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RECTIFIABILITY PROPERTIES OF VARIFOLDS IN l_∞^3

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Abstract. We prove the following theorem: Given a Varifold V in l_∞^3 with the property that $0 < \lim_{r\to 0} \frac{\mu_V(C_r(x))}{r^2} < \infty$ for μ_V a.e.$x \in SptV$, then V is rectifiable.

1. Introduction

The study of the tangential properties and rectifiable properties of measures has only a short but interesting history. It has been an active field over the past five decades. Much geometric information is carried by tangent measures of a measure. Especially if such tangent measures or measures are rectifiable, one can obtain some interesting further regularity results with respect to them. D. Preiss in [21] investigated to what extent the regular behavior of the measure of balls determine the tangential and rectifiability properties of measures. In other words, he compared the general measures with those of balls, namely, Hausdorff measures or the special measures acted on balls. He used the fundamental works of A. S. Besicovitch [5], [6] and the technique of H. Federer [9] to study the Geometry of measures in R^n, for instance, Distribution, Rectifiability, and Densities.

It is well known that a measure μ is $m-$rectifiable if it satisfies condition (BP)(one can see [21] for details). Up to now one knew that Case a: $m = 1$, $n = 2$ [6]; Case b: $m = 1$, n arbitrary [18]. On the other hand, when $m \geq 2$, there is not any comparable statement being hold. Up to 1987, the works of D. Preiss was published, then these questions just as above ($m \geq 2$) are solved, but all these results were considered in Euclidean spaces. It is natural to ask how to determine the tangential and rectifiability properties of a measure when $m \geq 2$ under the general norm spaces. In this paper, we only consider Case c: $m = 2$ and choose the model problem space in l_∞^3 which is the concrete space R^3 with sup norm. The grounds, we study this problem in l_∞^3, are the following two folds. The first fold is that many things associated with this problem can be

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computed directly; the second fold is the fact that any metric space is isometric to some subset of l_∞ [10]. So the first interesting thing related to this problem is the case when $m = 2$. For the finite dimensional normed vector spaces, We think that the problem proposed in this paper may be true, and that the corresponding result will be a stronger one. When $m \neq 2$, we will introduce a new method to study it in the next paper "A Marstrand Theorem for Cube in R^d with respect to Varifolds" [26]. Furthermore, we consider a Varifold as a Radon measure defined on Grassmannian $G(3,2)$. Because Varifolds vanish the orientation and provide the measure properties, many problems related to stationary of currents or measures may be considered there.

On the other hand, the rectifiability of a Radon measure with positive finite density in Euclidean space, was a central problem in Geometric Measure Theory for fifty years. Of course this problem was resolved by D. Preiss but it has been a well known open problem in Non-Euclidean spaces. Just as this we choose space l^3_∞ as a model space to study the rectifiability of Varifolds or general Radon measures. For Non-Euclidean spaces, L. Ambrosio and B. Kirchheim recently [3], [4] study the currents in metric spaces and rectifiable sets in metric and Banach spaces, respectively. Although they solved the generalized Plateau problem by using currents techniques in some Non-Euclidean space, these problems are closely related to the orientation. So, a natural problem is, for the non-orientation surfaces, for instance, Varifolds, whether the similar Plateau problem can be solved by virtue of the properties of Varifolds, and by the way, whether there is any chance to find the further interesting geometric and analysis properties of Varifolds.

Motivated by these statements and so on, we think that the materials selected here are of momentous current significance.

Throughout the paper, we take the notations and terminologies adopted by D. Preiss in [21]. For the sake of convenience, we will introduce some known concepts and results without proofs in the following sections.

We now can sketch this paper and consider the problem along the following procedure.

Section 2 gives the necessary preliminaries and some results. Some important results will considered in Section 3, and we will give the proofs of these results. Section 4 is devoted to some examples and Section 5 is devoted to the short conclusion about this subjects . On the other hand, we pose the train of thought of study adopted in this paper as follows.

(1) D. Preiss's results [21](for Radon measures)

Let μ be a Radon measure on R^n with property

$$0 < \lim_{r \to 0} \frac{\mu(B_r(x))}{r^m} < \infty$$

for μ a.e.$x \in Spt\mu$, then μ is rectifiable.

(2) Well known results (for Hausdorff measures)

Given a metric space M with metric d, let $\mathcal{H}^m_d$ denote Hausdorff measures where the diameter of the covering bodies is taken with respect to d. Suppose M has the property

$$\lim_{r\to 0} \frac{\mathcal{H}^m_d(B(x,r))}{\alpha(m)r^m} = 1$$

for $\mathcal{H}^m_d$ a.e. points $x \in M$, then M is rectifiable in the sense that $\mathcal{H}^m_d$ almost all of M is contained in countably many *Lipschitz* images of subsets of an m-dimensional Euclidean space.

From this result we know **Kirchheim**'s result [3], in briefly, that Rectifiability implies density is tenable in metric spaces

(3) A. Lorent's result [13]**(for Radon measures)**

Assume that μ measures l^3_∞. If μ is a locally 2-uniform measure in l^3_∞, then it is rectifiable.

