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ON POSITIVE SOLUTIONS FOR A CLASS OF INDEFINITE WEIGHT SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS WITH CRITICAL SOBOLEV EXPONENT

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ABSTRACT. By variational methods, we prove the existence of positive solutions of a class of indefinite weight semilinear elliptic boundary value problems on critical Sobolev exponent.

1. Introduction

We have known several famous results for the existence or the non-existence of positive solutions about semilinear elliptic boundary value problems in critical Sobolev exponent case ([5], [2]). As some different studies from that, we discuss the existence of positive solutions of the following indefinite weight semilinear elliptic boundary value problems:

$$(I_{\lambda\alpha}) \quad \begin{cases} -\Delta u = \lambda g(x) u(1+|u|^p) \text{ in } \Omega, \\ (1-\alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 \text{ on } \partial\Omega, \end{cases}$$

where λ and α are real parameters, Ω is an open bounded domain in $\mathbb{R}^N$, $N \geq 3$, with the smooth boundary $\partial\Omega$. We shall consider the critical Sobolev exponent case $p = \frac{4}{N-2}$ and the function $g : \overline{\Omega} \longrightarrow \mathbb{R}^1$ is smooth and changes sign.

We proved the existence of positive solutions of the case $0 < p < \frac{4}{N-2}$ ([4]). Here $\alpha \in (0,1)$ or $\int_\Omega g(x)dx \neq 0$ and $\alpha \in (\alpha_0, 0]$ for some constant $\alpha_0 < 0$. We used the constrained minimization method of the functional

$$E_\lambda(u) = \int_\Omega |\nabla u|^2 - \lambda \int_\Omega g u^2 + \frac{\alpha}{(1-\alpha)} \int_{\partial\Omega} u^2 dS_x$$

on the constrained set

$$\{u \in W^{1,2}(\Omega) \,:\, \lambda \int_\Omega g|u|^{p+2} = 1\}$$

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to prove the existence if $\alpha \neq 1$. The other case can be proved by the similar method on the Sobolev space $W_0^{1,2}(\Omega)$. In this paper, we assume that if $\alpha = 1$, the considering space is $W_0^{1,2}(\Omega)$.

If $p = \frac{4}{N-2}$, the above constrained set may not be weakly closed, and so we should find a different method to get positive solutions.

In Section 2, we show that a minimizing sequence of the functional which is induced by the weighted problem $(I_{\lambda\alpha})$: For $\alpha \in [0,1)$,

$$J_{\lambda\alpha}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega gu^2 - \frac{\lambda}{p+2}\int_\Omega g|u|^{p+2} + \frac{\alpha}{2(1-\alpha)}\int_{\partial\Omega} u^2 dS_x$$

on the Nehari manifold:

$$M_{\lambda\alpha} = \left\{u \in W^{1,2}(\Omega) \,:\, u \neq 0,\, \langle J'_{\lambda\alpha}(u), u\rangle = 0\right\},$$

where

$$\langle J'_{\lambda\alpha}(u), u\rangle = \int_\Omega |\nabla u|^2 - \lambda\int_\Omega gu^2(1+|u|^p) + \frac{\alpha}{1-\alpha}\int_{\partial\Omega} u^2 dS_x,$$

converges to a positive function in $W^{1,2}(\Omega)$ which is a classical positive solution of the problem $(I_{\lambda\alpha})$ if $\lambda_\alpha^- < \lambda < \lambda_\alpha^+$, and λ is near to either λ_α^- or λ_α^+, where λ_α^- and λ_α^+ are the principal eigenvalues of the following problem ([1]):

$$(L_\alpha)\ \begin{cases} -\Delta u = \lambda g(x) u \text{ in } \Omega, \\ (1-\alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 \text{ on } \partial\Omega. \end{cases}$$

Furthermore, we estimate the length of the intervals about λ in which the existence is guaranteed. We also have the similar result for $\alpha = 1$ using the following functional

$$J_{\lambda 1}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega gu^2 - \frac{\lambda}{p+2}\int_\Omega g|u|^{p+2}.$$

In the end of Section 2, we can show that $(I_{\lambda\alpha})$ has a positive solution for all $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+)$, except $\lambda \neq 0$ if $g(x) < 0$ for all $x \in \partial\Omega$. However, we note that if Ω is a ball, $g = 1, N = 3$ and $\alpha = 1$, then $(I_{\lambda\alpha})$ has a positive solution if and only if $\frac{1}{4}\lambda_1 < \lambda < \lambda_1$, where λ_1 is the principal eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition ([2]). As the application of the result, we can prove the existence of a positive solution of the following problem:

$$\begin{cases} -\Delta u = g(x) u^{\frac{N+2}{N-2}} \text{ in } \Omega, \\ (1-\alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 \text{ on } \partial\Omega, \end{cases}$$

if the function g satisfies the above same condition. We also note that, if Ω is an open ball, $g = 1$ in $\bar{\Omega}$ and $\alpha = 1$, we have had the nonexistence result of any positive solution ([5]). The above difference between the existence and the non-existence may be appeared from the special properties of the function g in the indefinite weight problems.

