

ON EXACT CONVERGENCE RATE OF STRONG NUMERICAL SCHEMES FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. We propose a simple and intuitive method to derive the exact convergence rate of global L_2 -norm error for strong numerical approximation of stochastic differential equations the result of which has been reported by Hofmann and Müller-Gronbach (2004). We conclude that any strong numerical scheme of order $\gamma > 1/2$ has the same optimal convergence rate for this error. The method clearly reveals the structure of global L_2 -norm error and is similarly applicable for evaluating the convergence rate of global uniform approximations.

1. Introduction

Let us consider a scalar diffusion process X_t satisfying the following stochastic differential equation

$$(1) \quad dX_t = a(t, X_t)dt + \sigma(t, X_t)dW_t$$

with 1-dimensional standard Brownian motion W_t , $t \geq 0$ and the continuously differentiable coefficient functions a and b satisfying the conditions

$$(2) \quad |a(t, x)| + |\sigma(t, x)| \leq K(1 + |x|),$$

$$(3) \quad |a(s, x) - a(t, x)| + |\sigma(s, x) - \sigma(t, x)| \leq K(1 + |x|)|s - t|^{1/2},$$

for some constant $K > 0$.

For time discrete strong approximation of (1) on the unit interval $[0, 1]$, several different notions of errors were analyzed in the literature. The most commonly considered one was the mean square error (MSE for short) at a time point. By using stochastic Ito-Taylor expansion, we can construct *Ito-Taylor approximation*

$$(4) \quad Y_t = \sum_{\alpha \in \mathcal{A}_\gamma} I_\alpha[f_\alpha(\tau_{n_t}, Y_{n_t})]_{\tau_{n_t}, t},$$

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where n_t is the maximum discretization point less than t , α is multi index, \mathcal{A}_γ is some hierarchical index set for strong order γ and I_α denotes multiple stochastic integral for the index α - see [6] for detailed explanation of notations, and this process has the uniform mean square convergence order γ , i.e.,

$$(5) \quad \left[E \left(\sup_{0 \leq t \leq 1} |X_t - Y_t|^2 \right) \right]^{1/2} \leq K(1 + E|X_0|^2) \delta^\gamma,$$

where δ is the maximum step-size - see also [6]. But Y_t is not implementable in practice since the full information of W_t is not available. We can implement Y_t only on $\{\tau_k\}_{1 \leq k \leq n_t}$ because Y_{τ_k} requires only finite numbers of information of W_t on each subinterval (τ_k, τ_{k+1}) , $k = 1, 2, \dots, n$, i.e., some multiple stochastic integrals of W_t . Usually, we use \hat{Y}_t , the linear interpolation of Y_t at the points $\{\tau_k\}_{1 \leq k \leq n_t}$ for numerical implementation. Therefore, the error analyses for \hat{Y}_t are practically important. For the last decade, the global L_2 -norm error for \hat{Y}_t defined by

$$(6) \quad e(X, \hat{Y}) = \left[\int_0^1 E(X_t - \hat{Y}_t)^2 dt \right]^{1/2}$$

was analyzed intensively - see [1], [2], [3], [4], [8], and [5]. See also [7] for the global uniform error analysis. We denote $e(Y) := e(X, Y)$.

In Proposition 1 of [4], the exact convergence rate of equidistant Platen-Wagner order 1.5 strong scheme \hat{X}_n^{equi} was evaluated w.r.t. the global L_2 -norm error (6);

$$(7) \quad \lim_{n \rightarrow \infty} n^{1/2} \cdot e(\hat{X}_n^{equi}) = C^{equi} / \sqrt{6},$$

where

$$C^{equi} = \left[\int_0^1 E[\sigma^2(t, X_t)] dt \right]^{1/2},$$

which was not better than Milstein scheme - see [3]. In [3] and [4], the Ito-Taylor approximation Y_t (4) was used in their error analyses, but we provide a much simpler proof for a generalized result. We use the linear interpolation of X_t instead of Y_t and prove that any strong order γ ($> 1/2$) numerical scheme has the same convergence rate as (7) w.r.t. the error (6) asymptotically. We note that our result was lately reported in Theorem 2 of [5], but we show an independent derivation, the idea of which will also be useful in the analyses of other types of numerical errors, e.g., global uniform error [7].