In this paper we wish to study the rectfiabilities of a measure in l^3_∞ by using density condition replacing the uniform measure condition, that is, we prove the following measure theorem

(4) Main Theorem (for Radon measures and Varifolds)

Given a Varifold V on l^3_∞ with the property that

$$0 < \lim_{r\to 0} \frac{\mu_V(C_r(x))}{r^2} < \infty$$

for μ_V a.e.$x \in SptV$, then V is rectifiable.

2. Preliminaries

Let R^n be the set of real valued $n-$ vectors, e_1, e_2, e_3 be orthonormal vectors in R^3 and $e_{j+3} = -e_j$ for $j \in \{1,2,3\}$. Let C_r be the open cube of radius r centered on x, where sides are perpendicular to the orthonormal vectors, and S be a finite dimensional normed vector space. We denote by $\|\cdot\|_E$ the Euclidean norm, and by $\|\cdot\|$ the sup norm. So that $\|x\| = \max\{|e_1\cdot x|, |e_2\cdot x|, |e_3\cdot x|\}$. Assume that l^3_∞ denotes the space with the sup norm. $G(m,n)$ denotes the collection of all the sets of m-dimensional linear subspaces of R^n. We also denote by $\mathrm{cl}(A) = \{x \in R^n : \text{there exists } z_n \in A \ s.t. z_n \to x \text{ as } n \to \infty \text{ for A} \subset R^n\}$ the closure of A, and $\partial A = \mathrm{cl}(A) \cap \mathrm{cl}(R^n \setminus A)$ for any $A \subset R^n$ the topological boundary of A. We again introduce some necessary notations here. Let $T^{(r)}_j$ be the side of $\mathrm{cl}(C_r(0))$ perpendicular to e_j and which intersects the line $< e_j >$ for $j = 1,2,\ldots,6$, and $S^{(0)}_j = \cup_{r\geq 0} T^{(r)}_j$ for $j = 1,2,\ldots,6$. We also denote by $S^{(0)}_{j+6} = S^{(0)}_j$ for $j = 1,2,\ldots,6$, and $S^{(x)}_j = S^{(0)}_j + x$ for $j = 1,2,\ldots,6$.

For $y \in Spt\mu$, let μ_r denote the induced measure of μ onto $\partial C_r(y)$, i.e., for any $A \subset l_\infty^3$ we have

$$\mu(A) = \int_{r>0} \mu_r(\partial C_r(y) \cap A) d\mathcal{L}^1 r$$

where $C_r(x) = \{z : \|z - x\| < r\}$. By using the symmetry of the pyramid we easily know that

$$\mu_r(S_j^{(x)} \cap \partial C_r(x)) = \mu_r(S_{j+3}^{(x)} \cap \partial C_r(x)).$$

Let $f_i^{(y)}(r) = \mu_r(\partial C_r(y) \cap S_i^{(y)})$, if $y \in G$, it is not hard to show that $f_i^{(y)}$ is a Lipschitz non-decreasing function. At the same time, we denote by $A(x, s, t) = C_t(x) \backslash \mathrm{cl}(C_s(x))$. By using Definition of the induced measure, we can write down μ_s as follows

$$f_j^{(y)}(s) = \lim_{h \to 0} \frac{\mu(S_j^{(y)} \cap A(y, s-h, s+h))}{2h} = \mu_s(S_j^{(y)} \cap \partial C_s(y))$$

for all such $s > 0$ and for any $y \in Spt\mu$ if this limit does exist. We denote by $L^{(y)}$ the set of points $s > 0$ for which the derivatives of $f_1^{(y)}$, $f_2^{(y)}$, $f_3^{(y)}$ all exist at s. Let $X(x, v, r) = \{y \in R^3 : |P_{v^\perp}(y - x)| \le s|P_{<v>}(y - x)|\}$ for any $x \in R^3$, $v \in S^2$, $s > 0$, where $P_\tau : R^3 \to \tau$ is the orthogonal projection onto τ for any linear subspace $\tau \subset R^3$. Furthermore, we denote by $\tilde{G} = \{x \in Spt\mu : \forall \delta > 0, \psi \in S^2, \limsup_{r \to 0} \frac{\mu(C_r(x) \cap X(x,\psi,\delta))}{r^2} > 0\}$ the set of points with positive cone density. At the same time, we also denote by $G = \{x \in \tilde{G} : \lim_{r \to 0} \frac{\mu(B_r(x) \backslash \tilde{G})}{r^2} = 0\}$ the set of density points of $\tilde{G}$, in $\tilde{G}$.

In this paper we say a measure over R^n is a map μ of the family of all subsets of R^n into $[0,\infty]$ s.t.

$$\mu(A) = \inf\{\sum_{B \in \mathcal{F}} \mu(B); A \subset R^n\}$$

for every set $A \subset R^n$ and $\mathcal{F}$ is a countable cover of A by Borel subsets of R^n. According to the theory of general Varifolds [1], [23], one knows that general Varifolds in U (U open in R^n) are simply Radon measures on $G_m(U) = \{(x, S) : x \in U, S \subseteq R^n\}$ where S is an m-dimensional subspace of R^n.

Definition 2.1. (Varifold) An m-Varifold V, briefly speaking, means a Radon measure on $G_m(R^n)$.