2. The main results

We first recall some facts about how the method of eigencurves can be used to define principal eigenvalues. We define $\mu(\lambda,\alpha)$ by

$$\mu(\lambda,\alpha)=\inf\left\{\int_\Omega(|\nabla u|^2-\lambda gu^2)dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u^2dS_x : u\in W^{1,2}(\Omega),\int_\Omega u^2=1\right\}.$$

It can be shown in [1] that $\mu(0,\alpha)>0$ on $\alpha\in[\alpha_0,1]$ where $\alpha_0\le 0$ for some small negative, and the function $\lambda\longrightarrow\mu(\lambda,\alpha)$ is a concave function such that $\mu(0,\lambda)\to-\infty$ as $\lambda\to\pm\infty$. So it follows that $\lambda\to\mu(\lambda,\alpha)$ has exactly two zeros λ_α^- and λ_α^+, and those are principal eigenvalues for (L_α). Furthermore, the eigencurves $\lambda\to\mu(\lambda,\alpha)$ can be used to produce an equivalent norm for $W^{1,2}(\Omega)$ if $\alpha\neq 1$. In this case $\alpha=1$, we use the following function μ:

$$\mu(\lambda)=\inf\left\{\int_\Omega\left[|\nabla|^2-\lambda gu^2\right]dx : u\in W_0^{1,2}(\Omega),\int_\Omega u^2=1\right\}.$$

Lemma 2.1 ([4]). *Suppose $\alpha\in(0,1)$ or that $\int_\Omega gdx\neq 0$ and $\alpha\in(\alpha_0,0]$ so that (L_α) has principal eigenvalues λ_α^- and λ_α^+. For any $\lambda\in(\lambda_\alpha^-,\lambda_\alpha^+)$,*

$$\|u\|_{\lambda\alpha}=\left\{\int_\Omega[|\nabla u|^2-\lambda gu^2]dx+\frac{\alpha}{1-\alpha}\int_{\partial\Omega}u^2dS_x\right\}^{\frac{1}{2}}$$

defines a norm in $W^{1,2}(\Omega)$ which is equivalent to the usual norm for $W^{1,2}(\Omega)$.

For the simplicity, we use the following function $K:\mathbb{R}\to\mathbb{R}$

$$K(\alpha)=\begin{cases}0 & \text{if } \alpha=1,\\ \dfrac{\alpha}{1-\alpha} & \text{otherwise.}\end{cases}$$

Lemma 2.2. *Let $\lambda\in(\lambda_\alpha^-,\lambda_\alpha^+)$, $\lambda\neq 0$ and let*

$$M_{\lambda\alpha}=\left\{u\in W^{1,2}(\Omega) : u\neq 0,\ \langle J'_{\lambda\alpha}(u),u\rangle=0\right\},$$

Then $M_{\lambda\alpha}$ is a nonempty subset of $W^{1,2}(\Omega)$.

Proof. Since g changes sign, we can choose a nonzero function $u_0\in W^{1,2}(\Omega)$ so that

$$\int_\Omega g|u_0|^{p+2}>0.$$

Let

$$t^p=\frac{\int_\Omega|\nabla u_0|^2-\lambda\int_\Omega gu_0^2+K(\alpha)\int_{\partial\Omega}u_0^2dS_x}{\lambda\int_\Omega g|u_0|^{p+2}}.$$

Then $u=tu_0\in M_{\lambda\alpha}$. □

Definition 2.3. We define the following functions:

$$K^+_{\lambda\alpha} = \inf\left\{\int_\Omega [|\nabla u|^2 - \lambda g u^2] + K(\alpha)\int_{\partial\Omega} u^2 dS_x : u \in W^{1,2}(\Omega), \int_\Omega g|u|^{p+2} = 1\right\},$$

$$K^-_{\lambda\alpha} = \inf\left\{\int_\Omega [|\nabla u|^2 - \lambda g u^2] + K(\alpha)\int_{\partial\Omega} u^2 dS_x : u \in W^{1,2}(\Omega), \int_\Omega g|u|^{p+2} = -1\right\},$$

$$K^+_{0\alpha} = \inf\left\{\int_\Omega |\nabla u|^2 + K(\alpha)\int_{\partial\Omega} u^2 dS_x : u \in W^{1,2}(\Omega), \int_\Omega g|u|^{p+2} = 1\right\}$$

and

$$K^-_{0\alpha} = \inf\left\{\int_\Omega |\nabla u|^2 + K(\alpha)\int_{\partial\Omega} u^2 dS_x : u \in W^{1,2}(\Omega), \int_\Omega g|u|^{p+2} = -1\right\}.$$

Lemma 2.4. *$K^-_{0\alpha} > 0$ and $K^+_{0\alpha} > 0$ if $\alpha \in (0,1]$.*

Proof. Let $\alpha \neq 1$. We show that $K^+_{0\alpha} > 0$. If not, there is a sequence $u_n \in W^{1,2}(\Omega)$ so that

$$\lim_{n\to\infty}\left[\int_\Omega |\nabla u_n|^2 + K(\alpha)\int_{\partial\Omega} u_n^2 dS_x\right] = 0 \quad \text{and} \quad \int_\Omega g|u_n|^{p+2} = 1.$$

By the Sobolev embedding : $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$, it is impossible.

