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ANALYTIC PROPERTIES OF THE q-VOLKENBORN INTEGRAL ON THE RING OF p-ADIC INTEGERS

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Abstract. In this paper, we consider the q-Volkenborn integral of uniformly differentiable functions on the p-adic integer ring. By using this integral, we obtain the generating functions of twisted q-generalized Bernoulli numbers and polynomials. We find some properties of these numbers and polynomials.

1. Introduction

Let p be an odd prime. $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will always denote, respectively, the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $v_p : \mathbb{C}_p \to \mathbb{Q} \cup \{\infty\}$ ($\mathbb{Q}$ the field of rational numbers) will denote the p-adic valuation of $\mathbb{C}_p$ normalized so that $v_p(p) = 1$. The absolute value on $\mathbb{C}_p$ will be denoted as $|\cdot|_p$, and $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{C}_p$. We let $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p \mid 1/x \in \mathbb{Z}_p\}$. A p-adic integer in $\mathbb{Z}_p^\times$ is sometimes called a p-adic unit. Let $UD(\mathbb{Z}_p, \mathbb{C}_p)$ denote the space of all uniformly (or strictly) differentiable $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p$. For each integer $N \geq 0$, C_{p^N} will denote the multiplicative group of the primitive p^N-th roots of unity in $\mathbb{C}_p^* = \mathbb{C}_p \setminus \{0\}$. Set $\mathbf{T}_p = \{\omega \in \mathbb{C}_p \mid \omega^{p^N} = 1 \text{ for some } N \geq 0\} = \bigcup_{N\geq 0} C_{p^N}$. The dual of $\mathbb{Z}_p$, in the sense of p-adic Pontrjagin duality, is $\mathbf{T}_p = C_{p^\infty}$, the direct limit (under inclusion) of cyclic groups C_{p^N} of order $p^N (N \geq 0)$, with the discrete topology. $\mathbf{T}_p$ admits a natural $\mathbb{Z}_p$-module structure which we shall write exponentially, *viz* ω^x for $\omega \in \mathbf{T}_p$ and $x \in \mathbb{Z}_p$. $\mathbf{T}_p$ can be embedded discretely in $\mathbb{C}_p$ as the multiplicative p-torsion subgroup and we now choose, for once and all, one such embedding. If $\omega \in \mathbf{T}_p$, then we denote by

$$\phi_\omega : (\mathbb{Z}, +) \longrightarrow (\mathbb{C}_p^\times, \cdot) \tag{1.1}$$

for the locally constant character $x \mapsto \omega^x$, which is locally analytic character if $\omega \in \{\omega \in \mathbb{C}_p \mid v_p(\omega - 1) > 0\}$. Then ϕ_ω has continuation to a continuous group homomorphism from $(\mathbb{Z}_p, +)$ to $(\mathbb{C}_p^\times, \cdot)$ (see [24]).

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The indefinite sum operator S is defined by $Sf(0) = 0$ and $Sf(n) = \sum_{i=0}^{n-1} f(i)$ for $n \geq 1$. It is well known that for $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, its Volkenborn integral is defined to be the limit of average

$$\frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) = \frac{Sf(p^N) - Sf(0)}{p^N}$$

as $N \to \infty$ (see [2], [19], [22]). We see that the uniform differentiability guarantees the existence of limits. Write down this integral as

$$I_0(f) = \int_{\mathbb{Z}_p} f(x)\, dx = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) = (Sf)'(0) \in \mathbb{Q}_p \tag{1.2}$$

(see [7] and [19] for more details). For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, the I_0-Fourier transform of f is the function $\widehat{f} : \mathbf{T}_p \to \mathbb{C}_p$ defined by

$$\widehat{f}_\omega = I_0(f\phi_\omega) \quad \text{for all } \omega \in \mathbf{T}_p. \tag{1.3}$$

This can be found in [24, Definition 3.1]. In fact we can extend $\widehat{f}$ to $\{\alpha \in \mathbb{C}_p \mid v_p(\alpha - 1) > 0\}$ by same formula, where it turns out to be an analytic function whose Taylor expansion has logarithmic growth (see [24, Proposition 5.3]). The analogue with the classical (complex) theory is substantially complicated by the absence of a p-adic valued Haar measure on $\mathbb{Z}_p$. So, C. F. Woodcock had to do various attempts to construct a satisfactory analogue of integral analysis for the spaces of functions on $\mathbb{Z}_p$. One, in [23] and [24], can find the fully detailed study of the I_0-Fourier transform on the space of all uniformly differentiable functions $f : \mathbb{Z}_p \to \mathbb{C}_p$.

