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ON STABILITY OF BANACH FRAMES

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Abstract. Some stability theorems (Paley-Wiener type) for Banach frames in Banach spaces have been derived.

1. Introduction

Duffin and Schaeffer [7] introduced frames for Hilbert spaces in 1952. Later on, in 1986, Daubechies, Grossmann and Meyer [6] found a fundamental new application to wavelet and Gabor's transforms in which frames play an important role. In fact, the theory of frames is a central tool in many areas such as signal processing, image processing, data compression etc. Coifman and Weiss [5] introduced the notion of *atomic decomposition* for function spaces. Later, Feichtinger and Gröchenig [9] extended the notion of atomic decomposition to certain Banach spaces. Gröchenig [10] introduced a more general concept for Banach spaces called a *Banach frame.* Banach frames were further studied in [2, 4, 8].

Stability theorems for frames in Hilbert spaces were studied in [1, 3, 8, 12] and for Banach frames were studied by Christensen and Heil [4].

In the present paper, we prove some stability theorems (Paley-Wiener type) for Banach frames in Banach spaces.

2. Preliminaries

Throughout this paper E will denote a Banach space over the scalar field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$, E^* and E^{**}, respectively, the first and second conjugate space of E, E_d an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$.

A sequence $\{f_n\} \subset E^*$ is said to be *total* if $\{x \in E : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$. A sequence $\{\alpha_n\} \subset \mathbb{R}$ is said to be *positively confined* if $0 < \inf_{1\leq n<\infty} \alpha_n \leq$

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$\sup_{1\leq n<\infty} \alpha_n < \infty$. For $x = \{x_n\}$, $y = \{y_n\}$ in E and $\alpha \in \mathbb{K}$, we define $x \pm y = \{x_n \pm y_n\}$, $x \cdot y = \{x_n y_n\}$ and $\alpha x = \{\alpha x_n\}$.

Definition. ([10]) Let E be a Banach space and E_d an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$. Let $\{f_n\} \subset E^*$ and $S : E_d \to E$ be given. Then the pair $(\{f_n\}, S)$ is called a *Banach frame for E with respect to E_d* if

(i) $\{f_n(x)\} \in E_d$, for each $x \in E$
(ii) there exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|x\|_E \leq \|\{f_n(x)\}\|_{E_d} \leq B\|x\|_E, \quad x \in E \tag{2.1}$$

(iii) S is a bounded linear operator such that

$$S(\{f_n(x)\}) = x, \quad x \in E.$$

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the Banach frame $(\{f_n\}, S)$. The operator $S : E_d \to E$ is called the *reconstruction operator* (or the *pre-frame operator*). The inequality (2.1) is called the *frame inequality.* It is easy to observe that frame bounds need not be unique. Further, if $T : E \to E_d$ is the coefficient map given by $T(x) = \{f_n(x)\}, x \in E$, then $(\|S\|)^{-1}$ and $\|T\|$ satisfying $A \leq \|S\|^{-1} \leq \|T\| \leq B$, are also frame bounds for the Banach frames $(\{f_n\}, S)$.

The Banach frame $(\{f_n\}, S)$ is called *tight* if $A = B$ and *normalized tight* if $A = B = 1$. If removal of one f_n renders the collection $\{f_n\} \subset E^*$ no longer a Banach frame for E, then $(\{f_n\}, S)$ is called an *exact Banach frame.*

3. Main results

We begin with a necessary and sufficient condition for the stability of a Banach frame.

Theorem 3.1. *Let $(\{f_n\}, S)$ $(\{f_n\} \subset E^*, S : E_d \to E)$ be a Banach frame for E with respect to E_d. Let $\{g_n\} \subset E^*$ be such that $\{g_n(x)\} \in E_d$, $x \in E$ and let $L : E_d \to E_d$ be a bounded linear operator such that $L\{g_n(x)\} = \{f_n(x)\}$, $x \in E$. Then these exists a reconstruction operator $U : E_d \to E$ such that $(\{g_n\}, U)$ is a Banach frame for E with respect to E_d if and only if there exists a constant $M > 1$ such that*

$$\|\{(f_n - g_n)(x)\}\|_{E_d} \leq M \min\{\|\{f_n(x)\}\|_{E_d}, \|\{g_n(x)\}\|_{E_d}\}, \quad x \in E.$$