The proof of $K^-_{0\alpha} > 0$ is exactly the same as the above.

By the similar method, we can prove that in the case $\alpha = 1$. □

Remark 2.5. Let $\alpha \in [0,1]$. We note that $K^-_{\lambda\alpha}$ and $K^+_{\lambda\alpha}$ are concave continuous curves on the interval $[\lambda^-_\alpha, \lambda^+_\alpha]$. Hence, $K^+_{\lambda\alpha} \le K^+_{0\alpha}$ for all $\lambda \in [0, \lambda^+_\alpha]$ and $K^-_{\lambda\alpha} \le K^-_{0\alpha}$ for all $\lambda \in [\lambda^-_\alpha, 0]$. Furthermore, by the Sobolev embedding, the equivalent norm, and the relations between the principal eigenvalues and the function g: $\lambda\int_\Omega g\phi^{p+1}dx > 0$ for all $p \ge 1$, where $\lambda \neq 0$ is a principal eigenvalue with corresponding positive principal eigenfunction ϕ ([4]), the following properties hold: (i) $K^-_{\lambda^-_\alpha} = K^+_{\lambda^+_\alpha} = 0$, (ii) $K^-_{\lambda\alpha} > 0$ if $\lambda \in (0, \lambda^+_\alpha)$ and $K^+_{\lambda\alpha} > 0$ if $\lambda \in (0, \lambda^+_\alpha)$.

Definitions and Remarks. Let $\lambda \in (\lambda^-_\alpha, \lambda^+_\alpha)$. We define the following sets:

$$H_\lambda = \left\{u \in W^{1,2}(\Omega) : \lambda\int_\Omega g|u|^{p+2} = 1\right\}.$$

Let $u \in H_\lambda$. Then $||u||^{\frac{2}{p}}_{\lambda\alpha} u \in M_{\lambda\alpha}$. If $u \in M_{\lambda\alpha}$, then $||u||^{-\frac{2}{p+2}}_{\lambda\alpha} u \in H_\lambda$. We define the functional $E_{\lambda\alpha} : H_\lambda \longrightarrow \mathbb{R}^1$ by

$$E_{\lambda\alpha}(u) = \int_\Omega |\nabla u|^2 - \lambda\int_\Omega g u^2 + K(\alpha)\int_{\partial\Omega} u^2 dS_x.$$

Then we obtain

$$E_{\lambda\alpha}(u) = \left[\frac{2(p+2)}{p} J_{\lambda\alpha}\left(||u||^{\frac{2}{p}}_{\lambda\alpha} u\right)\right]^{\frac{p}{p+2}}$$

and

$$J_{\lambda\alpha}(u) = \frac{p}{2(p+2)} E_{\lambda\alpha}\left(||u||_{\lambda\alpha}^{-\frac{2}{p+2}} u \right)^{\frac{p+2}{p}}.$$

If we let

$$Q_{\lambda\alpha} = \inf E_{\lambda\alpha}(H_\lambda) \text{ and } C_{\lambda\alpha} = \inf J_{\lambda\alpha}(M_{\lambda\alpha}),$$

then by the simple calculation it follows that

$$Q_{\lambda\alpha} = \left[\frac{2(p+2)}{p} C_{\lambda\alpha}\right]^{\frac{p}{p+2}}.$$

This implies that if $\{u_n\}$ is a minimizing sequence of $E_{\lambda\alpha}$ on H_λ, then $\{||u_n||_{\lambda\alpha}^{\frac{2}{p}} u_n\}$ is also a minimizing sequence of $J_{\lambda\alpha}$ on $M_{\lambda\alpha}$ and vice versa.

Remark 2.6. Let $\alpha \neq 1$. We can prove that $u = 0$ is not a limit point of $M_{\lambda\alpha}$ if $\lambda_\alpha^- < \lambda < \lambda_\alpha^+$. To show that, we assume there is a sequence $\{u_n\}$ in $M_{\lambda\alpha}$ so that $||u_n||_{\lambda\alpha} \to 0$ as $n \to \infty$. From the Sobolev embedding: $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$, the sequence $\{w_n\}$ which is defined by $w_n = \frac{u_n}{||u_n||_{\lambda\alpha}}$ is a bounded sequence in $L^{\frac{2N}{N-2}}(\Omega)$. We hence have the following result:

$$\begin{aligned} 0 =& \frac{\langle J'_{\lambda\alpha}(u_n), u_n\rangle}{||u_n||_\lambda^2} \\ =& \frac{\int_\Omega |\nabla u_n|^2 - \lambda \int_\Omega g u_n^2 + K(\alpha)\int_{\partial\Omega} u_n^2 dS_x}{||u_n||_{\lambda\alpha}^2} + (||u_n||_{\lambda\alpha})^{p+2} \int_\Omega g|w_n|^{p+2} \to 1 \end{aligned}$$

as $n \to \infty$, which leads to a contradiction.