Given an such m-Varifold V on U ($U \subset R^n$), there corresponds a Radon measure $\mu = \mu_V$ on U(called the weight of V) defined by

$$\mu(A) = V(\pi^{-1}(A)), A \subset U$$

where π is the projection $(x, S) \longmapsto x$ of $G_m(U)$ onto U.

Definition 2.2. (Rectifiable Sets) A set $U \subset R^n$ is called $m-$rectifiable if there exist a finite or countable set J and Lipschitz maps $f_j, j \in J$ from R^m to R^n such that $\mathcal{H}^m(E \setminus \bigcup_{j\in J} f_j(E_j)) = 0$, where $E_j \subset R^m$, for $j \in J$.

Definition 2.3. (Rectifiable Measures) The measure μ in $U \subset R^n$ is said to be m-rectifiable if it is absolutely continuous with respect to $\mathcal{H}^m$ and there exists an $m-$rectifiable Borel set $E \subset U$ with $\mu(E \setminus E) = 0$.

Definition 2.4. (m-rectifiable Varifolds) Given an $m-$Varifold V, we associated a Radon measure on U, μ_V, by setting $\mu_V = V(\pi^{-1}(A)), A \subset U$. Given an $m-$rectifiable measure μ_V, we can associate an $m-$rectifiable Varifold μ defining by $\mu(B) = \mu_V(\{x : (x, T_x) \in B\})$ for $B \subset U \times G(n,m)$, where T_x is the approximate tangent plane at x.

Similarly, the analogues of m-Varifolds and $m-$rectifiable Varifolds in l^3_∞ can be defined as above. It is well known that B. Kirchheim [5] studied the k-dimensional density $\lim_{r\to 0} \frac{\mathcal{H}^k(S\cap B_r(x))}{\omega_k r^k}$ of rectifiable sets with finite measure. L. Ambrosio and B. Kirchheim [4] studied the rectifiable sets in metric spaces and Banach spaces. In [4], the authors posed the inverse problem in metric spaces, that is, the following inverse problem: whether equality

$$\lim_{r\downarrow 0} \frac{\mathcal{H}^k(E \cap B_r(x))}{\omega_k r^k} = 1$$

for $\mathcal{H}^k$-$a.e.x \in E$ implies the so-called rectifiability for a general metric space E. Up to now we know that this problem is open.

In this paper we take the metric space E is l^3_∞. In fact we only want to use the behavior of l^3_∞ preserving the Lipschitz constant. At the same time, Varifolds as measures defined on $G_m(U)$ have a special structure Theorem (see Lemma 2.1 below).

Lemma 2.1. *Let V be an m-Varifold on U (U open in R^n). Then for μ_V-a.e.$x \in U$, there is a Radon measure η^x_V on $G(n,m)$ such that, for any continuous function $\beta(s)$ on $G(n,m)$,*

$$\int_{G(n,m)} \beta(S) d\eta^{(x)}_V(S) = \lim_{\rho\downarrow 0} \frac{\int_{G_m(B_\rho(x))} \beta(S) dV(y,S)}{\mu_V(B_\rho(x))}.$$

Furthermore for any Borel set $A \subset U$, if $\beta \geq 0$, then we arrive at

$$\int_{G_m(A)} \beta(S) dV(x,S) = \int_A \int_{G(n,m)} \beta(S) d\eta^{(x)}_V d\mu_V(x).$$

Proof. By virtue of the proof offered in [23], it is easy to derive that Lemma 2.4 is tenable in l^3_∞. □

For the needs of latter part of this paper, we now give out Definition of a locally 2-uniform measure as follows.

Definition 2.5. A locally 2-uniform measure μ is a measure with the property that for every $x \in Spt\mu$, $r \in (0,1)$ there holds $\mu(C_r(x)) = 4r^2$, where if $r > 0$ is arbitrary then the words "locally" will be omitted, namely the 2-uniform measure.

3. Main theorems and proofs

In this section we first study the properties of measures with respect to Problem 4 and then prove the useful Theorem 3.1 which will play an important role in developing D. Preiss's theory. For the later use, we state some Lemmas (see [10, 13]) as follows.

Lemma 3.1. ([13]) *Given a Radon measure μ on l_∞^3 with property that*

$$0 < \lim_{r\to 0} \frac{\mu(C_r(x))}{r^2} < \infty$$

for μ a.e.$x \in Spt\mu$. Let μ_r be the induced measure on $\partial C_r(x)$, and $f_i^{(x)}(r) = \mu_r(S_i^{(x)} \cap \partial C_r(x))$, then $f_i^{(x)}$ is a monotone non-decreasing Lipschitz map and $f_i^{(x)}(r) > 0$ for all $r \in (0,1) \cap L^{(x)}$.

Lemma 3.2. ([10]) *Assume that Y^* is the conjugate space of Y. If $h : X \to Y$ is a Lipschitzian map of metric spaces, $A \subset X, 0 \le k < \infty$ and $0 \le m < \infty$, then one arrives at the following formula*

$$\int_* \mathcal{H}^k(A \cap h^{-1}\{x\})d\mathcal{H}^m(x) \le (Liph)^m \frac{\alpha(k)\alpha(m)}{\alpha(k+m)}\mathcal{H}^{k+m}(A)$$

provided either $\{x : \mathcal{H}^k(A \cap h^{-1}\{x\}) > 0\}$ is the union of a countable family of sets with finite $\mathcal{H}^m$ measure, or Y is boundedly compact.