2. Evaluation of the convergence rate

For $T = [0, 1]$, consider the equidistant discretization $0 = t_0 \leq t_1 \leq \dots \leq t_n = 1$, where $t_i = i/n$, $i = 1, 2, \dots, n$. The equation (1) can be rewritten as

$$(8) \quad X_t = X_{t_m} + \int_{t_m}^t a(s, X_s) ds + \int_{t_m}^t \sigma(s, X_s) dW_s$$

for $t_m \leq t \leq t_{m+1}$. Let \bar{X} be the linear interpolation of X at the discretization nodes $\{t_i\}_{1 \leq i \leq n}$, i.e.,

$$(9) \quad \begin{aligned} \bar{X}_t &= \frac{1}{t_{m+1} - t_m} \left[(t_{m+1} - t)X_{t_m} + (t - t_m)X_{t_{m+1}} \right] \\ &= X_{t_m} + \left[\frac{t - t_m}{t_{m+1} - t_m} \right] \left[\int_{t_m}^{t_{m+1}} a(s, X_s) ds + \int_{t_m}^{t_{m+1}} \sigma(s, X_s) dW_s \right]. \end{aligned}$$

We let

$$\begin{aligned} A_m &= \int_{t_m}^t a(s, X_s) ds, & A'_m &= \int_t^{t_{m+1}} a(s, X_s) ds, \\ B_m &= \int_{t_m}^t \sigma(s, X_s) dW_s, & B'_m &= \int_t^{t_{m+1}} \sigma(s, X_s) dW_s, \end{aligned}$$

and denote \bar{A}_m , \bar{B}_m , \bar{A}'_m , and \bar{B}'_m the integrals with locally freezed integrands, i.e., we replace the integral time s to t_m for A_m , B_m , A'_m , and B'_m .

Let K be an unspecified positive constant.

Lemma 1.

$$(a) \quad \left[\sum_{m=1}^{n-1} \int_{t_m}^{t_{m+1}} \int_{t_m}^t E \left[\sigma(s, X_s) - \sigma(t_m, X_{t_m}) \right]^2 ds dt \right]^{1/2} \leq Kn^{-1}$$

$$(b) \quad \left[\sum_{m=1}^{n-1} \int_{t_m}^{t_{m+1}} \int_t^{t_{m+1}} E \left[\sigma(s, X_s) - \sigma(t_m, X_{t_m}) \right]^2 ds dt \right]^{1/2} \leq Kn^{-1}$$

$$(c) \quad \left[\sum_{m=1}^{n-1} \int_{t_m}^{t_{m+1}} E[|B_m| \cdot |B'_m|] dt \right]^{1/2} \leq Kn^{-3/4}.$$

Proof. It is easy to prove (a) and (b) using the coefficient conditions (2) and (3). The left hand side of (c) becomes

$$\begin{aligned} & \left[\sum_{m=1}^{n-1} \int_{t_m}^{t_{m+1}} E[|B_m| \cdot |B'_m|] dt \right]^{1/2} \\ & \leq \left[\sum_{m=1}^{n-1} \int_{t_m}^{t_{m+1}} E[(|B_m|^{1/2} \cdot (|B'_m|^{1/2} - |\bar{B}'_m|^{1/2}))^2] dt \right]^{1/2} \\ & \quad + \left[\sum_{m=1}^{n-1} \int_{t_m}^{t_{m+1}} E[(|\bar{B}'_m|^{1/2} \cdot (|B_m|^{1/2} - |\bar{B}_m|^{1/2}))^2] dt \right]^{1/2} \leq Kn^{-3/4}. \end{aligned}$$

Note that $E[\bar{B}_m \cdot \bar{B}'_m] = 0$. The last inequality in the above line is also proved easily by using the inequality $|\sqrt{|a|} - \sqrt{|b|}| \leq \sqrt{|a - b|}$ for $a, b \in \mathbb{R}$, Hölder's inequality, Ito's isometry, and the coefficient conditions (2) and (3). \square

Let us denote

$$\|f_m\|_{L_2} := \left[\sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} E(f_m(t))^2 dt \right]^{1/2}.$$

The terms for the drift coefficients are easily estimated as follows:

Lemma 2.

$$\begin{aligned} \|A_m\|_{L_2} &\leq Kn^{-3/2} \\ \|A'_m\|_{L_2} &\leq Kn^{-3/2}. \end{aligned}$$

We first estimate the error for the linear interpolation process \bar{X} .

Lemma 3. *Let $\bar{X}^{(n)}$ be the interpolation process (9) defined on the equi-distant time points $\{i/n\}_{1 \leq i \leq n}$. Then*

$$\lim_{n \rightarrow \infty} n^{1/2} \cdot e(\bar{X}^{(n)}) = C^{equi}/\sqrt{6}.$$

Proof. Let $C(t) := (t_{m+1} - t)/(t_{m+1} - t_m)$, and $D(t) := (t - t_m)/(t_{m+1} - t_m)$. Subtracting (9) from (8), we have

$$|X_t - \bar{X}_t^{(n)}| = |C(t) \cdot A_m - D(t) \cdot A'_m + C(t) \cdot B_m - D(t) \cdot B'_m|,$$

and since $|C(t)| \leq 1$, and $|D(t)| \leq 1$, we have

$$\begin{aligned} e(\bar{X}^{(n)}) &\leq \|[(C(t)\bar{B}_m)^2 + (D(t)\bar{B}'_m)^2]^{1/2}\|_{L_2} \\ &\quad + \|B_m - \bar{B}_m\|_{L_2} + \|B'_m - \bar{B}'_m\|_{L_2} + \|(B_m \cdot B'_m)^{1/2}\|_{L_2} \\ &\quad + \|A_m\|_{L_2} + \|A'_m\|_{L_2}, \end{aligned}$$

and

$$\begin{aligned} e(\bar{X}^{(n)}) &\geq \|[(C(t)\bar{B}_m)^2 + (D(t)\bar{B}'_m)^2]^{1/2}\|_{L_2} \\ &\quad - \|B_m - \bar{B}_m\|_{L_2} - \|B'_m - \bar{B}'_m\|_{L_2} - \|(B_m \cdot B'_m)^{1/2}\|_{L_2} \\ &\quad - \|A_m\|_{L_2} - \|A'_m\|_{L_2}. \end{aligned}$$