We can also have the same argument for the case $\alpha = 1$ by the similar method.

Lemma 2.7. *Let $\alpha \in (0,1]$ or $\int_\Omega g dx \neq 0$ if $\alpha = 0$. There are two positive numbers δ_1 and δ_2 such that for any $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^- + \delta_1) \cup (\lambda_\alpha^+ - \delta_2, \lambda_\alpha^+)$, if $\{u_n\}$ be a minimizing sequence of $J_{\lambda\alpha}$ on $M_{\lambda\alpha}$. Then*

$$\liminf_{n\to\infty} \left| \int_\Omega g u_n^2 \right| > 0.$$

Proof. Let φ^- and φ^+ be the corresponding eigenfunctions to the principal eigenvalues λ_α^- and λ_α^+, respectively. We can assume that

$$\int_\Omega g|\varphi^-|^{p+2} = -1, \quad \int_\Omega g|\varphi^+|^{p+2} = 1.$$

(Lemma 3.1 in [4]). We also note that

$$\int_\Omega g(\varphi^-)^2 < 0, \quad \int_\Omega g(\varphi^+)^2 > 0.$$

Let

$$\delta_2 = \lambda_\alpha^+ - \frac{\int_\Omega |\nabla\varphi^+|^2 - K_{0\alpha}^+ + K(\alpha)\int_{\partial\Omega} |\varphi^+|^2 dS_x}{\int_\Omega g|\varphi^+|^2} = \frac{K_{0\alpha}^+}{\int_\Omega g|\varphi^+|^2}.$$

Then for $\lambda \in (\lambda_\alpha^+ - \delta_2, \lambda_\alpha^+)$ and if $\{u_n\}$ is a minimizing sequence of $J_{\lambda\alpha}$ on $M_{\lambda\alpha}$, it is bounded in $W^{1,2}(\Omega)$, and then $u_n \to u$ weakly in $W^{1,2}(\Omega)$ and $u_n \to u$ strongly in $L^2(\Omega)$. We assume that $\lambda > 0$. By the previous equality about minimums we know that $\{\|u_n\|_{\lambda\alpha}^{-\frac{2}{p+2}} u_n\}$ is a minimizing sequence of $E_{\lambda\alpha}$ on H_λ, and so there is a positive number q such that

$$\lim_{n\to\infty} \lambda^{\frac{2}{p+2}} \|u_n\|_{\lambda\alpha}^{-\frac{4}{p+2}} \left[\int_\Omega |\nabla u_n|^2 - \lambda \int_\Omega g(u_n)^2 + K(\alpha)\int_{\partial\Omega} u_n^2 dS_x\right] < q < K_{0\alpha}^+$$

for some $g > 0$. Since $\|u_n\|_{\lambda\alpha} \not\to 0$ as $n \to \infty$, if $\int_\Omega g(u_n)^2 \to 0$ as $n \to \infty$, we get

$$K_{0\alpha}^+ \le q < K_{0\alpha}^+,$$

which leads to a contradiction. Therefore,

$$\lim_{n\to\infty} \int_\Omega g u_n^2 \neq 0.$$

Let

$$\delta_1 = \frac{\int_\Omega |\nabla\varphi^-|^2 - K_{0\alpha}^- + K(\alpha)\int_{\partial\Omega} |\varphi^-|^2 dS_x}{\int_\Omega g|\varphi^-|^2} - \lambda_\alpha^- = -\frac{K_{0\alpha}^-}{\int_\Omega g|\phi^-|^2}.$$

For the value $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^- + \delta_1)$ and $\lambda < 0$, we can get the same results by the above methods.

This completes the proof. □

We denote by $B_\varepsilon(X)$ the ball in a Hilbert space X centered at 0 and of radius ε. We state the following:

Proposition 2.8 ([3], pp. 6). *Let J be a C^1-functional on a Hilbert space X and let M be a closed subset of X verifying the following property:*

For any $u \in M$ with $J'(u) \neq 0$, there exists, for a small enough $\varepsilon > 0$, a Fréchet differentiable function $s_u : B_\varepsilon(X) \longrightarrow \mathbb{R}^1$ such that, by setting $t_u(\delta) = s_u\left(\delta \frac{J'(u)}{\|J'(u)\|}\right)$ for $0 \le \delta \le \varepsilon$, we have

$$t_u(0) = 1 \text{ and } t_u(\delta)\left(u - \delta\frac{J'(u)}{\|J'(u)\|}\right) \in M.$$

If J is bounded below on M, then for any minimizing sequence $\{v_n\}$ in M for J, there exists another minimizing sequence $\{u_n\}$ in M of J such that

$$J(u_n) \le J(v_n), \ \lim_{n\to\infty} \|u_n - v_n\| = 0$$

and

$$\|J'(u_n)\| \le \frac{1}{n}\left(1 + \|u_n\||t'_{u_n}(0)|\right) + |t'_{u_n}(0)||\langle J'(u_n), u_n\rangle|,$$

where $\langle\ ,\ \rangle$ is the inner product in X.