Lemma 3.3. ([10]) *If $f : R^m \to R^n$ is a Lipschitzian, s is a positive integer, and T is a purely unrectifiable Borel subset of R^n, then we know that*

$$\dim \operatorname{im} Df(x) < s$$

for $\mathcal{L}^m$ almost all x in $f^{-1}(T)$.

Remark 3.1. For Definition of a purely unrectifiable set, one can see [10], [11], [12] for details. On the other hand, We can also prove Lemma 3.3 by using the method offered in [22]. We omit the special proving procedure here. Of course, we here refer to [10] for details.

Definition 3.1. Let (E, d) be a metric space. If there exists a sequence $(\varphi_i) \subset Lip_1(E)$ such that $d(x, y) = \sup_{i\in N} |\varphi_i(x) - \varphi_i(y)|$ for $\forall x, y \in E$, then E is said to be weakly separable where $Lip_1(E)$ denotes the collection of Lipschitz functions with Lipschitz constant less than 1. A dual Banach space $Y = E^*$ is said to be w^*-separable if E is separable.

Lemma 3.4. *Let Y be a w^*- separable dual space. Assume that $A \subset Y$ is w^*- compact and $h : A \to R$ is Lipschitz and w^*- continuous. Thus there is a uniformly w^*-continuous map $\tilde{h} : Y \to R$ such that $\tilde{h}|_A = h$, $\sup|\tilde{h}| = \sup|h|$, and $Lip(\tilde{h}) = Lip(h)$.*

Proof. By Kirszbraun's theorem or [10], one can obtain Lemma 3.4. □

Theorem 3.1. *Given a Radon measure μ on l^3_∞ with property*

$$0 < \lim_{r\to 0} \frac{\mu(C_r(x))}{r^2} < \infty$$

for μ a.e.$x \in Spt\mu$. Let μ_r be the induced measure on $\partial C_r(x)$, and let W be an rectifiable and $\mathcal{H}^m$ measurable subset of R^m, $h : W \to l^3_\infty$ is a Lipschitz map, $\lambda > 0$ and

$$\mathcal{R} = l^3_\infty \cap \{x \in Spt\mu : \mu_r(h^{-1}\{x\}) \geq \lambda\}$$

then $\mathcal{R}$ is rectifiable.

Proof. Since W is an $(\mathcal{H}^m, m)$ rectifiable and measurable subset of R^m, then it is not hard to know that there exist compact subsets $K_1, K_2, \ldots$ of R^m and Lipschitz maps $\varphi_1, \varphi_2, \ldots$ of R^m into R^m such that $\varphi(K_1), \varphi(K_2), \varphi(K_3), \ldots$ are disjoint subsets of W with $\mathcal{H}^m[W \sim \cup_{i=1}^{\infty}\varphi_i(\mathrm{K}_i)] = 0$. At the same time, one also knows that for each positive integer i, the following inequalities are tenable:

$$Lip(\varphi_i) \leq \lambda, Lip[(\varphi_i|K_i)^{-1}] \leq \lambda, \lambda^{-1}|v| \leq |\langle v, D\varphi_i(a)\rangle| \leq \lambda|v|$$

for $a \in K_i$, $v \in R^m$. Just as these statements, we can consider that W is compact, then we can say that $\mathcal{R}$ is a Borel set according to [10] or [23]. On the other hand, we can extend h to a Lipschitz map $\tilde{h} : R^m \to l^3_\infty$ and take $V = W \cap \{x : \dim \mathrm{im}D\tilde{h}(x) < 2\}$. Since $\mathcal{R}$ is Borel set, we take $m = 3, k = 1$ in Lemma 3.2, then we can choose a countably 2-rectifiable Borel subset $\tilde{\mathcal{R}}$ of l^3_∞ such that $\mathcal{R} \setminus \tilde{\mathcal{R}}$ is purely unrectifiable. In other words, we should know that $\mathcal{L}^3(h^{-1}(\mathcal{R}|\tilde{\mathcal{R}}) \setminus V) = 0$ from Lemma 3.3. That is to say that $\mu_r(\partial C_r \cap C_r \cap h^{-1}\{x\} \setminus V) = 0$ for μ a.e.$x \in \mathcal{R}\setminus\tilde{\mathcal{R}}$ in terms of Lemma 3.2 and the conditions of Theorem 3.1. Then one arrives at $\mu_r(\partial C_r \cap h^{-1}(x) \cap V) \geq \lambda$ for μ a.e. $x \in \mathcal{R}\setminus\tilde{\mathcal{R}}$. In fact, if W is bounded compact, $A \subset W$, and $h : W \to Y$ is a Lipschitz map of metric spaces. If A is $\mu-$measurable and $\mu(A) < \infty$, then $\mu_r(\partial C_r(x) \cap A \cap h^{-1}\{x\})$ is μ-measurable. Namely, $\partial C_r(x) \cap A \cap h^{-1}\{x\}$ is μ_r-measurable for μ-a.e.x.

Case 1. For all $0 < \rho \leq r$, if $\mu_\rho \in MBV(A)$, then we can prove Theorem 3.1 as follows.

