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Numerical Computation of Ultra-High-Degree Legendre Function

Jay Hyoun Kwon[1] · Jong-Ki Lee[2]

Abstract

The computations of an ultra-high degree associated Legendre functions and its first derivative up to degree and order of 10800 are reported. Not only the magnitude of orders for the ultra-high degree calculation is presented but the numerical stability and accuracy of the computed values are described in detail. The accuracy on the order of 10^{-25} and 10^{-15} was obtained for the values of Legendre function and the first derivatives of Legendre functions, respectively. The computable highest degree and order of Legendre function in terms of latitudes and the linear relationship between the magnitude of the function with respect to degrees and orders is found. It is expected that the computed Legendre functions contribute in many geodetic and geophysical applications for simulations as well as theoretical verifications.

Keywords : Legendre function, numerical computation, ultra-high-degree

1. Introduction

Legendre function which forms the spherical harmonic expansions is one of the widely used functions in physics, especially in potential theory. Because infinite series of the spherical harmonic functions satisfy Laplace's equation of the gravity potential, the spherical harmonic expansions are intensively used in modeling earth gravity field. Indeed, the maximum degree and order of the expansion is limited in practice since it is impossible to expand the series to infinity. The maximum degree and order of the expansion is related to the spatial resolution in the gravity modeling. Higher degree and order implies more detailed signature of the gravity field is included in the expansion.

Thanks to the rapid development in computational technology, it is not difficult to watch recent activities using ultra-high-degree spherical harmonic expansions for geopotential/topography modeling. Among them, Wenzel (1998) calculated geopotential with the empirically derived harmonic coefficient up to degree 1800 and stated that the numerical instability in computing the associated Legendre functions is the main reason for limited calculation. Through the study on synthetic gravity modeling, Haagmans (2000) also calculated synthetic coefficient to degree and order 2160. Particularly, a numerical analysis on the calculation of associated Legendre functions is well described in Featherstone and Holmes (2002). Using the modified recursion that stabilizes the numerical problem, they successfully computed associated Legendre functions and its first derivatives to degree and order 2700.

Based on the analysis of the previous works, it is found that two major factors limit the maximum degree and order in the computation of the ultra-high-degree Legendre function. First, the associated Legendre functions in ultra-high degree and order ranges over thousands of orders of magnitude, and exceeds the capacity of the 64 bit arithmetic system. Second, the elapsed time of the computation is too long with conventional recursive routines. Obviously, these lead to utilize high performance

1) Department of Geoinformatics, The University of Seoul, Seoul 130-743, Korea(E-mail:jkwon@uos.ac.kr)
2) Department of Civil and Environmental Engineering and Geodetic Science, The Ohio State University, Columnus, OH 43210, USA(E-mail:lee.2608@osu.edu)

computing system which handles 128 bit arithmetic (for example, workstation cluster or Cray super computer).

In this paper, the computations of an ultra-high degree associated Legendre functions and its first derivative up to degree and order of 10800 are reported. To the authors' knowledge, this is the highest degree and order computation reported so far. Not only the magnitude of orders for the ultra-high degree calculation is presented but the numerical stability and accuracy of the computed values are described in detail. It is expected that this result contributes toward many geodetic applications such as in high precision gravity modeling and verifying the numerical accuracy of theories. For example, Kwon and Lee (2006) verified accuracy of upward continuation using simulated gravity field.

2. Legendre Function

As mentioned in the previous section, the maximum degree of the harmonic expansion of the disturbing potential determines the spatial resolution of the gravity field representation. That is, higher degree of the expansion provides more detailed signatures of the gravity field. The main problem in representing the gravity field with high degree is the numerical instability on the calculation of associated Legendre function which is defined by:

$$\overline{P}_{nm}(t) = \frac{1}{2^n n!} \cdot (1-t^2)^{m/2} \frac{d^{n+m}}{dt^{n+m}}(t^2-1)^n, n \geq 0, 0 \leq m \leq n \quad (1)$$

where n,m are the degree and order of the associated Legendre function; t is $\cos\theta$.

Numerically, associated Legendre function is usually calculated by a recursive formula, for example given as (Colombo, 1981):

$$\overline{P}_{nm}(\theta) = \alpha_{n-1,m} \cdot t \cdot \overline{P}_{n-1m}(\theta) - \beta_{n-2,m}\overline{P}_{n-2m}(\theta), 0 \leq m \leq n-2, \quad (2)$$

where

$\alpha_{nm} = \sqrt{\frac{(2n-1)(2n+1)}{(n-m)(n+m)}}$, and $\beta_{nm} = \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(n-m)(n+m)(2n-3)}}$.