Proof. Let $C = \inf J(M)$. Use Ekeland's variational principle ([3]) to get a minimizing sequence $\{u_n\}$ in M with the following properties:

(i): $J(u_n) \leq J(v_n) < C + \frac{1}{n}$,
(ii): $\lim_{n\to\infty} ||u_n - v_n|| = 0$,
(iii): $J(w) \geq J(u_n) - \frac{1}{n}||w - u_n||$ for all $w \in M$.

Let us assume $||J'(u_n)|| > 0$ for n large, since otherwise we are done. Apply the hypothesis on the set M with $u = u_n$ to find $t_n(\delta) = s_{u_n}\left(\delta \frac{J'(u_n)}{||J'(u_n)||}\right)$ such that $w_\delta = t_n(\delta)\left(u_n - \delta \frac{J'(u_n)}{||J'(u_n)||}\right) \in M$ for all small enough $\delta \geq 0$.

Use now the mean value theorem to get

$$\begin{aligned}\frac{1}{n}||w_\delta - u_n|| &\geq J(u_n) - J(w_\delta)\\ &= (1 - t_n(\delta))\langle J'(w_\delta), u_n\rangle + \delta t_n(\delta)\langle J'(w_\delta), \frac{J'(u_n)}{||J'(u_n)||}\rangle + o(\delta),\end{aligned}$$

where $\frac{o(\delta)}{\delta} \to 0$ as $\delta \to 0$. Dividing by $\delta > 0$ and passing to the limit as $\delta \to 0$ we derive

$$\frac{1}{n}\left(1 + |t_n'(0)|||u_n||\right) \geq -t_n'(0)\langle J'(u_n), u_n\rangle + ||J'(u_n)||,$$

which is our claim. □

Lemma 2.9. *Let $\alpha \in (0,1]$ or that $\int_\Omega g dx \neq 0$ for $\alpha = 0$. Given $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+), \lambda \neq 0$, $J_{\lambda\alpha}$ is bounded below on $M_{\lambda\alpha}$ and there exists a minimizing sequence $\{u_n\}$ of $J_{\lambda\alpha}$ on $M_{\lambda\alpha}$ so that*

$$\lim_{n\to\infty} ||J'_{\lambda\alpha}(u_n)||_{\lambda\alpha} = 0$$

and

$$\lim_{n\to\infty} J_{\lambda\alpha}(u_n) = \inf J_{\lambda\alpha}(M_{\lambda\alpha}).$$

Proof. Let $\lambda \in (\lambda_\alpha^-, \lambda_\alpha^+)$ and let $\lambda \neq 0$. We show that $J_{\lambda\alpha}$ is bounded below on $M_{\lambda\alpha}$. In fact, the following can be checked easily: if $u \in M_{\lambda\alpha}$, then

$$\lambda \int_\Omega g|u|^{p+2} > 0$$

and

$$J_{\lambda\alpha}(u) = \frac{p\lambda}{2(p+2)}\int_\Omega g|u|^{p+2}.$$

Let $u \in M_{\lambda\alpha}$. Define $G : \mathbb{R}^1 \times W^{1,2}(\Omega) \longrightarrow \mathbb{R}^1$ by $G(s,w) = \Phi_{\lambda\alpha}(s(u-w))$, where $\Phi_{\lambda\alpha} : W^{1,2}(\Omega) \longrightarrow \mathbb{R}^1$ is a functional defined by

$$\Phi_{\lambda\alpha}(u) = \int_\Omega |\nabla u|^2 - \lambda\int_\Omega gu^2 - \lambda\int_\Omega g|u|^{p+2} + K(\alpha)\int_{\partial\Omega} u^2 dS_x.$$

Then $G(1,0) = 0$ and

$$\begin{aligned}
&\frac{d}{ds}G(1,0)\\
= \;& 2\int_\Omega |\nabla u|^2 - 2\lambda\int_\Omega gu^2 - \lambda(p+2)\int_\Omega g|u|^{p+2} + 2K(\alpha)\int_{\partial\Omega} u^2 dS_x\\
= \;& -p\left(\int_\Omega [|\nabla u|^2 - \lambda g u^2] + K(\alpha)\int_{\partial\Omega} u^2 dS_x\right) \neq 0.
\end{aligned}$$

