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DIRECT PRODUCTED W^*-PROBABILITY SPACES AND CORRESPONDING AMALGAMATED FREE STOCHASTIC INTEGRATION

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Abstract. In this paper, we will define direct producted W^*-probability spaces over their diagonal subalgebras and observe the amalgamated freeness on them. Also, we will consider the amalgamated free stochastic calculus on such free probabilistic structure. Let $(A_j,\ \varphi_j)$ be a tracial W^*-probability spaces, for $j = 1, \ldots, N$. Then we can define the corresponding direct producted W^*-probability space $(A,\ E)$ over its N-th diagonal subalgebra $D_N \equiv \mathbb{C}^{\oplus N}$, where $A = \oplus_{j=1}^{N}\ A_j$ and $E = \oplus_{j=1}^{N}\ \varphi_j$. In Chapter 1, we show that D_N-valued cumulants are direct sum of scalar-valued cumulants. This says that, roughly speaking, the D_N-freeness is characterized by the direct sum of scalar-valued freeness. As application, the D_N-semicircularity and the D_N-valued infinitely divisibility are characterized by the direct sum of semicircularity and the direct sum of infinitely divisibility, respectively. In Chapter 2, we will define the D_N-valued stochastic integral of D_N-valued simple adapted biprocesses with respect to a fixed D_N-valued infinitely divisible element which is a D_N-free stochastic process. We can see that the free stochastic Itô's formula is naturally extended to the D_N-valued case.

Free Probability has been developed from 1980's. In this paper, we will follow the Speicher and Nica's combinatorial approach for studying Free Probability (See [8], [11] and [13]). Let $B \subset A$ be unital algebras with $1_A = 1_B$ and suppose that there exists a conditional expectation $E : A \to B$ satisfying the $\mathbb{C}$-linearity and

(i) $E(b) = b$, for all $b \in B$

(ii) $E(bab') = bE(a)b'$, for all $b, b' \in B$ and $a \in A$.

Then the algebraic pair $(A,\ E)$ is called a noncommutative probability space with amalgamation over B (or an amalgamated noncommutative probability space over B). When $B \subset A$ are $*$-algebras, the pair $(A,\ E)$ is called a $*$-probability space if (i) and (ii) holds and also it satisfies

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(iii) $E(a^*) = E(a)^*$, for all $a \in A$.

When $B \subset A$ are topological $*$-algebras, for instance, $B \subset A$ are C^*-algebras or von Neumann algebras, the pair $(A,\ E)$ is called a B-valued topological $*$-probability space, if $(A,\ E)$ satisfies (i), (ii), (iii) and

(iv) E is continuous with respect to the given topology on A.

In particular, if $B \subset A$ are C^*-algebras (or von Neumann algebras), $(A,\ E)$ is called a B-valued C^*-probability space (resp. a B-valued W^*-probability space). In this paper, we will concentrate on the case when $(A,\ E)$ is a B-valued W^*-probability space. All elements in $(A,\ E)$ are said to be B-valued random variables. When $B = \mathbb{C}$ and E is a linear functional, then we call this structure a (scalar-valued) noncommutative (resp. $*$- or topological $*$-) probability space and call its elements (free) random variables (See [8]). Suppose that $(A,\ E)$ is a topological $*$-probability space over B. Let $a_1, \ldots, a_s \in (A,\ E)$ be B-valued random variables, for $s \in \mathbb{N}$. Then they contain the following equivalent free distributional data,

∘ $(i_1, \ldots, i_n)$-th joint $*$-moment of $a_1, \ldots, a_s$: $E\left(a_{i_1}^{u_{i_1}} \cdots a_{i_n}^{u_{i_n}}\right)$,

∘ $(j_1, \ldots, j_m)$-th joint $*$-cumulant of $a_1, \ldots, a_s$:

$$k_m\left(a_{j_1}^{u_{j_1}}, \ldots, a_{j_m}^{u_{j_m}}\right) \overset{def}{=} \sum_{\pi \in NC(n)} E_\pi\left(a_{j_1}^{u_{j_1}}, \ldots, a_{j_m}^{u_{j_m}}\right) \mu(\pi, 1_n),$$

for $(i_1, \ldots, i_n) \in \{1, \ldots, s\}^n$ and $(j_1, \ldots, j_m) \in \{1, \ldots, s\}^m$, for $n, m \in \mathbb{N}$, where $u_{i_1}, \ldots, u_{i_n},\ u_{j_1}, \ldots, u_{j_m} \in \{1,\ *\}$ and $E_\pi\ (\cdots)$ is the partition-dependent B-valued moment of a and where $NC(n)$ is the lattice of all noncrossing partitions over $\{1, \ldots, n\}$ and μ is the Möbius functional in the incidence algebra, as the convolution inverse of the zeta functional ζ defined by

$$\zeta(\pi_1, \pi_2) \overset{def}{=} \begin{cases} 1 & \text{if } \pi_1 \leq \pi_2 \\ 0 & \text{otherwise,} \end{cases}$$

for all $\pi_1, \pi_2 \in NC(n)$, and $n \in \mathbb{N}$ (See [11] and [13]). Recall that the ordering on $NC(n)$, for $n \in \mathbb{N}$, is defined by

$$\pi \leq \theta \overset{def}{\Longleftrightarrow} \forall V \in \pi, \exists B \in \theta \text{ such that } V \subseteq B.$$

Here, "$\subseteq$" means the usual set-inclusion. Under this ordering $NC(n)$ is a lattice with its minimal element $0_n = \{(1), \ldots, (n)\}$ and its maximal element $1_n = \{(1, \ldots, n)\}$, for all $n \in \mathbb{N}$.

Let A_1 and A_2 be $*$-subalgebras of a topological unital $*$-algebra A. We say that they are free over B if all mixed $*$-cumulants of A_1 and A_2 vanish. Let S_1 and S_2 be subsets of A. We say these subsets S_1 and S_2 are free over B if the topological $*$-subalgebras $*\text{-}A\lg\ \{S_1,\ B\}$ and $*\text{-}A\lg\ \{S_2,\ B\}$ are free over B. In particular, the B-valued random variables a_1 and a_2 are free over B if the subsets $\{a_1\}$ and $\{a_2\}$ are free over B in $(A,\ E)$.

In [6], we considered Free Probability on direct product of noncommutative probability spaces, as a generalized model for (scalar-valued) noncommutative probability spaces. This new model is in fact not so general to cover amalgamated noncommutative probability theory, but it provides the extended model which has the same or similar properties with scalar-valued noncommutative probability spaces. Moreover, as we can see in this paper, the (scalar-valued) free stochastic calculus is naturally extendable to our amalgamated free stochastic calculus on this model.

Let $N \in \mathbb{N}$ and let

$$\mathcal{F} = \{(A_i, \varphi_i) : i = 1, \dots, N\}.$$

be the finite family of W^*-probability spaces $(A_1,\ \varphi_1), \dots, (A_N,\ \varphi_N)$, where $A_1, \dots, A_N$ are von Neumann algebras and $\varphi_1, \dots, \varphi_N$ are continuous linear functionals satisfying that $\varphi_i(a_i^*) = \overline{\varphi_i(a_i)}$, for all $a_i \in A_i$ and $i = 1, \dots, N$. Define the direct product A of $A_1, \dots, A_N$ by $A = \oplus_{j=1}^{N} A_j$, where "$\oplus$" means the topological direct product.

Now, define the diagonal matricial algebra D_N in the matricial algebra $M_N(\mathbb{C})$ by

$$D_N = \mathbb{C}[\{e_{11}, \dots, e_{NN}\}],$$

where $\{e_{ij} : i,\ j = 1, \dots, N\}$ is the canonical matrix units of $M_N(\mathbb{C})$. Then D_N is clearly a $*$-subalgebra of A and hence we can define the conditional expectation $E : A \to D_N$ by

$$E\left(\oplus_{j=1}^{N} a_j\right) = \begin{pmatrix} \varphi_1(a_1) & & 0 \\ & \ddots & \\ 0 & & \varphi_N(a_N) \end{pmatrix},$$

for all $\oplus_{j=1}^{N} a_j \in A$. Then the pair $(A,\ E)$ is a D_N-valued W^*-probability space. We call it a direct producted W^*-probability space over D_N. Notice that $D_N' \cap D_N = D_N$, where D_N' is defined to be a $*$-subalgebra $\{d \in D_N : ad = da,\ a \in A\}$.