Proof. Let A_f, B_f; A_g, B_g, respectively, be the frame bounds for Banach frames $(\{f_n\}, S)$ and $(\{g_n\}, U)$. Then, using frame inequalities for these frames, we get

$$\|\{(f_n - g_n)(x)\}\|_{E_d} \leq \left(1 + \frac{B_g}{A_f}\right) \|\{f_n(x)\}\|_{E_d}, \quad x \in E.$$

Similarly, we obtain

$$\|\{(f_n - g_n)(x)\}\|_{E_d} \le \left(1 + \frac{B_f}{A_g}\right) \|\{g_n(x)\}\|_{E_d}, \quad x \in E.$$

Choose $M = \left(1 + \dfrac{B_g}{A_f}\right)$ or $\left(1 + \dfrac{B_f}{A_g}\right)$ according as $\min\{\|\{f_n(x)\}\|_{E_d}, \|\{g_n(x)\}\|_{E_d}\}$ is $\|\{f_n(x)\}\|_{E_d}$ or $\|\{g_n(x)\}\|_{E_d}$. Conversely, by hypothesis, $\{g_n(x)\} \in E_d$, $x \in E$. If A_f and B_f are the frame bounds for the Banach frame $(\{f_n\}, S)$, then for each $x \in E$, we have

$$\begin{aligned} A_f\|x\|_E &\le \|\{f_n(x)\}\|_{E_d} \\ &\le \|\{(f_n - g_n)(x)\}\|_{E_d} + \|\{g_n(x)\}\|_{E_d} \\ &\le (1+M)\|\{g_n(x)\}\|_{E_d} \\ &\le (1+M)\left(\|\{(f_n - g_n)(x)\}\|_{E_d} + \|\{f_n(x)\}\|_{E_d}\right) \\ &\le (1+M)^2\|\{f_n(x)\}\|_{E_d} \\ &\le (1+M)^2 B_f \|x\|_E . \end{aligned}$$

Let $U = SL$. Then $U : E_d \to E$ be a bounded linear operator such that $U\{g_n(x)\} = x$, $x \in E$. Hence $(\{g_n\}, U)$ is a Banach frame for E with respect to E_d. □

Note. In the converse part of the Theorem 3.1 one may replace the condition $M > 1$ by $M > 0$.

The stability of Banach frame in Theorem 3.1 depends on the value of M since for large M, the frame inequality gets lost. Therefore, we still need stability conditions which gives optimal frame bounds. The following theorem gives such stability conditions.

Theorem 3.2. *Let $(\{f_n\}, S)$ be a Banach frame for E with respect to E_d. Let $\{g_n\} \subset E^*$ be such that $\{g_n(x)\} \in E_d$, $x \in E$ and let $V : E \to E_d$ be coefficient mapping given by $V(x) = \{g_n(x)\}$, $x \in E$. If there exist non-negative constants λ, μ, ν and ξ such that*

(i) $(\|T\| + \|V\| + 1)\sqrt{\max\{\lambda, \mu, \nu, \xi\}} < (\|S\|)^{-1}$

(ii) $\|\{(f_n - g_n)(x)\}\|_{E_d}^2 \le \lambda\|\{f_n(x)\}\|_{E_d}^2 + 2\mu\|\{f_n(x)\}\|_{E_d}\|\{g_n(x)\}\|_{E_d} + \nu\|\{g_n(x)\}\|_{E_d}^2 + \xi\|x\|_E^2$, $x \in E$,

then there exists a reconstruction operator U such that $(\{g_n\}, U)$ is a Banach frame for E with respect to E_d and with frame bounds

$$\left(\frac{(\|S\|)^{-1} - ((\|S\|)^{-1} + 1)\sqrt{\max\{\lambda, \mu, \nu, \xi\}}}{1 + \sqrt{\max\{\lambda, \mu, \nu, \xi\}}}\right)$$

and

$$\left(\frac{\|T\| + (\|T\| + 1)\sqrt{\max\{\lambda, \mu, \nu, \xi\}}}{1 - \sqrt{\max\{\lambda, \mu, \nu, \xi\}}}\right),$$

where T is the coefficient mapping given by $Tx = \{f_n(x)\}$, $x \in E$.