Hence, we can apply the Implicit Function Theorem at $(1,0)$ and get that for $\delta > 0$ small enough, there exists a differentiable function

$$s_u : B_\delta(W^{1,2}(\Omega)) \longrightarrow \mathbb{R}^1$$

such that $s_u(0) = 1, s_u(w)(u-w) \in M_{\lambda\alpha}$, and

$$\langle s_u'(0), w\rangle = \frac{\langle \Phi_{\lambda\alpha}'(u), w\rangle}{\langle \Phi_{\lambda\alpha}'(u), u\rangle}$$

for all $w \in B_\delta(W^{1,2}(\Omega))$. From the identification of duality to the Hilbert space $W^{1,2}(\Omega)$, we let

$$w_u = \frac{J_{\lambda\alpha}'(u)}{||J_{\lambda\alpha}'(u)||_{\lambda\alpha}} \text{ and } t_u(\rho) = s_u(\rho w_u)$$

for all $0 \le \rho \le \delta$. Then $t_u(0) = 1$ and

$$t_u(\rho)(u - \rho w_u) = s_u(\rho w_u)(u - \rho w_u) \in M_{\lambda\alpha}.$$

From Proposition 2.8, there is a minimizing sequence $\{u_n\}$ of $J_{\lambda\alpha}$ on $M_{\lambda\alpha}$ so that

$$J_{\lambda\alpha}(u_n) \le J_{\lambda\alpha}(v_n) < \inf J_{\lambda\alpha}(M_{\lambda\alpha}) + \frac{1}{n}, \lim_{n\to\infty} ||u_n - v_n||_{\lambda\alpha} = 0,$$

and

$$||J_{\lambda\alpha}'(u_n)||_{\lambda\alpha} \le \frac{1}{n}\left(1 + |t_{u_n}'(0)| ||u_n||_{\lambda\alpha}\right) + |t_{u_n}'(0)||\langle J_{\lambda\alpha}'(u_n), u_n\rangle|.$$

Since $J_{\lambda\alpha}(u_n) = \frac{\lambda p}{2(p+2)}||u_n||_{\lambda\alpha}^2$, so the sequence $\{u_n\}$ is bounded in $W^{1,2}(\Omega)$. Let $||u_n||_{\lambda\alpha} \le C_1$ for all n. Then

$$||J_{\lambda\alpha}'(u_n)||_{\lambda\alpha} \le \frac{1}{n}\left(1 + |t_{u_n}'(0)|C_1\right).$$

Since

$$|t_{u_n}'(0)| = \frac{|\langle \Phi_{\lambda\alpha}'(u_n), w_n\rangle|}{p||u_n||_{\lambda\alpha}^2},$$

where $w_n = w_{u_n}$, and $\lim_{n\to\infty} \inf ||u_n||_{\lambda\alpha} > 0$, if we show that $|t'_{u_n}(0)|$ is uniformly bounded on n, we are done. In fact, we have the following inequality

$$\begin{aligned} &|\langle \Phi'_{\lambda\alpha}(u_n), w_n\rangle| \\ \leq\ & 2\int_\Omega |\nabla u_n \cdot \nabla w_n| + 2\lambda \int_\Omega |u_n w_n| + \lambda(p+2)\int_\Omega |g||u_n|^{p+1}|w_n| \\ & +2K(\alpha)\int_{\partial\Omega} u_n w_n dS_x. \end{aligned}$$

From the well-known Sobolev embedding theorem, $||w_n||_{\lambda\alpha} = 1$ for all n, and Hölder inequality, we have two positive constants C_2 and C_3 so that

$$|\langle \Phi'_{\lambda\alpha}(u_n), w_n\rangle| \leq C_2||u_n||_{\lambda\alpha} + C_3.$$

Since $\{u_n\}$ is a bounded sequence in $W^{1,2}(\Omega)$, so is $\langle \Phi'_{\lambda\alpha}(u_n), w_n\rangle$ on n. Therefore, we can conclude that

$$\lim_{n\to\infty} ||J'_{\lambda\alpha}(u_n)||_{\lambda\alpha} = 0.$$

Clearly, we note that

$$\lim_{n\to\infty} J_{\lambda\alpha}(u_n) = \inf J_{\lambda\alpha}(M_{\lambda\alpha}).$$

□

Theorem 2.10. *Let $\alpha \in (0,1]$ or that $\int_\Omega g dx \neq 0$ for $\alpha = 0$. For any $\lambda \in (\lambda^-_\alpha, \lambda^-_\alpha + \delta_1) \cup (\lambda^+_\alpha - \delta_2, \lambda^+_\alpha), \lambda \neq 0$, the problem $(I_{\lambda\alpha})$ has a positive solution.*