In Chapter 1, we will define direct producted W^*-probability spaces and consider certain D_N-valued random variables in them. In particular, we will consider the D_N-semicircular elements and D_N-valued infinitely divisible elements. Direct producted W^*-probability spaces have very similar free probabilistic structure with (scalar-valued) W^*-probability spaces. The reason is that the subalgebra D_N commutes with the given direct producted von Neumann algebra $A = \oplus_{j=1}^{N} A_j$. We can observe that D_N-valued random variables $x = \oplus_{j=1}^{N} x_j$ and $y = \oplus_{j=1}^{N} y_j$ in $(A,\ E)$ are free over D_N if and only if random variables x_j and y_j are free in $(A_j,\ \varphi_j)$, for all $j = 1, \dots, N$. This result will be proven by the following cumulant computation,

$$k_n\left(\oplus_{j=1}^{N} x_j^{(1)}, \dots, \oplus_{j=1}^{N} x_j^{(n)}\right) = \oplus_{j=1}^{N} k_n^{(j)}\left(x_j^{(1)}, \dots, x_j^{(n)}\right),$$

for all $n \in \mathbb{N}$, where $k_n^{(j)}(\dots)$ means the cumulant with respect to φ_j on A_j, for $j = 1, \dots, N$. Also, by this computation, we can conclude that if a D_N-valued random variable $x = \oplus_{j=1}^{N} x_j$ in a direct producted W^*-probability space $(A,$

E) over D_N is D_N-semicircular (or D_N-valued infinitely divisible) if and only if all nonzero x_j's are semicircular (resp. infinitely divisible) in (A_j, φ_j), for $j = 1, \ldots, N$.

In Chapter 2, we will consider the D_N-valued free stochastic integration on (A, E) and the amalgamated Itô's formula. It is easily verified that we would have the similar results as in the scalar-valued case. Indeed, we have that if $u \in A \otimes_{D_N} A$ is a D_N-valued simple adapted biprocess and if x is a D_N-valued infinitely divisible element in A, then the D_N-valued stochastic integral $\int_0^\infty u \#_{D_N} dx$ is similarly defined like the classical free stochastic integrals and it satisfies that

$$\int_0^\infty u \#_{D_N} dx = \oplus_{j=1}^N \left(\int_0^\infty u_j \# dx_j \right),$$

where $u = \oplus_{j=1}^N u_j$ and $x = \oplus_{j=1}^N x_j$, and where u_j are simple adapted biprocesses in $A_j \otimes_{\mathbb{C}} A_j$ and x_j are infinitely divisible element in A_j, for $j = 1, \ldots, N$. Thus we have the D_N-valued Itô's formula

$$d(I_1 \cdot I_2) = dI_1 \cdot I_2 + I_1 \cdot dI_2 + dI_1 \cdot dI_2,$$

where $I_1 = \int_0^\infty u_1 \#_{D_N} dx$ and $I_2 = \int_0^\infty u_2 \#_{D_N} dx$, for D_N-valued simple adapted biprocesses u_1 and u_2 in $A \otimes_{D_N} A$. In the final section, we observe the D_N-valued free stochastic integral of simple adapted biprocesses with respect to a D_N-free Brownian motion. A D_N-free Brownian motion can be regarded as a D_N-valued free distribution of a D_N-semicircular infinitely divisible random variable in a certain D_N-valued W^*-probability space.

1. Direct producted noncommutative probability spaces

Throughout this chapter, let's fix $N \in \mathbb{N} \setminus \{1\}$ (possibly $N \to \infty$) and the collection $\mathcal{F}$ of (scalar-valued) W^*-probability spaces,

$$\mathcal{F} = \{(A_i, \varphi_i) : i = 1, \ldots, N\},$$

where $\varphi_j : A_j \to \mathbb{C}$ is a continuous linear functional on A_j satisfying that $\varphi_j(a^*) = \overline{\varphi_j(a)}$, for all $a \in A_j$ and for all $j = 1, \ldots, N$. Then, for the given von Neumann algebras $A_1, \ldots, A_N$, we can define the direct product von Neumann algebra

$$A = \oplus_{j=1}^N A_j = \{\oplus_{j=1}^N a_j : a_j \in A_j, j = 1, \ldots, N\}.$$

By the product topology, the direct producted algebra A of $A_1, \ldots, A_N$ is again a von Neumann algebra even though $N = \infty$. Define a W^*-subalgebra D_N of A by $D_N = \mathbb{C}^{\oplus N}$. We will call this subalgebra the N-th diagonal subalgebra of A.

Definition. Define the direct producted W^*-probability space A of $\mathcal{F}$ by the W^*-probability space (A, E) with amalgamation over the N-th diagonal subalgebra D_N, where $A = \oplus_{j=1}^N A_j$ is the direct product of $A_1, \ldots, A_N$ and E :

$A \to D_N$ is the conditional expectation from A onto D_N defined by

$$E\left(\oplus_{j=1}^{N} a_j\right) = \oplus_{j=1}^{N} \varphi_j(a_j),$$

for all $\oplus_{j=1}^{N} a_j \in A$. Sometimes, we will denote E by $\oplus_{j=1}^{N} \varphi_j$.

It is easy to see that the $\mathbb{C}$-linear map E is indeed a conditional expectation;

(i) $E\left(\oplus_{j=1}^{N} \alpha_j\right) = \oplus_{j=1}^{N} \varphi_j(\alpha_j) = \oplus_{j=1}^{N} \alpha_j,$

for all $\oplus_{j=1}^{N} \alpha_j \in D_N$.

(ii)

$$\begin{aligned} & E\left(\left(\oplus_{j=1}^{N} \alpha_j\right)\left(\oplus_{j=1}^{N} a_j\right)\left(\oplus_{j=1}^{N} \alpha_j'\right)\right) \\ &= E\left(\oplus_{j=1}^{N} \alpha_j a_j \alpha_j'\right) = \left(\oplus_{j=1}^{N} \varphi_j\right)\left(\oplus_{j=1}^{N} \alpha_j a_j \alpha_j'\right) \\ &= \oplus_{j=1}^{N} \varphi_j\left(\alpha_j a_j \alpha_j'\right) = \oplus_{j=1}^{N}\left(\alpha_j \varphi_j(a_j) \alpha_j'\right) \\ &= \left(\oplus_{j=1}^{N} \alpha_j\right) \cdot \left(\oplus_{j=1}^{N} \varphi_j(a_j)\right) \cdot \left(\oplus_{j=1}^{N} \alpha_j'\right) \\ &= \left(\oplus_{j=1}^{N} \alpha_j\right)\left(E\left(\oplus_{j=1}^{N} a_j\right)\right)\left(\oplus_{j=1}^{N} \alpha_j'\right), \end{aligned}$$

for all $\oplus_{j=1}^{N} \alpha_j$, $\oplus_{j=1}^{N} \alpha_j' \in D_N$ and $\oplus_{j=1}^{N} a_j \in A$.

(iii) $E\left(\left(\oplus_{j=1}^{N} a_j\right)^*\right) = E\left(\oplus_{j=1}^{N} a_j^*\right) = \oplus_{j=1}^{N} \varphi_j\left(a_j^*\right) = \oplus_{j=1}^{N} \overline{\varphi_j(a_j)}$
$= \left(\oplus_{j=1}^{N} \varphi_j(a_j)\right)^* = E\left(\oplus_{j=1}^{N} a_j\right)^*,$

for all $\oplus_{j=1}^{N} a_j \in A$. By (i), (ii) and (iii), the map E is a conditional expectation from $A = \oplus_{j=1}^{N} A_j$ onto D_N. Thus the algebraic pair (A, E) is a W^*-probability space with amalgamation over the N-th diagonal algebra D_N. Since $N \in \mathbb{N}$ and since φ_j's are continuous on A_j, for all $j = 1, \ldots, N$, the conditional expectation E is also continuous under the product topologies of $\oplus_{j=1}^{N} A_j$ and D_N.

1.1. D_N-freeness on direct producted W^*-probability spaces

Now, we will consider the D_N-freeness on the direct producted W^*-probability space (A, E). Notice that the N-th diagonal algebra D_N satisfies that

$$dx = xd, \text{ for all } d \in D_N \text{ and } x \in A. \tag{1.1}$$

The following proposition is the direct consequence from the definition of the conditional expectation $E = \oplus_{j=1}^{N} \varphi_j$ on A.

Proposition 1.1. *Let (A, E) be the direct producted W^*-probability space over the N-th diagonal algebra D_N and let $x_k = \oplus_{j=1}^{N} a_j^{(k)}$ be the D_N-valued random variables in (A, E), for $k = 1, \ldots, n$, for all $n \in \mathbb{N}$. Then*

$$E(x_1 \cdots x_n) = \oplus_{j=1}^{N} \varphi_j\left(a_1^{(j)} \cdots a_n^{(j)}\right), \tag{1.2}$$

for all $n \in \mathbb{N}$.