The q-extension of the I_0-Fourier transform has been constructed by T. Kim [11], called the I_q-Fourier transform. The I_q-Fourier transform has the form

$$(\widehat{f}_\omega)_q = I_q(f\phi_\omega) \quad \text{for all } \omega \in \mathbf{T}_p, \tag{1.4}$$

see Section 2, Eq. (2.1).

For any integer $n \geq 1$, we shall use the following standard notation

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}, \quad [0]_q = 0.$$

As $q \to 1, [n]_q \to n$, and this is the hallmark of a q-extension: the limit as $q \to 1$ recovers the classical object. If $q \in \mathbb{C}$, one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q - 1|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x \log q)$ for $x \in \mathbb{Z}_p$. For $N \geq 1$, the q-extension μ_q (originally introduced by T. Kim [6]) of the p-adic Haar distribution μ_{Haar}

$$\mu_q(a + p^N\mathbb{Z}_p) = \frac{q^a}{[p^N]_q} \tag{1.5}$$

is known as a distribution on $\mathbb{Z}_p$, where $\mu_{\text{Haar}}(a + p^N\mathbb{Z}_p) = \frac{1}{p^N}$ and $a + p^N\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x - a|_p \leq p^{-N}\}$. Note that $\lim_{q \to 1} \mu_q = \mu_{\text{Haar}}$. We shall write $d\mu_q(x)$

to remind ourselves that x is the variable of integration. This distribution μ_q yields an I_q-integral for $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x. \tag{1.6}$$

The I_q-integral for $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ was defined by T. Kim ([6], [7], [8], [10], [11]) and basic properties were studied by many authors. Also, by (1.2) and (1.6), it is well-known that the numbers B_n, $\beta_n(q)$ and $B_n(q)$ are connected with I_0-integrals and I_q-integrals as follows.

- For any $n \geq 0$, $I_0(x^n) = \int_{\mathbb{Z}_p} x^n dx = B_n$, where B_n is the ordinary Bernoulli numbers (see [19], [22]);
- For any $n \geq 0$, $I_q([x]_q^n) = \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \beta_n(q)$, where $\beta_n(q)$ is the Carlitz's q-Bernoulli numbers (see [6], [7], [8], [9], [11] [12]);
- For any $n \geq 0$, $I_q(x^n) = \int_{\mathbb{Z}_p} x^n d\mu_q(x) = B_n(q)$, where $B_n(q)$ is the modified Bernoulli numbers (see [3], [4]).

In this paper, we consider the q-Volkenborn integral of uniformly differentiable functions on $\mathbb{Z}_p$. By using this integral, we obtain the generating functions of twisted q-generalized Bernoulli numbers and polynomials. We find some properties of these numbers and polynomials.

2. The q-extension of the I_0-integral transform and related numbers and polynomials

Given $\omega \in \mathbf{T}_p$, we will denote by $\phi_\omega : \mathbb{Z}_p \to \mathbb{C}_p, x \mapsto \omega^x$, the locally constant extension of the power function from $\mathbb{Z}$ to $\mathbb{Z}_p$. In [11], the I_q-Fourier transform for $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ is the function $I_q(f) : \mathbf{T}_p \to \mathbb{C}_p$ defined by

$$I_q(f\phi_\omega) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} \phi_\omega(x) f(x) q^x. \tag{2.1}$$

This is an q-extension of the I_0-Fourier transform (1.3), that is, the I_0-Fourier transform of f is the case $q \to 1$ of $I_q(f\phi_\omega)$ in (2.1). In terms of integration, one can have the integral form

$$I_q(f\phi_\omega) = \int_{\mathbb{Z}_p} \phi_\omega(x) f(x)\, d\mu_q(x), \quad \omega \in \mathbf{T}_p. \tag{2.2}$$

Also its inverse I_q-Fourier transform is seem to be equivalent to the limit

$$f(x) q^x = \frac{\log q}{q-1} \lim_{N\to\infty} \sum_{\omega \in C_{p^N}} \phi_{\omega^{-1}}(x) I_q(f\phi_\omega) \tag{2.3}$$

for all $x \in \mathbb{Z}_p$ (see [11]).