Proof. Let

$$c = \inf J_{\lambda\alpha}(M_{\lambda\alpha})$$

and let $\{u_n\}$ be a sequence in $M_{\lambda\alpha}$ such that

$$\lim_{n\to\infty} J_{\lambda\alpha}(u_n) = c.$$

By Lemma 2.9, we can assume that

$$\lim_{n\to\infty} ||J'_{\lambda\alpha}(u_n)||_{\lambda\alpha} = 0.$$

Then $\{u_n\}$ is bounded and we can find a weak limit point u of the sequence in $W^{1,2}(\Omega)$. We can also assume that $\{u_n\}$ converges weakly to u and, by the Rellich-Kondrakov Theorem ([3]), that $u_n \to u$ strongly in $L^q(\Omega)$ for all $q < \frac{2N}{N-2}$. In particular, for any $v \in W^{1,2}(\Omega)$,

$$\langle J'_{\lambda\alpha}(u_n), v\rangle = \int_\Omega \nabla u_n \cdot \nabla v - \lambda\int_\Omega g u_n v - \lambda\int_\Omega g u_n |u_n|^p v + K(\alpha)\int_{\partial\Omega} u_n v dS_x,$$

which converges as $n \to \infty$ to

$$\int_\Omega (\nabla u \cdot \nabla v - \lambda g u v - \lambda g u |u|^p v)dx + K(\alpha)\int_{\partial\Omega} uv dS_x = \langle J'_{\lambda\alpha}(u), v\rangle.$$

Hence, $\langle J'_{\lambda\alpha}(u), v\rangle = 0$ for all $v \in W^{1,2}(\Omega)$ which means that u is a weak solution for $(I_{\lambda\alpha})$. In particular, $\langle J'_{\lambda\alpha}(u), u\rangle = 0$. Since $\liminf_{n\to\infty}\left|\int_\Omega gu_n^2\right| > 0$ by Lemma 2.7, we have that $u \neq 0$. Therefore, $u \in M_{\lambda\alpha}$.

Since $J_{\lambda\alpha}$ is weakly lower semi-continuous, we get

$$c \leq J_{\lambda\alpha}(u) \leq \lim_{n\to\infty} J_{\lambda\alpha}(u_n) = c.$$

It follows that $J_{\lambda\alpha}(u) = c$ and that $||u_n||_{\lambda\alpha} \to ||u||_{\lambda\alpha}$ which implies that $u_n \to u$ strongly in $W^{1,2}(\Omega)$. Since $J'_{\lambda\alpha}$ is continuous at u, we get $J'_{\lambda\alpha}(u) = 0$.

The positivity of u is clear from the equality $J_{\lambda\alpha}(u) = J_{\lambda\alpha}(|u|)$.

This completes the proof. □

Theorem 2.11. *Let $\alpha \in (0,1]$ or that $\int_\Omega gdx \neq 0$ for $\alpha = 0$. If $g(x) < 0$ for all $x \in \partial\Omega$, for any $\lambda \in (0, \lambda_\alpha^+)$, the problem $(I_{\lambda\alpha})$ has a positive solution.*

Proof. By Theorem 2.10, we have a positive solution u_λ of the problem:

$$\begin{cases} -\Delta u = \lambda g(x)u + g(x)u|u|^p \text{ in } \Omega, \\ (1-\alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 \text{ on } \partial\Omega, \end{cases}$$

for $\lambda \in (\lambda_\alpha^- + \delta_2, \lambda_\alpha^-) \cup (\lambda_\alpha^+ - \delta_1, \lambda_\alpha^+)$.

Let $0 < \lambda < \lambda_\alpha^+$ and let $\{u_n\}$ be a minimizing sequence of $J_{\lambda\alpha}$ in $M_{\lambda\alpha}$. If $K_{\lambda\alpha}^+ < K_{0\alpha}^+$ on $(0, \lambda_\alpha^+)$, we can have the same inequality about the limit $\liminf_{n\to\infty}\left|\int_\Omega gu_n^2\right| > 0$ by the same method in Lemma 2.7, and so using the same method in Theorem 2.10 we have the existence of a positive solution of $(I_{\lambda\alpha})$. We note that $K_{\lambda\alpha}^+ \leq K_{0\alpha}^+$ for all $\lambda \in (0, \lambda_\alpha^+)$.

Suppose that there is λ_0 so that $0 < \lambda_0 < \lambda_\alpha^+$ and $K_{\lambda_0\alpha}^+ = K_{0\alpha}^+$ and $K_{\lambda\alpha}^+ < K_{0\alpha}^+$ on $(\lambda_0, \lambda_\alpha^+)$. Let u_λ be the positive minimizer of the functional $J_{\lambda\alpha}$ on $M_{\lambda\alpha}$ for $\lambda \in (\lambda_0, \lambda_\alpha^+)$. Let

$$t_\lambda^p = \frac{\lambda \int_\Omega gu_\lambda^{p+2} + (\lambda - \lambda_0)\int_\Omega gu_\lambda^2 + K(\alpha)\int_{\partial\Omega} u^2 dS_x}{\lambda_0 \int_\Omega gu_\lambda^{p+2}}.$$

Then $t_\lambda u_\lambda \in M_{\lambda_0\alpha}$, and

$$J_{\lambda_0\alpha}(t_\lambda u_\lambda) = t_\lambda^2\left[J_{\lambda\alpha}(u_\lambda) + \frac{p}{2(p+2)}(\lambda - \lambda_0)\int_\Omega gu_\lambda^2\right].$$