Proof. Let $\eta = \max\{\lambda, \mu, \nu, \xi\}$. Then (ii) may be restated as:

$$\|\{f_n - g_n)(x)\}\|_{E_d} \leq \sqrt{\eta}(\|\{f_n(x)\}\|_{E_d} + \|\{g_n(x)\}\|_{E_d} + \|x\|_E), \quad x \in E.$$

Now

$$\begin{aligned}\|\{g_n(x)\}\|_{E_d} &\leq \|\{f_n(x)\}\|_{E_d} + \|\{(f_n - g_n)(x)\}\|_{E_d} \\ &\leq \|\{f_n(x)\}\|_{E_d} + \sqrt{\eta}(\|\{f_n(x)\}\|_{E_d} + \|\{g_n(x)\}\|_{E_d} + \|x\|_E).\end{aligned}$$

This gives

$$\begin{aligned}(1 - \sqrt{\eta})\|\{g_n(x)\}\|_{E_d} &\leq (1 + \sqrt{\eta})\|\{f_n(x)\}\|_{E_d} + \sqrt{\eta}\|x\|_E \\ &\leq [(1 + \sqrt{\eta})\|T\| + \sqrt{\eta}\ ]\ \|x\|_E.\end{aligned}$$

Also, since $ST : E \to E$ is an identity operator,

$$\|x\|_E = \|STx\|_E \leq \|S\|\ \|\{f_n(x)\}\|_{E_d}.$$

Thus

$$\begin{aligned}\|\{g_n(x)\}\|_{E_d} &\geq \|\{f_n(x)\}\|_{E_d} - \|\{(f_n - g_n)(x)\}\|_{E_d} \\ &\geq \|\{f_n(x)\}\|_{E_d} - \sqrt{\eta}(\|\{f_n(x)\}\|_{E_d} + \|\{g_n(x)\}\|_{E_d} + \|x\|_E)\end{aligned}$$

i.e.,

$$\begin{aligned}(1 + \sqrt{\eta})\|\{g_n(x)\}\|_{E_d} &\geq (1 - \sqrt{\eta})(\|S\|)^{-1}\ \|x\|_E - \sqrt{\eta}\ \|x\|_E \\ &= [(1 - \sqrt{\eta})(\|S\|)^{-1} - \sqrt{\eta}\ ]\ \|x\|_E.\end{aligned}$$

Therefore

$$\begin{aligned}&\left(\frac{(1 - \sqrt{\eta})(\|S\|)^{-1} - \sqrt{\eta}}{1 + \sqrt{\eta}}\right)\|x\|_E \\ &\leq \|\{g_n(x)\}\|_{E_d} \\ &\leq \left(\frac{(1 + \sqrt{\eta})\|T\| + \sqrt{\eta}}{1 - \sqrt{\eta}}\right)\|x\|_E, \ x \in E.\end{aligned}$$

Also $ST = I$ where I is an identity mapping on E. Therefore

$$\begin{aligned}\|I - SV\| &\leq \|S\|\|T - V\| \\ &\leq \|S\|\sqrt{\eta}\ (\|T\| + \|V\| + 1) \\ &< 1.\end{aligned}$$

Thus, SV is invertible. Put $U = (SV)^{-1}S$. Then $U : E_d \to E$ is a bounded linear operator such that $U(\{g_n(x)\}) = x$, $x \in E$. Hence $(\{g_n\}, U)$ is a Banach frame for E with respect to E_d and with desired frame bounds. □

We shall now show that Banach frames are stable under perturbation of frame elements by positively confined sequence of scalars.

Theorem 3.3. *Let $(\{f_n\}, S)$ be a Banach frame for E with respect to $E_d \subset \ell^\infty$. Let $\{g_n\} \subset E^*$ be such that $\{g_n(x)\} \in E_d$, $x \in E$ and let $L : E_d \to E_d$ be a bounded linear operator such that $L\{g_n(x)\} = \{f_n(x)\}$, $x \in E_d$. Let $\{\alpha_n\}$, $\{\beta_n\} \subset \mathbb{R}$ be two positively confined sequences. If there exist non-negative scalars λ, μ $(0 \leq \mu < 1)$ and γ such that*