Note that the distribution μ_q on $\mathbb{Z}_p$ has the property

$$\mu_q(ap + p^{N+1}\mathbb{Z}_p) = [p]_q^{-1} \mu_{q^p}(a + p^N\mathbb{Z}_p)$$

followed trivially from the definition (1.5). Since $\mathbb{Z}_p^\times = \mathbb{Z}_p \backslash p\mathbb{Z}_p$, this implies that for any $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ on $\mathbb{Z}_p^\times$

$$\int_{\mathbb{Z}_p^\times} \phi_\omega(x) f(x) d\mu_q(x) = I_q(f\phi_\omega) - [p]_q^{-1} I_{q^p}(f(px)\phi_{\omega^p}). \tag{2.4}$$

In the following proposition we obtain the shift versus integration of the I_q-Fourier transform for uniformly differentiable functions on $\mathbb{Z}_p$.

Proposition 2.1. *Let f be a uniformly differentiable function on $\mathbb{Z}_p$. For some fixed $s \in \mathbb{Z}_p$ and $\omega \in \mathbf{T}_p$*

$$I_q(f(x+s+1)\phi_\omega) - \frac{1}{\omega q} I_q(f(x+s)\phi_\omega) = \frac{q-1}{\omega q \log q}(f'(s) + f(s)\log q).$$

Proof. From the definition (2.1), it is easy to see that

$$\begin{aligned}
&\frac{\omega q \log q}{q-1} \int_{\mathbb{Z}_p} \phi_\omega(x) f(x+s+1) d\mu_q(x) \\
&= \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} \phi_\omega(x) f(x+s) q^x + \lim_{N\to\infty} \frac{\phi_\omega(p^N) f(p^N+s) q^{p^N} - f(s)}{p^N}.
\end{aligned}$$

It is easy to check that a uniformly differentiable function $\phi_\omega(x)f(x+s)q^x$ on $x \in \mathbb{Z}_p$ can be differentiated in the usual way:

$$\lim_{p^N\to 0} \frac{\phi_\omega(p^N) f(p^N+s) q^{p^N} - f(s)}{p^N} = f'(s) + f(s)\log q \quad \text{for } s \in \mathbb{Z}_p.$$

The result now follows easily. □

Corollary 2.2. 1. *Suppose that $\omega \in \mathbf{T}_p$. Let $I_q(e^{tx}\phi_\omega)$ be a power series about the origin as follows*

$$I_q(e^{tx}\phi_\omega) = \sum_{k=0}^{\infty} \frac{B_k(q,\omega)}{k!} t^k \in \mathbb{C}_p[[t]],$$

where $|t|_p < p^{-1/(p-1)}$ and $t \neq 0 \in \mathbb{C}_p$. Then the coefficients of expansion $\{B_k(q,\omega)\}$ can be written by

$$\begin{aligned}
B_k(q,\omega) &= I_q(x^k\phi_\omega) \\
&= \begin{cases} \frac{q-1}{\omega q-1}\left(H_k(\frac{1}{\omega q}) + \frac{k}{\log q} H_{k-1}(\frac{1}{\omega q})\right), & \text{if } k \geq 1 \\ \frac{q-1}{\omega q-1}, & \text{if } k = 0. \end{cases}
\end{aligned}$$

Here the generalized k-th Euler number $H_k(u)$ attached to an algebraic $u \neq 1$ has been defined by Frobenius (1910): $(1-u)/(e^t-u) = \sum_{k=0}^\infty H_k(u)t^k/k!$.

2. *For $\omega \in \mathbf{T}_p$ and $k \geq 0$,*

$$I_q(q^{kx}\phi_\omega) = \sum_{i=0}^{\infty} (q^k-1)^i I_q(\binom{x}{i}\phi_\omega) = \frac{k+1}{[k+1]_{q,\omega}},$$

where $[k]_{q,\omega} = (\omega q^k - 1)/(q-1)$ *and* $\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$. *Moreover, the sequence* $\{q^k\}$ *can be extended to a locally analytic function* q^x *for* $x \in \mathbb{Z}_p$.

