{T}_{[\cdot]_{ACP}}$ (resp. $\mathcal{T} = \mathcal{T}_{[\cdot]_{seq}}$) ; equivalently, for each subset A of X, $c(A) = [A]_{ACP}$ (resp. $c(A) = [A]_{seq}$).

Corollary 2.17. *If $(X, \mathcal{T})$ is a countably compact and countably AP space, then the Fréchet-Urysohn expansions $(X, \mathcal{T}_{[\cdot]_{seq}})$ and $(X, \mathcal{T}_{[\cdot]_{ACP}})$ of $(X, \mathcal{T})$ are homeomorphic ; i.e., $\mathcal{T}_{[\cdot]_{seq}} = \mathcal{T}_{[\cdot]_{ACP}}$.*

Proof. By Lemmas 2.11 and 2.12(2) and Theorem 2.16, $(X, \mathcal{T}_{[\cdot]_{seq}})$ and $(X, \mathcal{T}_{[\cdot]_{ACP}})$ are countably compact and Fréchet-Urysohn spaces with $\mathcal{T} \subset \mathcal{T}_{[\cdot]_{seq}}$ and $\mathcal{T} \subset \mathcal{T}_{[\cdot]_{ACP}}$. Note that for each subset A of X, $c_{\mathcal{T}_{[\cdot]_{seq}}}(A) = [A]_{seq}$, $c_{\mathcal{T}_{[\cdot]_{ACP}}}(A) = [A]_{ACP}$, and $[A]_{seq} \subset [A]_{ACP}$. Hence it is sufficient to prove that for each subset A of X, $[A]_{ACP} \subset [A]_{seq}$. Let A be a subset of X and $x \in [A]_{ACP}$. Then, by definition of $[\cdot]_{ACP}$, there exists a countable subset C of A such that $\{x\} = c(C) - C$. Since $(X, \mathcal{T})$ is countably compact and countably AP, by Theorem 2.7, it is countably Fréchet-Urysohn. It follows that there exists a sequence (x_n) of points of C such that (x_n) converges to x and so $x \in [C]_{seq}$. Thus $x \in [A]_{seq}$. □

Acknowledgement. The author wish to thank the referee for his/her very kind detailed comments in improving the exposition of the paper.

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