Consider the partition-depending moments of $x_k = \oplus_{j=1}^{N} a_j^{(k)}$, for $k = 1, \ldots, n$, for all $n \in \mathbb{N}$. Suppose $\pi \in NC(n)$. To compute the partition-depending moment $E_\pi(x_1, \ldots, x_n)$, we need to compute all block-depending moment $E_V(x_1, \ldots, x_n)$, for all blocks V in π. Since $E_V(x_1, \ldots, x_n)$ are contained in D_N, for all $V \in \pi$, we have that

$$E_\pi(x_1, \ldots, x_n) = \prod_{V \in \pi} E_V(x_1, \ldots, x_n),$$

by (1.1). In other words, we do not need to consider the insertion property for computing the partition-depending moments (See [13] for Insertion Property). Now, let's compute the n-th cumulant of an arbitrary D_N-valued random variable;

Proposition 1.2. *Let $(A,\ E)$ be the direct producted W^*-probability space over the N-th diagonal algebra D_N and let $x_k = \oplus_{j=1}^N a_j^{(k)}$ be D_N-valued random variables in $(A,\ E)$, for $n \in \mathbb{N}$. Then*

$$k_n(x_1, \ldots, x_n) = \oplus_{j=1}^N k_n^{(j)}\left(a_1^{(j)}, \ldots, a_n^{(j)}\right),$$

for all $n \in \mathbb{N}$, where $k_n^{(i)}(\cdots)$ is the n-th cumulant functional with respect to the noncommutative probability space $(A_i,\ \varphi_i)$, for all $i = 1, \ldots, N$.

Proof. Fix $n \in \mathbb{N}$. Then

$$\begin{aligned} k_n(x_1, \ldots, x_n) &= \sum_{\pi \in NC(n)} E_\pi(x_1, \ldots, x_n)\mu(\pi, 1_n) \\ &= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} E_V(x_1, \ldots, x_n)\right)\mu(\pi, 1_n) \end{aligned} \tag{1.3}$$

by (1.1), where $E_V(x, \ldots, x) = E(x_{i_1} \ldots x_{i_k})$, whenever $V = (i_1, \ldots, i_k)$ in

$$\begin{aligned} \pi &= \sum_{\pi \in NC(n)} \left(\prod_{V=(i_1,\ldots,i_k)\in\pi} \left(\oplus_{j=1}^N \varphi_j(a_{i_1}^{(j)} \ldots a_{i_k}^{(j)})\right)\right)\mu(\pi, 1_n) \\ &= \sum_{\pi \in NC(n)} \left(\oplus_{j=1}^N \left(\prod_{V=(i_1,\ldots,i_k)\in\pi} \varphi_j(a_{i_1}^{(j)} \ldots a_{i_k}^{(j)})\right)\right)\mu(\pi, 1_n) \\ &= \sum_{\pi \in NC(n)} \left(\oplus_{j=1}^N \left(\prod_{V=(i_1,\ldots,i_n)\in\pi} \varphi_j(a_{i_1}^{(j)} \ldots a_{i_k}^{(j)})\ \mu(\pi, 1_n)\right)\right) \\ &= \oplus_{j=1}^N \left(\sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \varphi_{j:V}(a_1^{(j)}, \ldots, a_n^{(j)})\right)\mu(\pi, 1_n)\right) \end{aligned}$$

where $\varphi_{j\,:\,V}(a_1^{(j)}, \ldots, a_n^{(j)})$ means the block-depending moment of $a_1^{(j)}, \ldots, a_n^{(j)}$, for

$$j = 1, \ldots, N = \oplus_{j=1}^N \left(k_n^{(j)}(a_1^{(j)}, \ldots, a_n^{(j)})\right), \tag{1.4}$$

where $k_n^{(j)}(\cdots)$ is the (scalar-valued) cumulant with respect to the linear functional φ_j on A_j, for all $j = 1, \ldots, N$. ☐

As a corollary, the n-th D_N-valued cumulant

$$k_n\left(\oplus_{j=1}^N a_j, \dots, \oplus_{j=1}^N a_j\right)$$

of a D_N-valued random variable $\oplus_{j=1}^N a_j$ in the direct producted W^*-probability space $(A,\ E)$, is nothing but the direct sum of n-th (scalar-valued) cumulants $k_n^{(j)}(a_j, \dots, a_j)$ of a_j, for $j = 1, \dots, N$,

$$\oplus_{j=1}^N \left(k_n^{(j)}(a_j, \dots, a_j)\right).$$

By (1.4), we can get the following D_N-freeness characterization on the direct producted W^*-probability space $(A,\ E)$.

Theorem 1.3. *Let (A, E) be the given direct producted W^*-probability space over the N-th diagonal algebra D_N and let $x_1 = \oplus_{j=1}^N a_j$ and $x_2 = \oplus_{j=1}^N b_j$ be the D_N-valued random variables in $(A,\ E)$. Then x_1 and x_2 are free over D_N in $(A,\ E)$ if and only if a_j and b_j are free in $(A_j,\ \varphi_j)$, for all $j = 1, \dots, N$.*

Proof. ($\Leftarrow$) Assume that random variables a_j and b_j are free in $(A_j,\ \varphi_j)$, for all $j = 1, \dots, N$. Then the W^*-subalgebras $vN(a_j)$ and $vN(b_j)$ are free in $(A_j,\ \varphi_j)$, for $j = 1, \dots, N$, where $vN(x)$ means the von Neumann algebra generated by $\{x\}$. Since

$$x_1^{u_1\ n} = \left(\oplus_{j=1}^N a_j\right)^{u_1\ n} = \oplus_{j=1}^N a_j^{u_1\ n}$$

and

$$x_2^{u_2\ n} = \left(\oplus_{j=1}^N b_j\right)^{u_2\ n} = \oplus_{j=1}^N b_j^{u_2\ n},$$

for all $n \in \mathbb{N}$ and $u_1,\ u_2 \in \{1, *\}$, we have that

$$k_n\left(x_{i_1}^{u_{i_1}\ m}, \dots, x_{i_n}^{u_{i_n}\ m}\right) = \oplus_{j=1}^N k_n^{(j)}\left(a_{j:i_1}^{u_{i_1}\ m}, \dots, a_{j:i_n}^{u_{i_n}\ m}\right)$$

for all $(i_1, \dots, i_n) \in \{1, 2\}^n$, $n, m \in \mathbb{N}$, where $a_{j:i_k} = a_j$ if $i_k = 1$, and $a_{j\,:\,i_k} = b_j$, if $i_k = 2$, for all $k = 1, \dots, n$. If the above D_N-valued cumulant is mixed, then each $k_n^{(j)}\ (a_{j:i_1}^{u_{i_1}}, \dots, a_{j:i_n}^{u_{i_n}})$ vanishes, by hypothesis, for $j = 1, \dots, N$. Moreover, if $d_1, \dots, d_n \in D_N$ are arbitrarily chosen, then

$$k_n\left(d_1 x_{i_1}^{u_{i_1}\ m}, \dots, d_n x_{i_n}^{u_{i_n}\ m}\right) = (d_1 \cdots d_n)\, k_n\left(x_{i_1}^{u_{i_1}\ m}, \dots, x_{i_n}^{u_{i_n}\ m}\right),$$

by (1.1) and (1.4). Therefore, all mixed D_N-valued cumulant of $vN\ (\{x_1\},\ D_N)$ and $vN\ (\{x_2\},\ D_N)$ vanish, too, and hence $vN(\{x_1\},\ D_N)$ and $vN(\{x_2\},\ D_N)$ are free over D_N in $(A,\ E)$.

($\Rightarrow$) Let's assume that the D_N-valued random variables $x_1 = \oplus_{j=1}^N a_j$ and $x_2 = \oplus_{j=1}^N b_j$ are free over D_N in $(A,\ E)$ and assume also that there exists j_0 in $\{1, \dots, N\}$ such that $a'_{j_0} \in vN(a_{j_0})$ and $b'_{j_0} \in vN(b_{j_0})$ are not free in $(A_{j_0},\ \varphi_{j_0})$. Suppose that there is $n \in \mathbb{N}$ such that $k_n(x'_{j_0:i_1}, \dots, x'_{j_0:i_n}) = \kappa \neq 0$ in $\mathbb{C}$, where $x'_{j_0:i_1}, \dots, x'_{j_0:i_n} \in \{a'_{j_0},\ b'_{j_0}\}$ are mixed. Then the mixed D_N-valued cumulants of $x'_1 = \oplus_{j=1}^N a'_j$ in $vN(\{x_1\},\ D_N)$ and $x'_2 = \oplus_{j=1}^N b'_j$ in $vN(\{x_2\},\ D_N)$, where the j_0-summands of x_1' and x_2' are a_{j_0}' and b_{j_0}', respectively, satisfying

$$\begin{aligned} k_n\left(x_{i_1}', \ldots, x_{i_n}'\right) &= k_n\left(\oplus_{j=1}^N a_{j:i_1}', \ldots, \oplus_{j=1}^N a_{j:i_n}'\right) \\ &= \oplus_{j=1}^N \left(k_n^{(j)}(a_{j:i_1}', \ldots, a_{j:i_n}')\right) \\ &= \kappa_1 \oplus \cdots \oplus \kappa_{j_0-1} \oplus \underset{j_0\text{-th}}{\kappa} \oplus \kappa_{j_0+1} \oplus \cdots \oplus \kappa_N \neq 0_{D_N}, \end{aligned}$$

where $\kappa_i \in \mathbb{C}$. Therefore, the D_N-valued cumulant of x_1' and x_2' does not vanish. This contradict our assumption that D_N-valued random variables x_1 and x_2 are free over D_N in $(A,\, E)$. □

The above theorem shows that the D_N-freeness of $\oplus_{j=1}^N\, a_j$ and $\oplus_{j=1}^N\, b_j$ in the direct producted W^*-probability space $(A,\, E)$ is completely characterized by the (scalar-valued) freeness of a_j and b_j in $(A_j,\, \varphi_j)$, for all $j = 1, \ldots, N$.