We define now the generating function of a new Bernoulli numbers by I_q-Fourier transform. The twisted q-extension of Bernoulli numbers is define by

$$I_q(e^{tx}\phi_\omega) = \frac{q-1}{\log q}\frac{t+\log q}{\omega q e^t - 1} = \sum_{k=0}^{\infty} B_k(q,\omega)\frac{t^k}{k!}, \quad \omega \in \mathbf{T}_p \tag{2.5}$$

for $|t|_p < p^{-1/(p-1)}$. If $q \to 1$, then from application of L'Hospital's rule the expression (2.5) is reduced to

$$\lim_{q\to 1} I_q(e^{tx}\phi_\omega) = \frac{t}{\omega e^t - 1}, \quad \omega \in \mathbf{T}_p.$$

The exciting properties of this formula were shown by T. Kim (see [5]) and C. F. Woodcock ([23, Proposition 7.1 (i)] and [24]). If $\omega = 1$, then

$$I_q(e^{tx}) = \frac{q-1}{\log q}\frac{t+\log q}{qe^t - 1} = \sum_{k=0}^{\infty} B_k(q,1)\frac{t^k}{k!},$$

where $|t|_p < p^{-1/(p-1)}$ (cf. [1], [3], [4], [9], [10], [12], [16], [20]).

Corollary 2.3. *Let* $\omega \in \mathbf{T}_p$ *with* $\omega^N = 1$, $\omega \neq 1$ *for* $N > 1$. *Set*

$$\lim_{q\to 1} I_q(e^{tx}\phi_\omega) = \sum_{k=0}^{\infty} B_k(1,\omega)t^k/k!.$$

Then

$$B_k(1,\omega) = N^{k-1}\sum_{i=0}^{N-1}\omega^i B_k(\frac{i}{N}),$$

where $B_k(\cdot)$ *is the usual Bernoulli polynomials and* $k \geq 1$.

Corollary 2.4. *Let* $k \geq 0$. *Then*

1. $\frac{q-1}{\log q}q^x x^k = \sum_{\omega\in\mathbf{T}_p,\,\omega\neq 1}\phi_{\omega^{-1}}(x)B_k(q,\omega) + B_k(q,1)$.
2. $\frac{q-1}{\log q}q^{(k+1)x} = \sum_{\omega\in\mathbf{T}_p,\,\omega\neq 1}\phi_{\omega^{-1}}(x)\frac{k+1}{[k+1]_{q,\omega}} + \frac{k+1}{[k+1]_{q,1}}$.

Proof. From (2.3) and Corollary 2.2, the series

$$\sum_{\omega\in C_{p^N}}\phi_{\omega^{-1}}(x)I_q(x^k\phi_\omega)$$

converges uniformly to $\frac{\log q}{q-1}q^x x^k$ as $N \to \infty$. So Part 1 follows directly. Part 2 follows by a similar method of Part 1. □

In Part 1 and Part 2 of Corollary 2.4, putting $k = 0$, we obtain the integral series expansion

$$\frac{q^x}{\log q} = \sum_{\omega\in\mathbf{T}_p}\phi_{\omega^{-1}}(x)\frac{1}{\omega q - 1},$$

whence, for $x=0$ in the above, we have $\frac{1}{\log q}=\sum_{\omega\in\mathbf{T}_p}\frac{1}{\omega q-1}$ for $q\neq 1$. This formula gives an explicit expression for $\frac{1}{\log q}$ in terms of $\frac{1}{\omega q-1}$ (see [23, p. 692]).

Now we consider the recursion formula for the sequence of numbers $\{B_k(q,\omega)\}$. From Proposition 2.1 we obtain the difference formula

$$I_q(f_1\phi_\omega)-\frac{1}{\omega q}I_q(f\phi_\omega)=\frac{q-1}{\omega q\log q}(f'(0)+f(0)\log q),$$

where $f_1(x)=f(x+1)$. From this expression, when $f(x)=x^k$ for $k\geq 0$, we easily deduce that

$$I_q((x+1)^k\phi_\omega)-\frac{1}{\omega q}I_q(x^k\phi_\omega)=\begin{cases}\frac{q-1}{\omega q}, & k=0,\\ \frac{q-1}{\omega q\log q}, & k=1,\\ 0, & k\geq 2.\end{cases} \tag{2.6}$$

We expand the left-hand side of Equation (2.6) by the binomial theorem. It may be stated as

$$I_q((x+1)^k\phi_\omega)=\sum_{i=0}^{k}\binom{k}{i}\int_{\mathbb{Z}_p}\phi_\omega(x)x^i d\mu_q(x). \tag{2.7}$$