For any $\lambda, \delta > 0$, we can define $Z_{\lambda\delta}$ as the collection of all points $\rho \in (0, 1)$ such that

$$\mu_\rho(\partial C_\rho \cap A \cap h^{-1}\{x\}) \geq \lambda \Rightarrow \mu_\rho(B_{3\delta}(x) \setminus \{x\}) \leq \frac{\lambda}{3}$$

for any $x \in K \subset Spt\mu$, where K is a compact set.

For the sake of convenience, we denote by $\mathcal{R}_{\lambda\delta}$ the following set:

$$\mathcal{R}_{\lambda\delta} = \{x \in K \subset Spt\mu : \mu_\rho(\partial C_\rho(x) \cap A \cap h^{-1}\{x\}) \geq \lambda\}$$

for any $\rho \in (0,1)$. We notice that $\mathcal{R} = \cup_{\lambda,\delta>0}\mathcal{R}_{\lambda\delta}$, hence it suffices to prove that $\mathcal{R}_{\lambda\delta}$ is countably $\mathcal{H}^2$–rectifiable.

In fact, we only derive for $\rho \in (0, r)$ that μ_ρ is a Lipschitz function *w.r.t.* ρ, and by a covering argument one can prove this conclusion immediately. Now denoting by O any subset of $\mathcal{R}_{\lambda\delta}$ with diameter less than δ, we now check that there exists a constant $c(k)$ such that

$$d(x, \acute{x}) \leq \frac{3c(k)}{\lambda^2}[\delta + \frac{1}{\delta}]|\rho - \acute{\rho}|$$

whenever $x, \acute{x} \in O$, $\mu_\rho(\partial C_\rho \cap h^{-1}\{x\}) \geq \lambda$ and $\mu_\rho(\partial C_\rho \cap h^{-1}\{\acute{x}\}) \geq \lambda$, and $\rho, \acute{\rho} \in (0, r)$. By the calculus of representation theory, we let in short $d = d(x, \acute{x}) \leq \delta$ and define $\phi(y) = d(y, x)$ in $B_d(x)$, and $\phi(y) = 0$ in $E \setminus B_{2d}(x)$ and $|\phi| = d, Lip(\phi) \leq \frac{1}{\delta}$. Since we can always stand for $\mu_r(\phi)$ as the integral $\int_E \phi d\mu_r$, then we know that

$$|\mu_r(\phi)(y)| = |\int_{B_d(x)\setminus\{x\}} \phi(y) d\mu_r| \leq d \int_{B_d\setminus\{x\}} d\mu_r \leq \frac{\lambda}{3} d.$$

On the other hand, we can derive by a direct computation that

$$\mu_{\acute{r}}(\phi) \geq \int_{\partial B_d(x)} \phi d\mu_{\acute{r}} = d(\int_{\{\acute{x}\}} + \int_{\partial B_d(x)\setminus\{\acute{x}\}} \phi d\mu_{\acute{r}}) \geq \lambda d - \frac{\lambda d}{3}.$$

At the same time, we can think measure μ_r as a linear functional by Riesz expression theorem, and then we know that Theorem 3.1 holds by also using the covering argument, Lemma 3.1 and Theorem 7.3 in [3].

Case 2. In general case, Case 1 is not tenable at this very moment. In this setting for given any $\epsilon > 0$, let Z be a countable, dense subset of the set of all linear symmetric automorphisms f of l^3_∞ with $|\det(f)| < \epsilon$ or $\|\wedge_2 f\| < \epsilon$, where $\wedge$ denotes the exterior product. We consider the automorphism $f \in Z$ and any positive integers i, j with the following properties, i.e., the subset $U(f, i, j)$ of V consisting of all points x s.t. $\|f^{-1} \circ D\tilde{h}(x)\| \leq 1 - i^{-1}$ and

$$|f^{-1}[\tilde{h}(y) - \tilde{h}(x) - < y - x, D\tilde{h}(x) >] \leq i^{-1}|y - x|$$

for any $y \in B(x, j^{-1})$.

Now we consider the special subset with $E \subset U(f, i, j)$ and $\text{diam}E \leq \frac{1}{j}$, then we derive, by using the statements above, that

$$\begin{aligned} |(f^{-1} \circ \tilde{h})y - (f^{-1} \circ \tilde{h})x| &\leq | < y - x, f^{-1} \circ D\tilde{h}(x) > | + i^{-1}|y - x| \\ &\leq (1 - i^{-1})|y - x| + i^{-1}|y - x| = |y - x| \end{aligned}$$

whenever $x, y \in E$, that is to say, $Lip(f^{-1} \circ h|E) \leq 1$. By Lemma 3.2 and Area formula [12], it is not hard to derive that for any subset $A \subset W$, there holds

$\mu(h(A)) \leq C\| \wedge_2 h\|\mu(A)$, where C is a constant. From these statements we see that the following inequalities are tenable.