Corollary 1.4. *Let* $e_{a_i} = 0 \oplus \cdots \oplus 0 \oplus a_i \oplus 0 \oplus \cdots \oplus 0$ *and* $e_{a_j} = 0 \oplus \cdots \oplus 0 \oplus a_j \oplus 0 \oplus \cdots \oplus 0$ *in* $(A,\, E)$, *where* $a_k \in (A_k,\, \varphi_k)$, *for* $k = i,\, j$. *Then* e_i *and* e_j *are free over* D_N *in* $(A,\, E)$, *whenever* $i \neq j$.

Define subalgebras $A_1', \ldots, A_N'$ of the direct product $A = \oplus_{j=1}^N\, A_j$ by

$$A_j' = \{(0 \oplus \cdots \oplus 0 \oplus a_j \oplus 0 \oplus \cdots \oplus 0) : a_j \in (A_j, \varphi_j)\},$$

for all $j = 1, \ldots, N$. Then A_j' is the embedding of A_j in A. By the previous corollary, we can easily get the following corollary.

Corollary 1.5. *The algebras* $A_1, \ldots, A_N$ *are free over* D_N *in the direct producted* W^**-probability space* $(A,\, E)$. *In particular,*

$$\underset{j=1}{\overset{N}{*_{D_N}}} A_j = \oplus_{j=1}^N A_j, \text{ in } (A, E),$$

where "$*_{D_N}$*" means the amalgamated free product over* D_N.

Proof. By the previous corollary, A_i' are free from each other, for $i = 1, \ldots, N$. Since A_i' is $*$-isomorphic to A_i in A, the von Neumann algebras A_i are free from each other over D_N in $(A,\, E)$. By Voiculescu,

$$\underset{j=1}{\overset{N}{*_{D_N}}} A_j = D_N \oplus \left(\oplus_{n=1}^\infty \left(\bigoplus_{i_1\neq i_2, i_2\neq i_3, \ldots, i_{n-1}\neq i_n} (A_{i_1}^o \otimes \cdots \otimes A_{i_n}^o)\right)\right),$$

where $A_{i_k}^o = A_{i_k} \ominus D_N$. Suppose $i \neq j$ in $\{1, \ldots, N\}$. Then $A_i^o \otimes A_j^o = O$, where O means the zero-space generated by 0_{D_N}. Therefore,

$$\underset{j=1}{\overset{N}{*_{D_N}}} A_j = D_N \oplus \left(\oplus_{j=1}^N A_j^o\right) = \oplus_{j=1}^N A_j.$$

□

1.2. Certain random variables in a direct producted W^*-probability spaces

Throughout this section, we also fix $N \in \mathbb{N}$ and let $(A_j,\ \varphi_j)$ be a W^*-probability space, for $j = 1, \ldots, N$. Recall that φ_j is called a trace if $\varphi_j\ (a\ b) = \varphi_j\ (b\ a)$, for all a, b in A_j, for $j = 1, \ldots, N$. In this case, a W^*-probability space $(A_j,\ \varphi_j)$ is said to be a tracial W^*-probability space. The direct producted W^*-probability space $(A,\ E)$ over the N-th diagonal algebra D_N is defined as before, where $A = \oplus_{j=1}^{N}\ A_j$ and $E = \oplus_{j=1}^{N}\ \varphi_j$. The conditional expectation E is said to be tracial if it satisfies that $E(xy) = E(yx)$, for all $x,\ y \in (A,\ E)$. In this case, we also say that this amalgamated W^*-probability space $(A,\ E)$ is tracial over D_N. It is easily verified that E is tracial if and only if linear functionals φ_j are traces, for all $j = 1, \ldots, N$. Indeed,

$$
\begin{aligned}
E\left((\oplus_{j=1}^{N} a_j)(\oplus_{j=1}^{N} b_j)\right) &= E\left(\oplus_{j=1}^{N} a_j b_j\right) = \oplus_{j=1}^{N} \varphi_j(a_j b_j) \\
&= \oplus_{j=1}^{N} \varphi_j(b_j a_j) = E\left(\oplus_{j=1}^{N} b_j a_j\right) \\
&= E\left((\oplus_{j=1}^{N} b_j)(\oplus_{j=1}^{N} a_j)\right).
\end{aligned}
$$

Equivalently, the direct producted W^*-probability space $(A,\ E)$ is tracial if and only if W^*-probability spaces $(A_j,\ \varphi_j)$ are tracial, for all $j = 1, \ldots, N$. Let's consider certain D_N-valued random variables in a tracial direct producted W^*-probability space.

Definition. Let $(A,\ E)$ be a tracial direct producted W^*-probability space over its N-th diagonal subalgebra D_N. A D_N-valued random variable $x \in (A,\ E)$ is said to be D_N-semicircular if it is self-adjoint and the only nonvanishing D_N-valued cumulant $k_n(x, \ldots, x)$ of x is the second one $k_2(x,\ x)$.

Let $B \subset A$ be von Neumann algebras with $1_A = 1_B$ and assume that there exists a (tracial) conditional expectation $F : A \to B$. Then we have an amalgamated W^*-probability space $(A,\ F)$ over B. In such general case, we can define the B-semicircularity of B-valued random variables as follows; a self-adjoint B-valued random variable x in $(A,\ F)$ is B-semicircular if the following B-valued cumulant relation holds,

$$
k_n^{(F)}(b_1 x, b_2 x, \ldots, b_n x) = \begin{cases} k_2^{(F)}(b_1 x,\ b_2 x) & \text{if } n = 2 \\ 0_B & \text{otherwise,} \end{cases}
$$

for all $n \in \mathbb{N}$, where $b_1, \ldots, b_n \in B$ are arbitrary, where $k_n^{(F)}(\cdots)$ is the B-valued cumulant with respect to F. When B commutes with A, we can have that

$$
k_n^{(F)}\left(b_1 x, \ldots, b_n x\right) = \left(b_1 \cdots b_n\right) k_n^{(F)}\left(x, \ldots, x\right),
$$

for all $x \in A$ and $n \in \mathbb{N}$, where $b_1, \ldots, b_n \in B$ are arbitrary. By (1.1), the N-th diagonal subalgebra D_N commutes with our direct producted von Neumann algebra $\oplus_{j=1}^{N} A_j$. Therefore, if $x \in (A,\, E)$, then

$$k_n\left(d_1 x, \ldots, d_n x\right) = \left(d_1 \cdots d_n\right) k_n(x, \ldots, x),$$

for all $n \in \mathbb{N}$, where $d_1, \ldots, d_n \in D_N$ are arbitrary. Thus the D_N-semicircularity defined in the above definition is well-defined.

Theorem 1.6. *Let $a_j \in (A_j,\, \varphi_j)$ be self-adjoint random variables, for all $j = 1, \ldots, N$. Then the self-adjoint D_N-valued random variable $x = \oplus_{j=1}^{N} a_j$ in $(A,\, E)$ is D_N-semicircular if and only if all nonzero a_j's are semicircular in $(A_j,\, \varphi_j)$, for $j = 1, \ldots, N$.*

Proof. ($\Leftarrow$) By (1.4), it is clear.