From (2.5), (2.7) and Part 1 of Corollary 2.2, we derive

$$\begin{aligned}&I_q((x+1)^k\phi_\omega)-\frac{1}{\omega q}I_q(x^k\phi_\omega)\\ &=\sum_{i=0}^{k}\binom{k}{i}B_i(q,\omega)-\frac{1}{\omega q}B_k(q,\omega)\\ &=\begin{cases}\frac{\omega q-1}{\omega q}B_0(q,\omega), & k=0,\\ \sum_{i=0}^{k-1}\binom{k}{i}B_i(q,\omega)+\frac{\omega q-1}{\omega q}B_k(q,\omega), & k\geq 1.\end{cases}\end{aligned} \tag{2.8}$$

As a consequence of the above formulae (2.6) and (2.8) we deduce the recurrence relation for the sequence of numbers $\{B_k(q,\omega)\}$ as follows

Proposition 2.5. *The numbers $\{B_k(q,\omega)\}$ satisfies*

$$B_0(q,\omega)=\frac{1}{[1]_{q,\omega}},\quad B_1(q,\omega)=\frac{1}{[1]_{q,\omega}\log q}-\frac{\omega q}{[1]_{q,\omega}(\omega q-1)}$$

and

$$B_k(q,\omega)=\frac{\omega q}{1-\omega q}\sum_{i=0}^{k-1}\binom{k}{i}B_i(q,\omega)\quad \text{for } k\geq 2.$$

3. Twisted p-adic q-L-functions

Let d be a fixed integer and p be a fixed prime number. We set

$$(3.1)\qquad \begin{aligned} &X = \varprojlim_{N} (\mathbb{Z}/dp^N\mathbb{Z}), \quad X^* = \bigcup_{\substack{0<a<dp\\(a,p)=1}} a + dp\mathbb{Z}_p,\\ &a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},\end{aligned}$$

where $a \in \mathbb{Z}$ with $0 \le a < dp^N$. Let $D = \{q \in \mathbb{C}_p \mid |q-1|_p < 1\}$, and let $\overline{D} = \mathbb{C}_p \backslash D$ be the complement of the open unit disc around 1. Note that if $q \in \overline{D}$ and $\mathrm{ord}_p(1-q) \ne -\infty$, then $\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$ is the measure (cf. [7]). Hereafter, we assume that $q \in \overline{D}$ and $\mathrm{ord}_p(1-q) \ne -\infty$.

Let χ be a primitive Dirichlet character with conductor d. Defining the generalized numbers of $B_k(q,\omega)$ by the formula

$$(3.2)\qquad \begin{aligned} &B_{k,\chi}(q,\omega)\\ &= k! \cdot \text{coefficient of } t^k \text{ in } \frac{q-1}{\log q} \sum_{a=1}^{d} \frac{\chi(a)\phi_\omega(a) q^a (t + \log q) e^{ta}}{\phi_\omega(d) q^d e^{dt} - 1}.\end{aligned}$$

Thus we deduce the integral of the generalized numbers

$$(3.3)\qquad B_{k,\chi}(q,\omega) = \int_X \phi_\omega(x)\chi(x) x^k d\mu_q(x) \quad \text{for } k \ge 0.$$

To see that (3.3) follows from

$$(3.4)\qquad \int_X \phi_\omega(x)\chi(x) e^{tx} d\mu_q(x) = \sum_{k=0}^{\infty} \int_X \phi_\omega(x)\chi(x) x^k d\mu_q(x) \frac{t^k}{k!},$$

we note

$$\begin{aligned} &\int_X \phi_\omega(x)\chi(x) e^{tx} d\mu_q(x)\\ &= \frac{q-1}{\log q} \lim_{N\to\infty} \frac{1}{dp^N} \sum_{x=1}^{dp^N} \chi(x)\phi_\omega(x) e^{tx} q^x\\ &= [d]_q^{-1} \sum_{a=1}^{d} \chi(a)\phi_\omega(a) q^a e^{ta} \frac{q^d - 1}{\log q^d} \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} \phi_{\omega^d}(x) e^{tdx} q^{dx}\\ &= [d]_q^{-1} \sum_{a=1}^{d} \chi(a)\phi_\omega(a) q^a e^{ta} \int_{\mathbb{Z}_p} \phi_{\omega^d}(x) e^{tdx} d\mu_{q^d}(x)\\ &= \frac{q-1}{\log q} \sum_{a=1}^{d} \frac{\chi(a)\phi_\omega(a) q^a (t+\log q) e^{ta}}{\phi_\omega(d) q^d e^{dt} - 1} \quad \text{(by (2.5))}.\end{aligned}$$

Proposition 3.1. *Let χ be a primitive Dirichlet character with conductor d and $x \in \mathbb{Z}_p$. Then*

$$\frac{q-1}{\log q} q^x \chi(x) x^k = \sum_{\omega \in \mathbf{T}_p,\, \omega \neq 1} \phi_{\omega^{-1}}(x) B_{k,\chi}(q,\omega) + B_{k,\chi}(q,1)$$

for $k \geq 0$.