$$\begin{aligned}&\int_* \mu_r(\partial C_r \cap E \cap h^{-1}\{x\})d\mu(x)\\ &= \int_* \mu_r[\partial C_r \cap E(f^{-1} \circ h)^{-1}\{f^{-1}(x)\}]d\mu(x)\\ &\leq C_0\| \wedge_2 f\| \int_* \mu_r(\partial C_r \cap E \cap (f^{-1} \circ h)\{y\})d\mu(y)\\ &\leq C_1\mathcal{L}^3(E)\end{aligned}$$

where C_0,C_1 are two constants. The last inequality is tenable because of the hypotheses of Theorem. On the other hand, V is the union of sets $U(f,i,j)$ and we can represent V as the union of a countable disjointed family $\mathcal{F}$ consisting of $\mathcal{L}^3$ measurable sets E for which

$$\int \mathcal{H}(E \cap h^{-1}\{x\})d\mathcal{H}^2(x) \leq c\epsilon\mathcal{L}^3(E)$$

where c is a constant. Next we are going to considering the summation over $\mathcal{F}$ and see that

$$\lambda\mathcal{H}^2(\mathcal{R} \setminus \widetilde{\mathcal{R}}) \leq c\epsilon\mathcal{L}^3(V).$$

This implies that $\mathcal{H}^2(\mathcal{R} \setminus \widetilde{\mathcal{R}}) = 0$. In other words, we prove the rectifiability of $\partial C_r(x) \cap Spt\mu$ *w.r.t.* μ_r for arbitrary real number $r > 0$.

By synthesizing the relevant results of Case 1 and Case 2, the remanent work for us is to prove the rectifiability of $Spt\mu \cap C_r$. In fact, we represent $C_r \cap Spt\mu$ by the bundle with the analogue of a Grassmann. That is to say, we have the following expression $\bigcup_{0<\rho<r} \partial C_\rho \cap Spt\mu$, and then rewrite down the measure μ as the product measure $\mu_r \times \nu|_{[0,t]}$ for $t > 0$, where measure ν is defined on $[0,t]$. Since interval $[0,t]$ can be regarded as a continuum by using the results in [20], one arrives at that it has a finite $\mathcal{H}^1-$ measure consisting of a countable union of rectifiable curves. From the views of Hausdorff measure definition and the theorem posed in [23], one can obtain this conclusion. In other words, measure μ can be regarded as the measure $\mu_r \times \mathcal{H}^1$. It is immediately that Theorem 3.1 is tenable. This completes the proof of Theorem 3.1. □

Remark 3.2. Theorem 3.1 shows that a Radon measure μ, needless to be a Varifold, satisfying the conditions offered in Theorem 3.1, possess the rectifiability. It is obvious that Theorem 3.1 is certainly tenable for Varifolds. By virtue of this special properties of Varifolds, we think about that the proof of Theorem 3.1 for Varifolds must be being in another way. With regard to the knowledge of Varifolds, one can refer to [1] for details. We can now write down an interesting result for Varifolds as follows.

Theorem 3.2. *Given a 2-Varifold V defined on $G_2(l^3_\infty)$ with property*

$$0 < \lim_{r\to 0} \frac{\mu_V(C_r(x))}{r^2} < \infty$$

for μ_V a.e.$x \in Spt\mu_V$, where $\mu \hat{=} \mu_V$ is a Radon measure on l^3_∞ (called the weight of V) defined by $\mu(A) = V(\pi^{-1}(A)), A \subset U \subset l^3_\infty$, and π is the projection of $G_2(U)$ onto U. Then μ is rectifiable.

Proof. Let $G(3,2)$ denote the collection of all 2-dimensional subspaces of R^3. For a subset $A \subset l^3_\infty$, we define $G_2(A) = A \times G(3,2)$. Considering $B \subset G_2(A)$, then for a 2-Varifold V, we have the following convention:

$$\mu(A) = \mu_V(A) \hat{=} V(\pi^{-1}(A)).$$

We now consider the unit ball $C_1(0)$ centered at region in l^3_∞. It is not hard to prove that $C_r(x) = \{x + ry : y \in C_1(0)\}$. Notice that the linearity of a projection $\pi : (x, S) \longmapsto x$ of $G_2(A)$ onto A, where $S \in G(3,2)$. We can rewrite $\pi^{-1}(C_r(x))$ by $\{(x + ry, S) : (y, S) \in B\}$ for arbitrary $S \in G(3,2)$. In other words, we know that

$$\frac{\mu(C_r(x))}{r^2} = \frac{V(\pi^{-1}(C_r(x)))}{r^2} = \frac{V(\{(x+ry, S) | (y, S) \in B\}) \cap G_2(\pi^{-1}(C_r(x)))}{r^2}.$$

Then we have the formula as follows

$$\frac{\mu(C_r(x))}{r^2} = V_{x,r}(\pi^{-1}(C_r(x)).$$

Since the limit

$$\lim_{r\to 0} \frac{\mu(C_r(x))}{r^2}$$

exists, we see that the limit

$$\lim_{r\to 0} V_{x,r}(\pi^{-1}(C_r(x)))$$

exists simultaneously. On the other hand, by virtue of Lemma 38.4 in [23], one arrives at for any Borel set $A \subset U \subset l^3_\infty$ there exists a Radon measure $\eta^{(x)}_V$ on $G(3,2)$ s.t. for any continuous function $\beta \geq 0$ on $G(3,2)$ the following formula is tenable.

$$\int_{G_2(A)} \beta(S) dV(x, S) = \int_A \int_{G(3,2)} \beta(S) d\eta^{(x)}_V(S) d\mu_V(x).$$