($\Rightarrow$) Without loss of generality, assume that $a_j \neq 0_{A_j}$, in $(A_j,\, \varphi_j)$, for all $j = 1, \ldots, N$, and suppose that the D_N-valued random variable $x = \oplus_{j=1}^{N} a_j$ is D_N-semicircular in $(A,\, E)$. Let's assume that there exists $j_0 \in \{1, \ldots, N\}$ such that a_{j_0} is not semicircular in $(A_{j_0},\, \varphi_{j_0})$. Then there exists nonvanishing n_0-th cumulant $k_{n_0}^{(j_0)}(a_{j_0}, \ldots, a_{j_0}) = \kappa_{n_0} \neq 0$ of a_{j_0}, in $\mathbb{C}$, where $n_0 \neq 2$ in $\mathbb{N}$. observe that

$$\begin{aligned} k_{n_0}\left(x, \ldots, x\right) &= \oplus_{j=1}^{N} k_{n_0}^{(j)}(a_j, \ldots, a_j) \\ &= 0 \oplus \cdots \oplus 0 \oplus \underset{j_0\text{-th}}{\kappa_{n_0}} \oplus 0 \oplus \cdots \oplus 0 \\ &\neq 0_{D_N}. \end{aligned}$$

This contradict our assumption that x is D_N-semicircular. □

We will say that D_N-valued random variables x and y in $(A,\, E)$ are identically distributed over D_N if

$$k_n\left(x^{u_1}, \ldots, x^{u_n}\right) = k_n(y^{u_1}, \ldots, y^{u_n}), \text{ for all } n \in \mathbb{N},$$

or equivalently, $E\left(x^{u_1} \cdots x^{u_n}\right) = E(y^{u_1} \cdots y^{u_n})$, for all $n \in \mathbb{N}$, where $u_1, \ldots, u_n \in \{1,\, *\}$.

Definition. Let $(A,\, E)$ be a tracial direct producted W^*-probability space over D_N, where $A = \oplus_{j=1}^{N} A_j$ and $E = \oplus_{j=1}^{N} \varphi_j$ with respect to tracial W^*-probability spaces $(A_j,\, \varphi_j)$, for $j = 1, \ldots, N$. The self-adjoint D_N-valued random variable $x = \oplus_{j=1}^{N} a_j$ is called a D_N-valued infinitely divisible random variable if, for any $m \in \mathbb{N}$, there always exist nonzero self-adjoint D_N-valued random variables $x_{m,1}, \ldots, x_{m,m}$ in $(A,\, E)$ such that (i) $x_{m,1}, \ldots, x_{m,m}$ are free from each other over D_N in $(A,\, E)$, for fixed $m \in \mathbb{N}$, (ii) they are identically distributed over D_N and (iii) x satisfies the following D_N-valued cumulant relation,

$$k_n\left(\underbrace{x,\ldots\ldots,x}_{n\text{-times}}\right) = k_n\left(\underbrace{\sum_{k=1}^{m} x_{m,k},\ldots,\sum_{k=1}^{m} x_{m,k}}_{n\text{-times}}\right)$$

$$= m\cdot k_n\left(\underbrace{x_{m,k_0},\ldots,x_{m,k_0}}_{n\text{-times}}\right),$$

for all $n \in \mathbb{N}$, where $k_0 = 1$ or 2 or $\cdots$ or m.

The second equality of (iii) in the previous definition holds true automatically by (i) and (ii). i.e., if $x = \oplus_{j=1}^{N} a_j$ is D_N-valued infinitely divisible, then

$$\begin{aligned} k_n(x,\ldots,x) &= k_n\left(\sum_{k=1}^{m} x_{m,k},\ldots,\sum_{k=1}^{m} x_{m,k}\right) \\ &= \sum_{k=1}^{m} k_n(x_{m,k},\ldots,x_{m,k}) \end{aligned}$$

(by the D_N-valued infinitely divisibility)

by the D_N-freeness of $x_{m,1},\ldots,x_{m,m}$, for $m = m\cdot k_n(x_{m,1},\ldots,x_{m,1})$ by the identically distributedness over D_N of $x_{m,1},\ldots,x_{m,m}$, for all $m \in \mathbb{N}$. By this observation, we can characterize the D_N -valued infinitely divisibility on (A, E), in terms of the (scalar-valued) infinitely divisibility on (A_j, φ_j), for $j = 1,\ldots,N$.

Theorem 1.7. *The D_N-valued random variable $x = \oplus_{j=1}^{N} a_j$ in (A, E) is D_N-valued infinitely divisible if and only if all nonzero a_j are infinitely divisible in (A_j, φ_j), for $j = 1,\ldots,N$.*

Proof. ($\Leftarrow$) Suppose all nonzero a_j's are infinitely divisible in (A_j, φ_j), for $j = 1,\ldots,N$. Then, by the (scalar-valued) infinitely divisibility, the free distribution μ_{a_j} of a_j is infinitely divisible and hence, for any $m \in \mathbb{N}$, there exists a free distribution $\mu_{m:j}$ such that $\mu_{a_j} = m\cdot\mu_{m:j}$, for all $j = 1,\ldots,N$. Thus, there exists a random variable $a_{m:j}$ in (A_j, φ_j) such that $k_n^{(j)}(a_j,\ldots,a_j) = m\cdot k_n^{(j)}(a_{m:j},\ldots,a_{m:j})$, for all $n \in \mathbb{N}$.

So, there exists $a_{m:j}^{(1)},\ldots,a_{m:j}^{(m)}$ such that they are free from each other in (A_j, φ_j) and they are identically distributed (over $\mathbb{C}$) in (A_j, φ_j) and $k_n^{(j)}(a_j,\ldots,a_j) = \sum_{k=1}^{m} k_n^{(j)}\left(a_{m:j}^{(k)},\ldots,a_{m:j}^{(k)}\right)$, for all $n \in \mathbb{N}$.

By (1.4), we have that

$$\begin{aligned} k_n(x, \ldots, x) &= \oplus_{j=1}^{N} k_n^{(j)}(a_j, \ldots, a_j) \\ &= \oplus_{j=1}^{N} \left(\sum_{k=1}^{m} k_n^{(j)}(a_{m:j}^{(k)}, \ldots, a_{m:j}^{(k)}) \right) \\ &= \sum_{k=1}^{m} \left(\oplus_{j=1}^{N} k_n^{(j)}(a_{m:j}^{(k)}, \ldots, a_{m:j}^{(k)}) \right) \\ &= \sum_{k=1}^{m} k_n \left(\oplus_{j=1}^{N} a_{m:j}^{(k)}, \ldots, \oplus_{j=1}^{N} a_{m:j}^{(k)} \right), \end{aligned}$$

for all m, $n \in \mathbb{N}$. Notice that the D_N-valued random variables $\oplus_{j=1}^{N} a_{m:j}^{(1)}, \ldots, \oplus_{j=1}^{N} a_{m:j}^{(m)}$ are free over D_N in (A, E), and also they are identically distributed over D_N, by the identically distributedness of $a_{m:j}^{(1)}, \ldots, a_{m:j}^{(m)}$ in (A_j, φ_j), for all $j = 1, \ldots, N$. Therefore, the D_N-valued random variable x is D_N-valued infinitely divisible in (A, E), by the existence of $\oplus_{j=1}^{N} a_{m:j}^{(1)}, \ldots, \oplus_{j=1}^{N} a_{m:j}^{(m)}$, for all $m \in \mathbb{N}$.

($\Rightarrow$) Suppose that $x = \oplus_{j=1}^{N} a_j$ is D_N-valued infinitely divisible in (A, E) and assume that there exists at least one $k \in \{1, \ldots, N\}$ such that a nonzero a_k is not infinitely divisible. It is easy to check that x cannot be D_N-valued infinitely divisible in (A, E). □

In Chapter 2, we will consider certain D_N-valued infinitely divisible elements having their D_N-valued free distributions as the D_N-valued free Brownian motions. In fact, the D_N-valued infinitely divisible semicircular elements in the direct producted W^*-probability space (A, E) have their D_N-free distributions as D_N-valued free Brownian motions.

2. D_N-valued stochastic calculus

Throughout this chapter, we let (A_j, φ_j) be tracial W^*-probability spaces, for all $j = 1, \ldots, N$, and let (A, E) be the direct producted W^*-probability space with $A = \oplus_{j=1}^{N} A_j$ and $E = \oplus_{j=1}^{N} \varphi_j$. In Section 1.2, we showed that a self-adjoint D_N-valued random variable $x = \oplus_{j=1}^{N} a_j$ is D_N-valued infinitely divisible if and only if all nonzero a_j are (scalar-valued) infinitely divisible in (A_j, φ_j), for $j = 1, \ldots, N$. In this chapter, we will consider the D_N-valued stochastic calculus of simple adapted biprocesses in $A \otimes A$, for the fixed stochastic process, as a D_N-valued infinitely divisible element in A.

2.1. Definitions

In this section, we will define D_N-valued simple adapted biprocesses and the D_N-valued stochastic integral. Also, we will consider their basic properties. Throughout this section, we will let $A = \oplus_{j=1}^{N} A_j$ be a direct producted von

Neumann algebra and let $A \otimes_{D_N} A$ be the D_N-tensor product of two copies of A. For convenience, we denote $A \otimes_{D_N} A$ just by $A \otimes A$, if there is no confusion.