Let p be odd rational prime and let $\omega_p : X^* \to X$ be the function defined by (see [2], [15], [19], [22])

$$\omega_p(x) = \lim_{\substack{n\to\infty \\ p-\text{adically}}} x^{p^n}.$$

The function ω_p is called the Teichmüller character, and it appears quite frequently in many different guises. For $s \in \mathbb{Z}_p$ and $\omega \in \mathbf{T}_p$, we define

$$L_{p,q}(s,\chi,\omega) = \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{\substack{0 \leq x \leq dp^N-1 \\ (p,x)=1}} q^x \phi_\omega(x) \chi(x) \left(\frac{x}{\omega_p(x)}\right)^{1-s}. \tag{3.5}$$

For $k \geq 0$, we set $\chi_k = \chi\omega_p^{-k}$. Since $\mu_q(pU) = [p]_q^{-1}\mu_{q^p}(U)$ for $U \subset X$, the value of the function $L_{p,q}(s,\chi,\omega)$ at non-positive integers are given by

$$L_{p,q}(1-k,\chi,\omega) = B_{k,\chi_k}(q,\omega) - p^k[p]_q^{-1}\chi_k(p) B_{k,\chi_k}(q^p,\omega^p) \tag{3.6}$$

for $k \geq 1$. We thus obtain the following

Theorem 3.2. *Let χ be a primitive Dirichlet character with conductor d and $s \in \mathbb{Z}_p, \omega \in \mathbf{T}_p$. Then the function $L_{p,q}(s,\chi,\omega)$ interpolates the values $B_{k,\chi_k}(q,\omega) - p^k[p]_q^{-1}\chi_k(p)B_{k,\chi_k}(q^p,\omega^p)$ when $s = 1-k$ for $k \geq 1$.*

For $q \in \overline{D}$, we have

$$\left|\frac{\mu_q(a+dp^N\mathbb{Z}_p)}{1-q}\right|_p = \left|\frac{q^a}{(1-q)[dp^N]}\right|_p = \left|\frac{q^a}{1-q^{dp^N}}\right|_p \leq 1.$$

By [13, p. 31, Eq. (3.4)], if $k \equiv k' \pmod{(p-1)p^N}$, then we obtain the assertion that

$$\left|x^k - x^{k'}\right|_p \leq \frac{1}{p^{N+1}} \quad \text{for } x \in X^*.$$

Using the corollary at the end of [14, Chapter II, §5] and (3.5), their integrals over the compact set X^* are also close together, and in fact, it is easy to see that for $k \geq 1$

$$\begin{aligned}
&(1-q)^{-1}L_{p,q}(1-k,\chi_{-k},\omega) \\
&= \lim_{N\to\infty}\frac{1}{1-q^{dp^N}}\sum_{\substack{0\leq x\leq dp^N-1\\ (p,x)=1}} q^x\phi_\omega(x)\chi\omega_p^k(x)\left(\frac{x}{\omega_p(x)}\right)^k \\
&= \int_{X^*}\chi(x)\phi_\omega(x)x^k\,\frac{d\mu_q(x)}{1-q} \\
&\equiv \int_{X^*}\chi(x)\phi_\omega(x)x^{k'}\,\frac{d\mu_q(x)}{1-q} \pmod{p^{N+1}} \\
&= (1-q)^{-1}L_{p,q}(1-k',\chi_{-k'},\omega).
\end{aligned}$$

Hence we can prove the following congruence.