(i) $\gamma < (1-\lambda)\|S\|^{-1}\left(\inf_{1\leq n<\infty} \alpha_n\right)$

(ii) $$\|\{(\alpha_n f_n - \beta_n g_n)(x)\}\|_{E_d} \leq \lambda\|\{(\alpha_n f_n)(x)\}\|_{E_d} + \mu\|\{(\beta_n g_n)(x)\}\|_{E_d} + \gamma\|x\|_E, \; x \in E,$$

then there exists a reconstruction operator U such that $(\{g_n\}, U)$ is a Banach frame for E with respect to E_d and with frame bounds

$$\left(\frac{(1-\lambda)(\|S\|)^{-1}\left(\inf_{1\leq n<\infty} \alpha_n\right) - \gamma}{(1+\mu)\left(\sup_{1\leq n<\infty} \beta_n\right)}\right)$$

and

$$\left(\frac{(1+\lambda)\|T\|\left(\sup_{1\leq n<\infty} \alpha_n\right) + \gamma}{(1-\mu)\left(\inf_{1\leq n<\infty} \beta_n\right)}\right),$$

where T is the coefficient mapping given by $Tx = \{f_n(x)\}$, $x \in E$.

Proof. The operator $ST : E \to E$ is an identity operator such that

$$\|x\|_E = \|ST(x)\|_E \leq \|S\| \; \|\{f_n(x)\}\|_{E_d}, \quad x \in E.$$

Now

$$\begin{aligned}\|\{(\beta_n g_n)(x)\}\|_{E_d} &\leq \|\{(\alpha_n f_n)(x)\}\|_{E_d} + \|\{(\alpha_n f_n - \beta_n g_n)(x)\}\|_{E_d} \\ &\leq \|\{(\alpha_n f_n)(x)\}\|_{E_d} + \lambda\|\{(\alpha_n f_n)(x)\}\|_{E_d} \\ &\quad + \mu\|\{(\beta_n g_n)(x)\}\|_{E_d} + \gamma\|x\|_E, \quad x \in E.\end{aligned}$$

This gives

$$\begin{aligned}&(1-\mu)\|\{(\beta_n g_n)(x)\}\|_{E_d} \\ &\leq \left((1+\lambda)\|T\|\left(\sup_{1\leq n<\infty} \alpha_n\right) + \gamma\right)\|x\|_E, \quad x \in E.\end{aligned}$$

Since $E_d \subset \ell^\infty$, we get

$$\begin{aligned}&(1-\mu)\left(\inf_{1\leq n<\infty} \beta_n\right)\|\{g_n(x)\}\|_{E_d} \\ &\leq (1-\mu)\|\{(\beta_n g_n)(x)\}\|_{E_d} \\ &\leq \left((1+\lambda)\|T\|\left(\sup_{1\leq n<\infty} \alpha_n\right) + \gamma\right)\|x\|_E, \quad x \in E.\end{aligned}$$

Also, by condition (ii), we get

$$\begin{aligned}(1+\mu)\|\{(\beta_n g_n)(x)\}\|_{E_d} &\geq (1-\lambda)\|\{(\alpha_n f_n)(x)\}\|_{E_d} - \gamma\|x\|_E \\ &\geq \left((1-\lambda)(\|S\|)^{-1}\left(\inf_{1\leq n<\infty} \alpha_n\right) - \gamma\right)\|x\|_E, \ x \in E.\end{aligned}$$

Therefore

$$\begin{aligned}&(1+\mu)\left(\sup_{1\leq n<\infty} \beta_n\right)\|\{g_n(x)\}\|_{E_d} \\ &\geq (1+\mu)\|\{(\beta_n g_n)(x)\}\|_{E_d} \\ &\geq \left((1-\lambda)(\|S\|)^{-1}\left(\inf_{1\leq n<\infty} \alpha_n\right) - \gamma\right)\|x\|_E, \qquad x \in E.\end{aligned}$$

Hence

$$\begin{aligned}&\left(\frac{(1-\lambda)(\|S\|)^{-1}\left(\inf_{1\leq n<\infty} \alpha_n\right) - \gamma}{(1+\mu)\left(\sup_{1\leq n<\infty} \beta_n\right)}\right)\|x\|_E \\ &\leq \|\{g_n(x)\}\|_{E_d} \\ &\leq \left(\frac{(1+\lambda)\|T\|\left(\sup_{1\leq n<\infty} \alpha_n\right) + \gamma}{(1-\mu)\left(\inf_{1\leq n<\infty} \beta_n\right)}\right)\|x\|_E, \qquad x \in E.\end{aligned}$$