By applying Ito's isometry, we obtain

$$\begin{aligned} &\int_{t_m}^{t_{m+1}} E[(C(t)\bar{B}_m)^2 + (D(t)\bar{B}'_m)^2] dt \\ &= \frac{E[\sigma^2(t_m, X_{t_m})]}{(t_{m+1} - t_m)} \cdot \int_{t_m}^{t_{m+1}} (t_{m+1} - t)(t - t_m) dt \\ &= \frac{1}{6} (t_{m+1} - t_m) E[\sigma^2(t_m, X_{t_m})], \end{aligned}$$

and

$$\begin{aligned} & \|[(C(t)\bar{B}_m)^2 + (D(t)\bar{B}'_m)^2]^{1/2}\|_{L_2} \\ &= \frac{1}{\sqrt{6}} \left[\sum_{m=0}^{n-1} E[\sigma^2(t_m, X_{t_m})] \cdot n^{-1/2} \right]^{1/2} \cdot n^{-1/2}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} n^{1/2} \cdot e(\bar{X}^{(n)}) &\leq C^{equi}/\sqrt{6}, \\ \liminf_{n \rightarrow \infty} n^{1/2} \cdot e(\bar{X}^{(n)}) &\geq C^{equi}/\sqrt{6}, \end{aligned}$$

by Lemma 1 and Lemma 2. □

Remark 1. Every numerical scheme aims to approximate \bar{X} on the discretization points. But, Lemma 3 addresses \bar{X} still has a positive error in the global L_2 approximation. The error $e(\bar{X})$ is the essential part of this L^2 approximations and the error caused by numerical scheme is negligible if we have the condition that $\gamma > 1/2$ for the uniform MSE, as one will see in the next main result.

Theorem 1. *Let $\hat{Y}^{(n)}$ be a numerical scheme of strong order $\gamma > 1/2$ for uniform MSE defined on the time points $\{i/n\}_{1 \leq i \leq n}$. Then*

$$\lim_{n \rightarrow \infty} n^{1/2} \cdot e(\hat{Y}^{(n)}) = C^{equi}/\sqrt{6}.$$

Remark 2. We only have the assumption on the strong order γ for $\hat{Y}_t^{(n)}$ for uniform MSE (on discretization points). It can be a stochastic Ito-Taylor numerical scheme for example. This means that if the numerical process converges to X_t fast enough at discretization points i.e., $\gamma > 1/2$, it converges to X_t as fast as \bar{X}_t globally. Observe that the global L^2 convergence order is always $1/2$ when $\gamma > 1/2$.

Proof. We have

$$\begin{aligned} e(\hat{Y}^{(n)}) &\leq e(X, \bar{X}) + e(\bar{X}, \hat{Y}^{(n)}), \\ e(\hat{Y}^{(n)}) &\geq e(X, \bar{X}) - e(\bar{X}, \hat{Y}^{(n)}). \end{aligned}$$

Since

$$e(\bar{X}, \hat{Y}^{(n)}) \leq \left[E \left(\sup_{t_k} |\bar{X}_{t_k} - \hat{Y}_{t_k}^{(n)}|^2 \right) \right]^{1/2} \leq Kn^{-\gamma},$$

we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} n^{1/2} \cdot e(\hat{Y}^{(n)}) &\leq C^{equi}/\sqrt{6}, \\ \liminf_{n \rightarrow \infty} n^{1/2} \cdot e(\hat{Y}^{(n)}) &\geq C^{equi}/\sqrt{6}, \end{aligned}$$

by Lemma 3. □

Remark 3. In a similar way, we can show that the optimal convergence rate for global uniform error [7] is invariant if the numerical scheme used has the MSE strong order $(\log n/n)^{1/2}$, i.e., Euler scheme also has the optimal performance asymptotically in this case.

3. Conclusion

We proposed to use the linear interpolation process, denoted by \bar{X} , in evaluating the convergence rates of numerical schemes. In the literature ([2], [3], [4], [7], [5]), the *Ito-Taylor approximation* (4) has been used for evaluating the convergence rates with complicated proofs. Our method is simple and more intuitive than the previous approach such that one can clearly grasp the essential part of the numerical errors. We expect our method to provide a useful approach for numerical error analysis of stochastic processes.

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