Definition. Let $(A,\, E)$ be the direct producted W^*-probability space over its N-th diagonal subalgebra D_N. Let $u = (u_t)_{t \geq 0} \subset A \otimes A$, where $t \in \mathbb{R}^+ \cup \{0\}$. We call u a D_N-valued simple biprocess if $t \in \mathbb{R}^+ \mapsto u_t \in A \otimes A$ is a piecewise D_N-valued constant map such that $u_s = 0_{D_N} \otimes 0_{D_N}$, for $s > 0$ large enough.

Definition. A D_N-valued simple biprocess $u = (u_t)_{t \geq 0} \subset A \otimes A$ is adapted if each u_t is contained in $A_t \otimes A_t$, for each $t \geq 0$, where A_t are filters (W^*-subalgebras) of A having their common subalgebra D_N. The interval in $\mathbb{R}^+$ making $u_t \neq 0$ is called the adaptedness of u.

To have a D_N-valued simple biprocess, the D_N-tensor product $A \otimes A$ should be large enough to have its (at least finitely many) filters $A_t \otimes A_t$, for $t \in \mathbb{R}^+$. Equivalently, the von Neumann algebra $A = \oplus_{j=1}^N A_j$ should be large enough to have its filters A_t and hence A_j should be large enough to have their filters $A_{j\,:\,t}$, for $j = 1, \ldots, N$, where $t \in \mathbb{R}^+$. Notice that a filter A_t of A, for the fixed $t \in \mathbb{R}^+$, is $*$-equivalent to $\oplus_{j=1}^N A_{j\,:\,t}$, for all $j = 1, \ldots, N$. If $u = (u_t)_t \subset A \otimes A$ is a D_N-valued simple adapted biprocess, then it can be expressed by

$$u = \sum_{m=1}^{M} \left(\sum_{k=0}^{n} y_{t_m}^{(k)} \otimes w_{t_m}^{(k)} \right),$$

for $0 = t_0 < t_1 < \cdots < t_m$.

Definition. Define the action $\#_{D_N}$ of $A \otimes A$ on A by

$$(y \otimes w) \;\#_{D_N}\; a = yaw,$$

for all $a \in A$.

From now on, we will consider a D_N-valued infinitely divisible random variable x in $(A,\, E)$ as a net $(x_t)_t$. As we have seen in Section 1.2, if x is D_N-valued infinitely divisible, then there exists a sequence $(x_n)_{n=1}^{\infty}$ such that

$$k_m \left(\underbrace{x, \ldots\ldots, x}_{m\text{-times}} \right) = n \cdot k_m \left(\underbrace{x_n, \ldots\ldots, x_n}_{m\text{-times}} \right), \text{ for all } n, m \in \mathbb{N}.$$

Thus we can regard the D_N-valued infinitely divisible element x as a net $(x_t)_{t \geq 0}$, where $x_t = x_n$, if $t \in [n,\, n+1)$ and where $x_0 = 0_A$.

Definition. Let $u = (u_t)_t \subset A \otimes A$ be a D_N-valued simple adapted biprocess and let $x = (x_t)_t \in (A,\, E)$ be D_N-valued infinitely divisible. Define the D_N-valued stochastic integral $\int_0^\infty u \;\#_{D_N}\; dx$ of u in terms of x by

$$\int_0^\infty u \,\#_{D_N}\, dx = \sum_{m=1}^{M} \left(\sum_{k=0}^{n} y_{t_m}^{(k)} (x_{t_m} - x_{t_{m-1}}) w_{t_m}^{(k)} \right),$$

where $u = \sum_{m=1}^{M} \left(\sum_{k=0}^{n} y_{t_m}^{(k)} \otimes w_{t_m}^{(k)} \right)$.

The following theorem shows that $u = (u_t)$ is a D_N-valued simple adapted biprocess in $A \otimes_{D_N} A$ if and only if u is $\oplus_{j=1}^{N} u_j$ with scalar-valued simple adapted processes u_j in $A_j \otimes_{\mathbb{C}} A_j$, for all $j = 1, \ldots, N$.

Theorem 2.1. *The element $u = (u_t)_t$ is a D_N-valued simple adapted biprocess in $A \otimes A$, where $A = \oplus_{j=1}^{N} A_j$, if and only if $u = \oplus_{j=1}^{N} u_j$ with simple adapted biprocesses u_j in $A_j \otimes_{\mathbb{C}} A_j$, having the same adaptedness, for $j = 1, \ldots, N$.*

Proof. ($\Rightarrow$) Without loss of generality, if u is a D_N-valued simple adapted biprocess in $A \otimes A$, then we can regard it as $\sum_{m=1}^{M} (y_{t_m} \otimes w_{t_m})$. Then

$$
\begin{aligned}
u &= \sum_{m=1}^{M} (y_{t_m} \otimes w_{t_m}) \\
&= \sum_{m=1}^{M} \left(\left(\oplus_{j=1}^{N} y_{j:t_m} \right) \otimes \left(\oplus_{j=1}^{N} w_{j:t_m} \right) \right) \\
&= \sum_{m=1}^{M} \left(\oplus_{j=1}^{N} (y_{j:t_m} \otimes_{\mathbb{C}} w_{j:t_m}) \right) \\
&= \oplus_{j=1}^{N} \left(\sum_{m=1}^{M} (y_{j:t_m} \otimes_{\mathbb{C}} w_{j:t_m}) \right).
\end{aligned}
$$

We can define $u_j = \sum_{m=1}^{M} (y_{j:t_m} \otimes_{\mathbb{C}} w_{j:t_m})$ in A_j, for $j = 1, \ldots, N$. It is easy to see that since u is a D_N-valued simple adapted biprocess in $A \otimes A$, each u_j is a simple adapted biprocess in $A_j \otimes_{\mathbb{C}} A_j$, for $j = 1, \ldots, N$. Moreover, u_j's have the same adaptedness, for all $j = 1, \ldots, N$.

($\Leftarrow$) Since u_j in $A_j \otimes_{\mathbb{C}} A_j$ are simple adapted biprocesses having the same adaptedness, for $j = 1, \ldots, N$, the element $u = \oplus_{j=1}^{N} u_j$ is a D_N-valued simple adapted biprocess in $A \otimes A$. Also, the adaptedness of u and those of u_j's are identical. □

By the previous theorem, we can regard a D_N-valued simple adapted biprocess in $A \otimes A$ as a direct sum of (scalar-valued) simple adapted biprocesses u_j's in $A_j \otimes_{\mathbb{C}} A_j$, for $j = 1, \ldots, N$.

Notice that if u_j are simple adapted biprocesses in $A_j \otimes A_j$, for $j = 1, \ldots, N$, and if they have the nonempty intersection $J = \cap_{k=1}^{N} J_k$ of adaptednesses, where J_k are the adaptedness of u_k, then the operator $u = \oplus_{j=1}^{N} u_j$ is a D_N-valued simple adapted biprocess in $A \otimes A$ with its adaptedness J.

2.2. D_N-valued stochastic integrals

In this section, we will consider the D_N-valued version of Itô's stochastic integral product formula. We can recognize that the classical free stochastic integration of simple adapted biprocesses can be regarded as the D_N-free stochastic integration of them with $N = 1$, by Section 2.1. Let (M, τ) be a tracial (scalar-valued) W^*-probability space with its trace $\tau : M \to \mathbb{C}$. Assume that

x is a (scalar-valued) infinitely divisible element in (M, τ), in the sense that it is a D_N-valued infinitely divisible element when $N = 1$ (See Section 1.2). Suppose $u \in M \otimes M$ is a simple adapted biprocess, in the sense that u is D_N-valued simple adapted biprocess when $N = 1$. Then we can define the stochastic integral $\int_0^\infty u \# dx$ of u, as in the previous section. Here the action "#" is nothing but "$\#_{D_1}$". The Itô's free stochastic integral product formula is well-known.

Proposition 2.2. (Itô's Product Formula) *Let $x = (x_t)_t$ be an infinitely divisible element in a tracial W^*-probability space (M, τ) and let $u = (u_t)_t$ and $v = (v_t)_t$ be simple adapted biprocesses in $M \otimes M$. Then*

$$\begin{aligned} &\left(\int_0^\infty u \# dx\right)\left(\int_0^\infty v \# dx\right) \\ &= \int_0^\infty \left(\int_0^\infty u \# dx\right) v \# dx \\ &+ \int_0^\infty u \left(\int_0^\infty v \# dx\right) \# dx + \int_0^\infty (u \# dx)(v \# dx). \end{aligned} \tag{2.1}$$

Equivalently, we have that;

Corollary 2.3. *Let u, v, x be given as before. Let $I_1 = \int_0^\infty u \# dx$ and $I_2 = \int_0^\infty v \# dx$. Then*

$$d(I_1 I_2) = dI_1 \cdot I_2 + I_1 \cdot dI_2 + dI_1 \cdot dI_2,$$

where "d" means the differentiation.