As the previous calculation in Remark, we note that

$$\inf_{\lambda\to\lambda_0}\int_\Omega gu_\lambda^{p+2} > 0,$$

and we also note that

$$\liminf_{\lambda\to\lambda_0} J_{\lambda\alpha}(u_\lambda) < \infty$$

implies that

$$\liminf_{\lambda\to\lambda_0}\left|\int_\Omega gu_\lambda^2\right| \neq \infty,$$

and hence, $t_\lambda \to 1$ as $\lambda \to \lambda_0$. Since $\{t_\lambda u_\lambda\}$ is a minimizing sequence of $J_{\lambda_0\alpha}$ as $\lambda \to \lambda_0$, we get the weak limit u_{λ_0} of u_λ so that

$$\lim_{\lambda\to\lambda_0} t_\lambda u_\lambda = u_{\lambda_0} \quad \text{in} \quad L^2(\Omega).$$

If $u_{\lambda_0} \neq 0$, we know that it is the minimizer of $J_{\lambda_0\alpha}$ and is the positive solution of the above boundary value problem with respect to λ_0. Let

$$v_\lambda = \lambda^{\frac{1}{p+2}}\|u_\lambda\|_{\lambda\alpha}^{-\frac{2}{p+2}}u_\lambda.$$

Then

$$K_{\lambda\alpha}^+ = \int_\Omega |\nabla v_\lambda|^2 - \lambda\int_\Omega g(v_\lambda)^2 + K(\alpha)\int_{\partial\Omega} v_\lambda^2 dS_x,$$

$$K_{\lambda_0\alpha}^+ = \int_\Omega |\nabla v_{\lambda_0}|^2 - \lambda_0\int_\Omega g(v_{\lambda_0})^2 + K(\alpha)\int_{\partial\Omega} v_{\lambda_0}^2 dS_x,$$

and

$$-\int_\Omega g(v_\lambda)^2 \le \frac{K_{\lambda\alpha}^+ - K_{\lambda_0\alpha}^+}{\lambda-\lambda_0} \le -\int_\Omega g(v_{\lambda_0})^2.$$

Taking the limit on the both sides as $\lambda \to \lambda_0$, we get

$$\frac{dK_{\lambda\alpha}^+}{d\lambda}(\lambda_0) = -\int_\Omega g(v_{\lambda_0})^2.$$

Hence, $K_{\lambda\alpha}^+$ is differentiable at $\lambda = \lambda_0$, and so $\int_\Omega g(v_{\lambda_0})^2 = 0$. Since

$$K_{\lambda_0\alpha}^+ = K_{0\alpha}^+ = \int_\Omega |\nabla v_{\lambda_0}|^2 + K(\alpha)\int_{\partial\Omega} v_{\lambda_0}^2 dS_x,$$

so v_{λ_0} is also a positive solution of the problem:

$$\begin{cases} -\Delta u = g(x)u|u|^p \text{ in } \Omega, \\ (1-\alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 \text{ on } \partial\Omega, \end{cases}$$

which leads to a contradiction.

Let $u_{\lambda_0} = 0$. The $u_\lambda \to 0$ a.e. in Ω as $\lambda \to \lambda_0$. By Harnack Inequality we can argue that $u_\lambda \to 0$ uniformly on any compact subset of Ω. Also by the Maximum Principle $\{u_\lambda\}$ is uniformly bounded on any neighborhood of $\partial\Omega$. Therefore, $u_\lambda \to 0$ uniformly bounded in $\overline{\Omega}$, and then by the Lebesgue dominated convergence Theorem

$$1 = \lim_{\lambda\to\lambda_0}\|u_\lambda\|_{\lambda\alpha}^{-2}\int_\Omega g|u_\lambda|^{p+2} = 0$$

since $\|u_\lambda\|_{\lambda\alpha} \not\to 0$ as $\lambda \to \lambda_0$, which also leads to a contradiction.

This completes the proof. □

Corollary 2.12. *Let $\alpha \in (0,1]$ or that $\int_\Omega g dx \neq 0$ for $\alpha = 0$. If g satisfies the condition in Theorem 2.11, then the following problem:*

$$\begin{cases} -\Delta u = g(x)u|u|^{\frac{4}{N-2}} \text{ in } \Omega, \\ (1-\alpha)\dfrac{\partial u}{\partial n} + \alpha u = 0 \text{ on } \partial\Omega, \end{cases}$$

has a positive solution.

Proof. With the result of Theorem 2.11 if we let $\lambda_0 = 0$, the proof for the convergence of a minimizing sequence of the functional:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+2}\int_\Omega g|u|^{p+2}dx + 2K(\alpha)\int_{\partial\Omega} u^2 dS_x$$

on the Nehari manifold

$$\{u \in W^{1,2}(\Omega) : u \neq 0, \int_\Omega |\nabla u|^2 - \int_\Omega g|u|^{p+2} + K(\alpha)\int_{\partial\Omega} u^2 dS_x = 0\}$$

can be produced by the one of Theorem 2.11. □

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