Since the arbitrary of $\beta \geq 0$ on $G(3,2)$ and use the hypothesis of Theorem 3.2, that is, the limit $\lim_{r\to 0} \frac{\mu_V(C_r(x))}{r^2}$ is finite, then we know that there exist some positive number θ and $T \in G(3,2)$ such that

$$\lim_{r\to 0} V_{x,r} = \theta \eta^{(x)}_V(T).$$

By using Definition with respect to tangent spaces in [23], we see that Theorem 3.2 is tenable. This completes the proof of Theorem 3.2. □

Remark 3.3. We note that the proof of Theorem 3.2 is much easier than that of Theorem 3.1. We think that Varifold itself has the associated with the "bundle structure" along the subset offered in the research setting. The "bundle structure" replacing the necessary tangent spaces plays an important role in the way of rectifiability criterion *w.r.t.* measures or sets. Just as this we say that one will obtain some fine and deep results for Varifolds under the same conditions as above. In fact, Theorem 3.2 is only one of all possible results. It is well known that the description of rectifiability of a measure by virtue of the approaches of densities in higher dimension depends mainly on the locally version of Besicovith-Federer Projection Theorem [9], or on the variants of this Projection Theorem [10]. But in this setting, we have not any extant projection theorem in l_∞^3 for us being used to derive our main problem. On the other hand, [14] shows that one can only consider the case of $0 < s \leq 2$ with respect to Criterions of the rectifiability of a measure in terms of $s-$density.

In addition, for Varifolds, since we adopt a geometric analysis method, and also use the induced measure by μ which depends on the structure and dimension of spaces, to derive the rectifiability of Varifolds, and notice that the paper [14], it is natural to choose integer "2" not arbitrary number "s" to study this classical problem. For the case of arbitrary number "s", we will study it in our next paper "A Marstrand Theorem for Cube in R^d with respect to Varifolds " [26]. In that paper we first follow [7] to construct the projection theorem and then adopt the idea of tangent measures to derive the rectifiability of a Radon measure μ, in particular, to derive the rectifiability of Varifolds.

Finally, we can replace space l_∞^3 by R^3 because of equivalence up to Theorem 3.2. For the sake of convenience of representing cube in the proof of Theorem 3.2, we still adopt the space l_∞^3 as our model space. Of cause, we can adopt the space R^3 replacing l_∞^3 in sense of equivalence. Just as being out of considering for consistency, we still adopt l_∞^3 as our model problem space.

Remark 3.4. By using the definition of contact manifolds we see that the so-called Varifolds are just the corresponding analogue in general set setting. From these criterion we think that many approaches posed in contact manifolds will be translated into our setting for studying the properties of Varifolds.

Theorem 3.3. *Given a 2-rectifiable Varifold V defined on $G_2(l_\infty^3)$ with property*

$$0 < \lim_{r\to 0} \frac{\mu_V(C_r(x))}{r^2} < \infty$$

for μ_V a.e.$x \in Spt\mu_V$. Then μ is rectifiable.

Proof. By the proof of Theorem 3.2 we can write measure μ as $\mu = \mu_V = \mathcal{H}^n \lfloor \theta$. Then we see by a direct derivation that there exists a approximate tangent space T_xA satisfying the relationship 11.4 in [23]. Thus, by virtue of Theorem 38.3 in [23], it is not hard to find that Theorem 3.3 is tenable. □

Remark 3.5. In fact, a measure $\mu = \mathcal{H}^n\lfloor\theta$ implies that the studied set has the natural tangent bundle structure. That is to say, we can use the Hausdorff measure to replace the requirement of the tangency of a set at point x. Then, a measure or a set will possess the rectifiability because of the "flatness" ([3]) of Hausdorff measures. On the other hand, it is well known that the tangent measure will possess the "flatness" of a measure which is used to promulgate the rectifiability of measures. Similarly, we can study the rectifiability of sets with the same techniques. With regard to Theorem 3.1 or Theorem 3.2, if we modify the conditions proposed as above by replacing it with density property, then we can prove the following;

Theorem 3.4. *Let μ be a Radon measure in U. We define the so-called 2-carrying set as follows:*

$$set_2(\mu)\hat{=}U \cap \{x \in Spt\mu : \Theta^2(\mu, x) < \infty\}$$

Then $\mu\lfloor set_2(\mu) = \Theta^2(\mu, \cdot)\mathcal{H}^2\lfloor set_2(\mu)$.

Proof. By using the proof of Theorem 3.1, we know that the 2-carrying set $set_2(\mu)$ of μ is rectifiable. Then, one arrives at $\mu\lfloor set_2(\mu)$ is absolutely continuous with respect to $\mathcal{H}^2$ by virtue of a covering argument. Now, we consider the Borel set as follows for each j: $S_j\hat{=}set_2(\mu) \cap \{x \in Spt\mu : \Theta^2(\mu, x) > \frac{1}{j}\}$ and we observe that $\mathcal{H}^2\lfloor S_j$ is a Radon measure and that $\mu\lfloor S_j$ is absolutely continuous with respect to $\mathcal{H}^2\lfloor S_j$. According to Theorem 2.9.1 and Theorem 2.8.18 in [10] we have that $\mu\lfloor S_j = h_j(x)\mathcal{H}^2\lfloor S_j$ with property