By Section 2.1 and by the above proposition, we can have the D_N-valued analogue of Itô's free stochastic integral product formula.

Theorem 2.4. (D_N-valued Itô's Formula) *Let x be a D_N-valued infinitely divisible element in a tracial W^*-probability space (A, E) over its N-th diagonal subalgebra D_N, where $A = \oplus_{j=1}^N A_j$ and $E = \oplus_{j=1}^N \varphi_j$. If $u = (u_t)_t$ and $v = (v_t)_t$ are D_N-valued simple adapted biprocess in $A \otimes A$, and if $I_1 = \int_0^\infty u \#_{D_N} dx$ and $I_2 = \int_0^\infty v \#_{D_N} dx$, then*

$$d(I_1 I_2) = dI_1 \cdot I_2 + I_1 \cdot dI_2 + dI_1 \cdot dI_2.$$

Proof. In Section 2.1, we showed that if u is a D_N-valued simple adapted biprocess in $A \otimes A$, where $A = \oplus_{j=1}^N A_j$, then there exist (scalar-valued) simple adapted biprocesses u_j in $A_j \otimes_{\mathbb{C}} A_j$, for $j = 1, \ldots, N$, such that $u = \oplus_{j=1}^N u_j$. So, if $u = (u_t)_t$ and $v = (v_t)_t$ are D_N-valued simple adapted biprocesses in $A \otimes A$, then there exist simple adapted biprocesses u_j and v_j in $A_j \otimes_{\mathbb{C}} A_j$, for $j = 1, \ldots, N$, such that $u = \oplus_{j=1}^N u_j$ and $v = \oplus_{j=1}^N v_j$, where $u_j = (u_{j:t})_t$ and $v_j = (v_{j:t})_t$, for all $j = 1, \ldots, N$. Remark that u_j's (resp. v_j's) have the same adaptedness, for each j. Thus we can have that

$$
\begin{aligned}
& I_1 \cdot I_2 \\
&= \left(\int_0^\infty u \ \#_{D_N} \ dx\right)\left(\int_0^\infty v \ \#_{D_N} \ dx\right) \\
&= \left(\int_0^\infty (\oplus_{j=1}^N u_j) \ \#_{D_N} \ dx\right)\left(\int_0^\infty (\oplus_{j=1}^N v_j) \ \#_{D_N} \ dx\right) \\
&= \left(\int_0^\infty (\oplus_{j=1}^N u_j) \ \#_{D_N} \ d\left(\oplus_{j=1}^N x_j\right)\right)\left(\int_0^\infty (\oplus_{j=1}^N v_j) \ \#_{D_N} \ d\left(\oplus_{j=1}^N x_j\right)\right)
\end{aligned}
$$

by Section 1.2, where $x_j \in (A_j, \varphi_j)$ are infinitely divisible elements making

$$
\begin{aligned}
x = & \ \oplus_{j=1}^N x_j, \text{ for } j = 1, \ldots, N \\
= & \left(\oplus_{j=1}^N \left(\int_0^\infty u_j \ \# \ dx_j\right)\right)\left(\oplus_{j=1}^N \left(\int_0^\infty v_j \ \# \ dx_j\right)\right)
\end{aligned}
$$

by the definition of $\#_{D_N} = \oplus_{j=1}^N \left(\left(\int_0^\infty u_j \ \# \ dx_j\right)\left(\int_0^\infty v_j \ \# \ dx_j\right)\right)$.

By the classical Itô's product formula, each

$$
\left(\int_0^\infty u_j \ \# \ dx_j\right)\left(\int_0^\infty v_j \ \# \ dx_j\right)
$$

satisfies (2.1). Therefore, we have that

$$
\begin{aligned}
& I_1 \cdot I_2 \\
&= \oplus_{j=1}^N \left(\left(\int_0^\infty u_j \ \# \ dx_j\right)\left(\int_0^\infty v_j \ \# \ dx_j\right)\right) \\
&= \oplus_{j=1}^N \left(\int_0^\infty \left(\int_0^\infty u_j \ \# \ dx_j\right) v_j \ \# \ dx_j\right. \\
&\quad + \int_0^\infty u_j \left(\int_0^\infty v_j \ \# \ dx_j\right) \# \ dx_j \\
&\quad \left. + \int_0^\infty (u_j \ \# \ dx_j)(v_j \ \# \ dx_j)\right) \text{ by (2.1)} \\
&= \oplus_{j=1}^N \left(\int_0^\infty \left(\int_0^\infty u_j \ \# \ dx_j\right) v_j \ \# \ dx_j\right) \\
&\quad + \oplus_{j=1}^N \left(\int_0^\infty u_j \left(\int_0^\infty v_j \ \# \ dx_j\right) \# \ dx_j\right) \\
&\quad + \oplus_{j=1}^N \left(\int_0^\infty (u_j \ \# \ dx_j)(v_j \ \# \ dx_j)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty \left(\oplus_{j=1}^N \left(\int_0^\infty u_j \# dx_j \right) v_j \# dx_j \right) \\
&\quad + \int_0^\infty \oplus_{j=1}^N \left(u_j \left(\int_0^\infty v_j \# dx_j \right) \# dx_j \right) \\
&\quad + \int_0^\infty \oplus_{j=1}^N \left((u_j \# dx_j)(v_j \# dx_j) \right) \\
&= \int_0^\infty \left(\left(\int_0^\infty \left(\oplus_{j=1}^N u_j\right) \#_{D_N} d\left(\oplus_{j=1}^N x_j\right) \right) \left(\oplus_{j=1}^N v_j\right) \#_{D_N} d\left(\oplus_{j=1}^N x_j\right) \right) \\
&\quad + \int_0^\infty \left(\left(\oplus_{j=1}^N u_j\right) \left(\int_0^\infty \left(\oplus_{j=1}^N v_j\right) \#_{D_N} d\left(\oplus_{j=1}^N x_j\right) \right) \#_{D_N} d\left(\oplus_{j=1}^N x_j\right) \right) \\
&\quad + \int_0^\infty \left(\left(\left(\oplus_{j=1}^N u_j\right) \#_{D_N} d(\oplus_{j=1}^N x_j) \right) \left(\left(\oplus_{j=1}^N v_j\right) \#_{D_N} d(\oplus_{j=1}^N x_j) \right) \right) \\
&= \int_0^\infty \left(\int_0^\infty u \#_{D_N} dx \right) v \#_{D_N} dx \\
&\quad + \int_0^\infty u \left(\int_0^\infty v \#_{D_N} dx \right) \#_{D_N} dx + \int_0^\infty (u \#_{D_N} dx)(v \#_{D_N} dx).
\end{aligned}
$$

Therefore, equivalently, we can get that

$$d(I_1 \cdot I_2) = dI_1 \cdot I_2 + I_1 \cdot dI_2 + dI_1 \cdot dI_2.$$

□

2.3. D_N-valued stochastic integration for D_N-free Brownian motions

In this section, we will consider a special case of the D_N-valued free stochastic integration. As before, let (A, E) be a direct producted W^*-probability space over its N-th diagonal subalgebra D_N with $A = \oplus_{j=1}^N A_j$ and $E = \oplus_{j=1}^N \varphi_j$. Like in Section 2.2, we assume that E is a tracial conditional expectation, equivalently, all continuous linear functionals φ_j are trace on A_j, for all $j = 1, \ldots, N$. The D_N-free Brownian motion is defined by the D_N-free distribution of D_N-semicircular infinitely divisible element.

Definition. The D_N-free distribution μ is called a D_N-free Brownian motion if $\mu = \mu_x$, where μ_x means the D_N-free distribution of x and where x is a D_N-semicircular infinitely divisible element.

Notice that the choice of a D_N-semicircular infinitely divisible element x in a D_N-valued W^*-probability space (A, E) is not fixed for the given D_N-free Brownian motion μ. If x is a D_N-semicircular infinitely divisible element in a direct producted tracial W^*-probability space (A, E) over D_N with $A = \oplus_{j=1}^N A_j$ and $E = \oplus_{j=1}^N \varphi_j$, where (A_j, φ_j) are tracial W^*-probability spaces, then the D_N-valued free distribution μ_x is a D_N-free Brownian motion. And if $u = (u_t)_t$ and $v = (v_t)_t$ are D_N-valued simple adapted biprocesses in $A \otimes A$, then we have

$$d(I_1 \cdot I_1) = dI_1 \cdot I_2 + I_1 \cdot dI_2 + dI_1 \cdot dI_2,$$

by the D_N-valued Itô's product formula, where $I_1 = \int_0^\infty u \#_{D_N} dx$ and $I_2 = \int_0^\infty v \#_{D_N} dx$.