Put $U = SL$. Then $U : E_d \to E$ be a bounded linear operator such that $U\{g_n(x)\} = x$, $x \in E$. Hence $(\{g_n\}, U)$ is a Banach frame for E with respect to E_d and with desired frame bounds. □

Remark 1. Positive confinedness of sequence $\{\alpha_n\}$, $\{\beta_n\}$ in $\mathbb{R}$ is necessary. Indeed, if $\{\alpha_n\}$ is not positively confined, then either $\inf_{1\leq n<\infty} \alpha_n = 0$ or $\sup_{1\leq n<\infty} \alpha_n$ is infinite. So we get either negative lower frame bounds or an infinite upper frame bounds for the Banach frame $(\{g_n\}, U)$. Also, if $\{\beta_n\}$ is not positively confined, then either upper frame bound is infinite or lower frame bound in zero. In both the cases the frame inequality is lost.

Let $E = \ell^\infty$ and $E_d = E$. Let $(\{f_{1,n}\}, S_1)$ $(\{f_{1,n}\} \subset E^*, S_1 : E_d \to E)$ be a Banach frame for E with respect to E_d. Define $\{f_{2,n}\} \subset E^*$ by

$$\begin{cases} f_{2,1} = f_{1,1} \\ f_{2,2} = f_{1,1} \\ f_{2,n} = f_{1,n-1}, \ n = 3, 4, 5, \dots, \end{cases}$$

then there exists a reconstruction operator S_2 such that $(\{f_{2,n}\}, S_2)$ is a Banach frame for E.

Define $\{g_{1,n}\}$ and $\{g_{2,n}\}$ in E^* by

$$\begin{cases} g_{1,1} = 0 \\ g_{1,n} = f_{1,n}, \ n = 2,3,4,\dots, \end{cases}$$
$$\begin{cases} g_{2,1} = 0 \\ g_{2,2} = 0 \\ g_{2,n} = f_{1,n-1}, \ n = 3,4,\dots. \end{cases}$$

Then, for suitable choice of λ and μ,

$$\|\{(f_{i,n} - g_{i,n})(x)\}\|_{E_d} \leq \lambda \|\{f_{i,n}(x)\}\|_{E_d} + \mu\|x\|_E, x \in E, \quad i = 1,2$$

is satisfied. But there exists, in general, no reconstruction operator $U : E_d \to E$ such that $\left(\left\{\sum_{i=1}^{2} g_{i,n}\right\}, U\right)$ is a Banach frame for E. So it is natural to ask the question that under what sufficient conditions, $\left(\left\{\sum_{i=1}^{2} g_{i,n}\right\}, U\right)$ is a Banach frame for E. The following theorem gives such sufficient conditions in a more general setup.

Theorem 3.4. *For $i \in \Lambda_k = \{1,2,3,\dots,k\}$, let $(\{f_{i,n}\}, S_i)$ $(\{f_{i,n}\} \subset E^*, S_i : E_d \to E)$ be a Banach frame for E with respect to E_d. Let $\{g_{i,n}\} \subset E^*$ be such that $\{g_{i,n}(x)\} \in E_d$, $x \in E$, $i \in \Lambda_k$ and let $L : E_d \to E_d$ be a bounded linear operator such that $L\left\{\left(\sum_{i\in\Lambda_k} g_{i,n}\right)(x)\right\} = \{f_{p,n}(x)\}$, for some $p \in \Lambda_k$. If there exist non negative constants λ, μ such that*

(a) $\lambda \sum_{i\in\Lambda_k} \|T_i\| + k\mu < (\|S_j\|)^{-1} - \sum_{\substack{i\in\Lambda_k\\ i\neq j}} \|T_i\|$, *for some* $j \in \Lambda_k$

(b) $\|\{(f_{i,n} - g_{i,n})(x)\}\|_{E_d} \leq \lambda\|\{f_{i,n}(x)\}\|_{E_d} + \mu\|x\|_E, x \in E,\ i \in \Lambda_k,$

then there exists a reconstruction operator U such that $\left(\left\{\sum_{i\in\Lambda_k} g_{i,n}\right\}, U\right)$ is a Banach frame for E with respect to E_d and with frame bounds

$$\left((\|S_j\|)^{-1} - \left[\sum_{\substack{i\in\Lambda_k\\ i\neq j}} \|T_i\| + \lambda \sum_{i\in\Lambda_k} \|T_i\| + k\mu\right]\right)$$

and

$$\left((1+\lambda)\sum_{i\in\Lambda_k} \|T_i\| + k\mu\right),$$

where T_i is the coefficient mapping given by $T_i x = \{f_{i,n}(x)\}$, $x \in E$, $i \in \Lambda_k$.