$$h_j(x) = \lim_{r\to 0} \frac{\mu(S_j \cap C_r(x))}{\mathcal{H}^2(S_j \cap C_r(x))}$$

for $\mathcal{H}^2$ a.e. $x \in S_j$. On the other hand, S_j is 2-rectifiable, Borel, and have locally finite $\mathcal{H}^2$ measures so that $\Theta^2(\mathcal{H}^2\lfloor S_j, x) = 1$ for $\mathcal{H}^2$ a.e.$x \in S_j$(see [10]), then we know that $h_j(x) = \Theta^2(\mu\lfloor S_j, x)$ for $\mathcal{H}^2$ a.e. $x \in S_j$. We refer to [10] again and derive that $\Theta^2(\mu\lfloor S_j, x) = \Theta^2(\mu, x)$ for $\mathcal{H}^2$ a.e.$x \in S_j$. Finally it is not hard to see that $\mu\lfloor set_2(\mu) = \Theta^2(\mu, \cdot)\mathcal{H}^2\lfloor set_2(\mu)$. This ends the proof of Theorem 3.4. □

Remark 3.6. It is well known that Theorem 3.4 is tenable for Varifolds. Of course it is also tenable for 2-rectifiable Varifolds. In fact, for Theorem 3.4, T. D. Pauw in [20] studied the same problem but the theorem posed by T. D. Pauw needs the hypothesis with rectifiability of 2-carrying set $set_2(\mu)$.

4. Examples

Example 4.1. We consider 1-set (see [8] for details) E. Assume that 1-set E is regular, i.e., $\lim_{r\to 0} \frac{\mathcal{H}^1(E\cap B(x,r))}{2r} = 1$. By using the summation of measures and the property hypothesis of Theorem 3.1 we can derive that the Radon measure $\mu = \mathcal{H}^1$ for a.e.$x \in E$. In other words, we show that μ is rectifiable. Furthermore, we can consider the continuum $E \subset R$. By virtue of Theorem

3.2 and [20], we see that a Varifold V in this setting is rectifiable, i.e., V is a 1-rectifiable Varifold.

Example 4.2. For convenience we continue to consider the $\mu-$ measurable set $E \subset R$. In this setting: $(l^1_\infty, \|\cdot\|) \equiv (R, \|\cdot\|_E)$ by Definition. Let f be an Lebesgue-integrable function in R, $x_0 \in R$. We observe that for integral $F(x) = \int_0^x f(t)dt$, the following relationship

$$F'(x_0) = \lim_{r\to 0} \frac{F(x_0 + r) - F(x_0)}{r} = f(x_0)$$

is tenable.

We denote by $\mu(E) = \int_E f(t)dt$. Then,

$$\lim_{r\to 0} \frac{\mu([x_0, x_0 + r])}{r} = \lim_{r\to 0} \frac{1}{r} \int_{x_0}^{x_0+r} f(t)dt = F'(x_0)$$

for a.e.$x_0 \in U$(closed interval $U \subset R$).

Now, we take function $f(x) = 1$ if $x \in E$; Otherwise $f(x) = 0$ if $x \in R \setminus E$. Thus we have $\lim_{r\to 0} \frac{\mu(E\cap[x_0,x_0+r])}{r} = 1$ for a.e. $x_0 \in E$. In other words, if E is $\mu-$ measurable set, then E is rectifiable from Theorem 5.2 and Theorem 5.6 in [21].

Example 4.3. Let $\Omega \subset l^3_\infty$ be fixed open set, and $\mathcal{C}$ be a class of closed subsets of Ω. Take $E \subset \Omega$, and consider a Lipschitz mapping f from Ω to itself. We require that $f(E) \in \mathcal{C}$ whenever $E \in \mathcal{C}$, $W = W_f = \{x \in \Omega; f(x) \neq x\}$ and $W_f \cup f(W_f) \subset B$ for some ball $C_r(x) \subset\subset \Omega$ of l^3_∞. Roughly speaking, we can consider the following functional on $\mathcal{C}$

$$\mathcal{J}(E) = \int_E \theta(x) d\mu_V(x)$$

for 2-Varifolds V, where θ is continuous on Ω and $1 \leq \theta \leq C$ everywhere. We would like to find $E \in \mathcal{C}$ s.t. $\mathcal{J}(E) = \Im$, where $\Im = \inf\{\mathcal{J}(F; F \in \mathcal{C})\}$.

The approach would be to take a minimizing sequence, i.e., a sequence $\{E_k\}$ in $\mathcal{C}$, with $\lim_{k\to\infty} \mathcal{J}(E_k) = \Im$. It is obvious that if one takes Varifolds to be $Curvature\ Varifolds$, then the functional must have some interesting geometric properties. All these subjects will be studied in our next paper(in preparation)"Some Properties of Curvature Varifolds".

5. Conclusion

We have given a family of new sufficient conditions for Radon measure and Varifolds being rectifiability. For instance, by using the corresponding sufficient conditions, one studied the rectifiability of measure or sets. In particular, one can use these conditions to prove the rectifiable property of Varifolds. Of important thing is that they can be applied to design the smoothness of sets with low dimension. In addition, the methods of this paper may be applied to some other cases such as the measures or Varifolds given in Refs.[2], [11], [13], [15], [16], [19], [24], [25] and so on.

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