Theorem 3.3. *Let χ be a primitive Dirichlet character with conductor d, and let $k \equiv k' \pmod{(p-1)p^N}$ and $\omega \in \mathbf{T}_p$. Then*

$$(1-q)^{-1}L_{p,q}(1-k,\chi_{-k},\omega) \equiv (1-q)^{-1}L_{p,q}(1-k',\chi_{-k'},\omega) \pmod{p^{N+1}}.$$

Finally, we shall also want to consider modified twisted L-functions in the complex field $\mathbb{C}$. Let $q \in \mathbb{C}$ with $0 < |q| < 1$, and let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. We set

$$L_q(s,\chi,\omega) = \frac{q-1}{1-s}\sum_{n=1}^{\infty}\frac{\omega^n q^n\chi(n)}{n^{s-1}} + \frac{q-1}{\log q}\sum_{n=1}^{\infty}\frac{\omega^n q^n\chi(n)}{n^s}, \tag{3.7}$$

the series being absolutely convergent (cf. [3], [4], [6], [9], [12], [17], [18], [21]). In particular, if we replace s by $1-k$, one then sees easily that

$$\begin{aligned}
&L_q(1-k,\chi,\omega) \\
&= \frac{q-1}{k}\sum_{n=1}^{\infty}\omega^n q^n\chi(n)n^k + \frac{q-1}{\log q}\sum_{n=1}^{\infty}\omega^n q^n\chi(n)n^{k-1} \\
&= \frac{q-1}{k}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k\left(\sum_{n=1}^{\infty}\omega^n q^n\chi(n)e^{nt} + \frac{t}{\log q}\sum_{n=1}^{\infty}\omega^n q^n\chi(n)e^{nt}\right)\Bigg|_{t=0}.
\end{aligned}$$

We consider the function

$$\Psi_q(t) = (1-q)\sum_{n=1}^{\infty}\omega^n q^n\chi(n)e^{nt} + \frac{(1-q)t}{\log q}\sum_{n=1}^{\infty}\omega^n q^n\chi(n)e^{nt}.$$

Since χ is a character mod d we rearrange the terms in the series for $\Psi_q(t)$ according to the residue classes mod d. Then we have

$$\begin{aligned}\Psi_q(t) &= \left((1-q)+\frac{(1-q)t}{\log q}\right)\sum_{a=1}^{d}\sum_{b=0}^{\infty}\omega^{a+bd}q^{a+bd}\chi(a+bd)e^{(a+bd)t}\\ &= \frac{q-1}{\log q}\sum_{a=1}^{d}\frac{\chi(a)\omega^a q^a(t+\log q)e^{ta}}{\omega^d q^d e^{dt}-1},\end{aligned}$$

which is equal to the formula in (3.2). We will apply the recipe above. Then we see that for $k \geq 1$

$$L_q(1-k,\chi,\omega) = -\frac{1}{k}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k (\Psi_q(t))\bigg|_{t=0}. \tag{3.8}$$

Comparing (3.2) and (3.8), we arrive at the following

Proposition 3.4. *Let χ be a primitive Dirichlet character with conductor d, and let $q \in \mathbb{C}$ with $0 < |q| < 1$. Then*

$$L_q(1-k,\chi,\omega) = -\frac{1}{k}B_{k,\chi}(q,\omega) \quad \textit{for } k \geq 1.$$

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$. Using (3.6) and Proposition 3.4 we have

$$-\frac{L_{p,q}(1-k,\chi,\omega)}{k} = L_q(1-k,\chi_k,\omega) - p^k[p]_q^{-1}\chi_k(p)L_{q^p}(1-k,\chi_k,\omega^p)$$

for $k \geq 1$. Here the right-hand side is the value of the complex L-function which the left-hand side is the values of the p-adic L-function and the value are equal in the field $\overline{\mathbb{Q}}$ common to $\mathbb{C}_p$ and $\mathbb{C}$.

Theorem 3.5. *Let $\omega \in \mathbf{T}_p$ and $q \in \overline{D}$ with $ord_p(1-q) \neq -\infty$. Let ω_p be the Teichmüller character. For χ a primitive Dirichlet character with conductor d, the function from $\mathbb{Z}_p \setminus \{1\}$ to $\mathbb{C}_p$*

$$\frac{L_{p,q}(s,\chi,\omega)}{s-1} = \frac{1}{s-1}\lim_{N\to\infty}\frac{1}{[dp^N]_q}\sum_{\substack{0\leq x\leq dp^N-1\\(p,x)=1}} q^x\phi_\omega(x)\chi(x)\left(\frac{x}{\omega(x)}\right)^{1-s}$$

interpolates the values $L_q(1-k,\chi_k,\omega) - p^k[p]_q^{-1}\chi_k(p)L_{q^p}(1-k,\chi_k,\omega^p)$ when $s = 1-k$.

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