Proof. For each $i \in \Lambda_k$, $S_i T_i$ is an identity operator on E. Therefore

$$\|x\|_E = \|S_i T_i(x)\|_E \leq \|S_i\| \, \|\{f_{i,n}(x)\}\|_{E_d}, \quad x \in E. \tag{3.1}$$

Also

$$\left\| \sum_{i \in \Lambda_k} \{f_{i,n}(x)\} \right\|_{E_d} \leq \left(\sum_{i \in \Lambda_k} \|T_i\| \right) \|x\|_E, \quad x \in E. \tag{3.2}$$

Now

$$\begin{aligned}
&\left\| \left\{ \left(\sum_{i \in \Lambda_k} g_{i,n} \right)(x) \right\} \right\|_{E_d} \\
&= \left\| \sum_{i \in \Lambda_k} \{(f_{i,n} - (f_{i,n} - g_{i,n}))(x)\} \right\|_{E_d} \\
&\geq \left\| \sum_{i \in \Lambda_k} \{f_{i,n}(x)\} \right\|_{E_d} - \left\| \sum_{i \in \Lambda_k} \{(f_{i,n} - g_{i,n})(x)\} \right\|_{E_d} \\
&\geq \left\| \{f_{j,n}(x)\} + \sum_{\substack{i \in \Lambda_k \\ i \neq j}} \{f_{i,n}(x)\} \right\|_{E_d} - \sum_{i \in \Lambda_k} \|\{(f_{i,n} - g_{i,n})(x)\}\|_{E_d}.
\end{aligned}$$

By using (3.1) and (3.2), we get

$$\begin{aligned}
&\left\| \left\{ \left(\sum_{i \in \Lambda_k} g_{i,n} \right)(x) \right\} \right\|_{E_d} \\
&\geq \left((\|S_j\|)^{-1} - \left(\sum_{\substack{i \in \Lambda_k \\ i \neq j}} \|T_i\| + \lambda \sum_{i \in \Lambda_k} \|T_i\| + k\mu \right) \right) \|x\|_E, \qquad x \in E.
\end{aligned}$$

Also, using (3.2), we obtain

$$\left\| \left\{ \left(\sum_{i \in \Lambda_k} g_{i,n} \right)(x) \right\} \right\|_{E_d} \leq \left[(1+\lambda) \sum_{i \in \Lambda_k} \|T_i\| + k\mu \right] \|x\|_E, \quad x \in E.$$

Put $U = S_p L$. Then $U : E_d \to E$ is a bounded linear operator such that $U\left(\left\{\left(\sum_{i \in \Lambda_k} g_{i,n}\right)(x)\right\}\right) = x$, $x \in E$. Hence $\left(\left\{\sum_{i \in \Lambda_k} g_{i,n}\right\}, U\right)$ is a Banach frame for E with respect to E_d and with desired frame bounds. □

Remark 2. The condition (a) in Theorem 3.4 is not necessary. Indeed, if $(\{f_{1,n}\}, S_1)(\{f_{1,n}\} \subset E^*, S_1 : E_d \to E)$ is a normalized tight Banach frame for E with respect to E_d, then, for $f_{2,n} = g_{1,n} = g_{2,n} = f_{1,n}$, $n \in \mathbb{N}$,

$\sum_{i=1}^{2} g_{i,n} = 2f_{1,n}$. So there exists a reconstruction operator $U : E_d \to E$ such that $\left(\left\{\sum_{i=1}^{2} g_i\right\}, U\right)$ is a Banach frame for E with respect to E_d. Further, since $(\{f_{1,n}\}, S_1)$ is a normalized tight Banach frame, it is easy to conclude that the condition (a) in Theorem 3.4 is not satisfied.

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