Suppose that (B, τ) is a tracial W^*-probability space and let $b \in (B, \tau)$ be a semicircular infinitely divisible element, with $k_2^\tau(b, b) = \kappa$ in $\mathbb{C}$, where $k_n^\tau(\cdots)$ means the cumulant with respect to the given trace τ on B. Then the free distribution μ_b is identical with the (scalar-valued) free Brownian motion μ_1. It is well-known that if we have an operator

$$T = \frac{\kappa}{\sqrt{2}}(l_e + l_e^*) \quad \text{in } B(H),$$

where H is a Fock space $\oplus_{n=0}^\infty K^{\otimes n}$ with $K^{\otimes 0} = \mathbb{C}$, and e is an element in the Hilbert basis of K, which is a separable Hilbert space, and where l and l^* are the creation and annihilation operators, respectively, on H, then T is identically distributed with $b \in (B, \tau)$, under the canonical trace tr on $B(H)$. Here, the trace tr on $B(H)$ is defined by

$$tr(X) = \langle \Omega, X\Omega \rangle, \text{ for all } X \in B(H),$$

where $\langle , \rangle$ is the inner product on H and Ω is the vacuum vector. So, instead of studying the free Brownian motion $\mu_1 = \mu_b$, directly, we can consider the identical free distribution μ_T, where μ_T is the free distribution of $T \in B(H)$ with respect to tr.

From now, we will let D_N-semicircular infinitely divisible element x in the direct producted tracial W^*-probability space (A, E) over D_N satisfy $k_2(x, x) = \oplus_{j=1}^N \kappa_j$ in D_N, where $\kappa_j \in \mathbb{C}$, for $j = 1, \ldots, N$. Also, without loss of generality, assume that $\kappa_j \neq 0$ in $\mathbb{C}$, for all $j = 1, \ldots, N$. Since $k_2(x, x) = \oplus_{j=1}^N \kappa_j$ in D_N, we can understand the D_N-valued random variable x satisfies $x = \oplus_{j=1}^N x_j$, with semicircular infinitely divisible random variables $x_j \in (A_j, \varphi_j)$ having their second cumulants,

$$k_2^{(j)}(x_j, x_j) = \kappa_j \in \mathbb{C}, \text{ for all } j = 1, \ldots, N,$$

where $k_n^{(i)}(\cdots)$ means the cumulant with respect to the given trace φ_i on A_i, for $i = 1, \ldots, N$. By the previous discussion, there exist

$$T_j = \frac{\kappa_j}{\sqrt{2}}\left(l_{e_j} + l_{e_j}^*\right) \text{ in } B(H_j), \text{ for } j = 1, \ldots, N,$$

such that the corresponding free distributions μ_{x_j} of x_j and μ_{T_j} of T_j are identical, where e_j's are basis elements of a Hilbert space K_j, where $H_j = \oplus_{n=0}^\infty K_j^{\otimes n}$, for $j = 1, \ldots, N$. Therefore, we can get the following lemma.

Lemma 2.5. *Let x be a D_N-semicircular infinitely divisible element in a direct producted tracial W^*-probability space (A, E) over its N-th diagonal subalgebra D_N, where $k_2(x, x) = \oplus_{j=1}^N \kappa_j$ in D_N. Then there exists $T_j = \frac{\kappa_j}{\sqrt{2}}(l_{e_j} + l_{e_j}^*)$ in $B(H_j)$, for $j = 1, \ldots, N$, such that the D_N-valued random variables x in A and the operator $T_x = \oplus_{j=1}^N T_j$ in $B(\oplus_{j=1}^N H_j)$ are identically distributed over D_N.*

Proof. Let's assume that $k_2(x, x) = \oplus_{j=1}^N \kappa_j$ in D_N with $\kappa_j \neq 0$ in $\mathbb{C}$, for all $j = 1, \ldots, N$. Then, by the D_N-semicircularity and the D_N-valued infinitely divisibility of $x \in (A, E)$, there are semicircular infinitely divisible random variables x_j in (A_j, φ_j) such that $x = \oplus_{j=1}^N x_j$ and $k_2^{(j)} (x_j, x_j) = \kappa_j$, for all $j = 1, \ldots, N$. Since each x_j is semicircular and infinitely divisible with $k_2^{(j)}(x_j, x_j) = \kappa_j$, there exists an operator $T_j = \frac{\kappa_j}{\sqrt{2}} (l_{e_j} + l_{e_j}^*)$ in $B(H_j)$, such that x_j and T_j are identically distributed, where H_j is a Fock space $\oplus_{n=0}^\infty K_j^{\otimes n}$, with a separable Hilbert space K_j and $K_j^{\otimes 0} = \mathbb{C}$, where l and l^* are the creation and the annihilation operators, respectively and e_j is an element in the Hilbert basis of K_j. Then, the operator

$$T_x = \oplus_{j=1}^N T_j \text{ in } B\left(\oplus_{j=1}^N H_j\right)$$

and x are identically distributed over D_N. Indeed, we can define the canonical tracial conditional expectation Tr on $B(\oplus_{j=1}^N H_j)$ by

$$Tr\left(\oplus_{j=1}^N X_j\right) = \oplus_{j=1}^N tr_{H_j}(X_j) = \oplus_{j=1}^N \langle \Omega_j, X_j \Omega_j \rangle_{H_j} \text{ in } D_N,$$

or all $\oplus_{j=1}^N X_j \in B(\oplus_{j=1}^N H_j)$, where $\langle , \rangle_{H_j}$ are the inner products on H_j and Ω_j are the vacuum vectors of H_j, for $j = 1, \ldots, N$. For this tracial conditional expectation Tr, the D_N-valued random variable T_x is identically distributed with $x \in (A, E)$ over D_N. □

By the previous lemma, we can get the following theorem.

Theorem 2.6. *Let $u = (u_t)_t$ be a D_N-valued simple adapted biprocess in $A \otimes A$ and let x be a D_N-semicircular infinitely divisible element in (A, E), with $k_2(x, x) = \oplus_{j=1}^N \kappa_j$ in D_N. If $U = (\pi_0 \otimes \pi_0)(u)$, where $\pi_0 : A \to B(H)$ is a faithful representation, then*

$$\int_0^\infty u \#_{D_N} dx = \int_0^\infty u \#_{D_N} d\mu = \int_0^\infty U \#_{D_N} dT_x,$$

where μ is a D_N-free Brownian motion and $H = \oplus_{j=1}^N H_j$ is a direct sum of Fock spaces H_j and $T_x = \oplus_{j=1}^N T_j$ with $T_j = \frac{\kappa_j}{\sqrt{2}}(l_{e_j} + l_{e_j}^*)$ in $B(H_j)$.

Proof. Since each A_j is a von Neumann algebra, for $j = 1, \ldots, N$, we can have a faithful representation $\pi_j : A_j \to B(K_j)$, for $j = 1, \ldots, N$, for some Hilbert spaces K_j. Thus there exists a direct product of these representations $\pi = \oplus_{j=1}^N \pi_j : A \to \oplus_{j=1}^N B(K_j)$. So, the direct product von Neumann algebra $A = \oplus_{j=1}^N A_j$ has its faithful representation $\pi' : A \to B(\oplus_{j=1}^N K_j)$, where $\pi' = \iota \circ \pi$, where $\iota : \oplus_{j=1}^N B(K_j) \to B\left(\oplus_{j=1}^N K_j\right)$ is the canonical embedding defined by $\iota\left(\oplus_{j=1}^N X_j\right) = \oplus_{j=1}^N X_j$, for all $X_j \in B(K_j)$, for $j = 1, \ldots, N$. The faithfulness of π' is guaranteed by that of π_j's and that of ι. Consider the Fock space $H_j = \oplus_{n=0}^\infty K_j^{\otimes n}$ and the Hilbert space $H = \oplus_{j=1}^N H_j$. Then there is an extension $\pi_0 : A \to B(H)$ of π', as a faithful representation of A.

Assume that $T_x = \oplus_{j=1}^{N} \left(\frac{\kappa_j}{\sqrt{2}} (l_{e_j} + l_{e_j}^*) \right)$ in $B(H)$, where $e_j \in K_j$, for $j = 1, \ldots, N$. Then $\mu_{T_x} = \mu = \mu_x$, where μ is the D_N-free Brownian motion satisfying $\mu(z^2) = \oplus_{j=1}^{N} \kappa_j$ in D_N, where $z^2 \in D_N[[z]]$, by the previous lemma, where D_N $[[z]]$ is the ring of all formal series. If $u \in A \otimes A$ is a D_N-valued simple adapted biprocess, then we have $U = (\pi_0 \otimes \pi_0)$ (u) in $B(H) \otimes B(H)$. Therefore, we can have that

$$\int_0^\infty u \#_{D_N} dx = \int_0^\infty U \#_{D_N} dT_x.$$

□

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