The seed for the above equation is served by sectoral (i.e. $n=m$) $\overline{P}_{mm}$ which is also computed using a recursive formula (ibid).

$$\overline{P}_{mm}(\theta) = u\sqrt{\frac{2m+1}{2m}}\overline{P}_{m-1m-1}(\theta), \forall m > 1, \quad (3)$$

where $u = \sin\theta$.

The main numerical problem in above equation comes from u^m since it approaches to zero as m increases so that it causes underflow in the calculation. In addition, it has been shown that the magnitude of orders for $\overline{P}_{mm}$ ranges from 0 to -5000 as the latitude for the computation approaches to pole when the maximum expansion is *2700* (Holmes and Featherstone, 2002). Obviously, the order of -5000 cannot be handled in 64 bit arithmetic causing underflow, since the double precision only handles range of $10^{-310} \sim 10^{310}$.

To overcome the underflow and broad range problem, Holmes and Featherstone (2002) modified the equation and developed a recursive formula for $\overline{P}_{nm}(\theta)/u^m$ of which the magnitudes of orders ranges less than 600:

$$\frac{\overline{P}_{nm}(\theta)}{u^m} = \frac{1}{\sqrt{j}}\left(g_{nm} t \frac{\overline{P}_{nm+1}(\theta)}{u^{m+1}} - h_{nm}u^2\frac{\overline{P}_{nm+2}(\theta)}{u^{m+2}}\right), \quad (4)$$

where $g_{nm} = \frac{2(m+1)}{\sqrt{(n-m)(n+m+1)}}$, $h_{nm} = \sqrt{\frac{(n+m+2)(n-m-1)}{(n-m)(n+m+1)}}$, $j=2$ for $m=0$ and $j=1$ for $m>0$.

The equation for the sectoral is also given as:

$$\frac{\overline{P}_{mm}(\theta)}{u^m} = \sqrt{\frac{2m+1}{2m}}\frac{\overline{P}_{m-1m-1}(\theta)}{u^{m-1}}, \forall m > 1. \quad (5)$$

In addition, the recursive formula for the first derivative of associated Legendre functions are derived as (Abramowitz and Stegan, 1972):

$$\frac{\overline{P}^{(1)}_{nm}(\theta)}{u^m} = m\frac{t}{u}\frac{\overline{P}_{nm}(\theta)}{u^m} - e_{nm}u\frac{\overline{P}_{nm+1}(\theta)}{u^{m+1}}, \forall n \geq m, \quad (6)$$

where $e_{nm} = \sqrt{\frac{(n+m+1)(n-m)}{j}}$

Applying a scale factor of 10^{-280} to the modified equations to accommodate the ranges of 600, associated Legendre functions and the first derivatives are successfully computed up to degree and order 2700 with IEEE double precision data type (ibid).

3. Calculation of the Ultra High Degree Legendre Function

The degree and order of 2700 corresponds to the spatial resolution about 6.6 km which is not sufficient to represent the high frequency component of gravity field. Even the modified routines of the recursive Legendre function is used, still the numerical 'overflow' is caused in 64 bit arithmetic. As can be seen in Table 1, the overflow occurs in degree of 773 when the latitude of the computation point is 80 degrees.

Table 1 Degrees from which overflow occurs in the computation of Legendre function with respect to polar distance in 64 bit arithmetic.

Table 1. Degrees from which overflow occurs in the computation of Legendre function with respect to polar distance in 64 bit arithmetic.

Latitude	Polar distance	NaN Start (n)
0	90	
...	...	...
32	58	8207
33	57	7688
34	56	7214
35	55	6781
36	54	6382
37	53	6016
38	52	5678
39	51	5365
...	...	...
50	40	3061
60	30	1952
70	20	1261
80	10	773

A straightforward approach to compute associate Legendre functions to higher degree and orders is to use higher bit data type, i.e. 128 bit, since the range of the 128 bit double data type is $2^{-8188} \le n < 2^{8189}$ or $10^{-4932} \le n < 10^{4932}$. It should be noted that the range of 128 bit double data type is supported by combinations of the system and compiler that supports specific system. That is, a workstation cluster or CRAY, HP, and Intel, CRAY FORTRAN are the hardware and software, respectively with which 128 bit computation is possible. After careful examinations on the performances and capabilities for hardware and software, The Itanium 2 Cluster and Intel FORTRAN compiler was selected as the platform for the computation. Itanium 2 is distributed/shared memory hybrid of commodity systems based on the new Intel Itanium 2 architecture processor. The cluster is built using HP zx6000 workstations, an SGI Altix 3000 and several Altix 350s.

Using the cluster and Intel FORTRAN, associated Legendre functions and the first derivatives up to degree and order of 10800 is computed. Since the magnitudes and orders of Legendre functions ranges over 9000, the modified recursive formula (10) and (12) are used in the computation, which reduces the range of the magnitudes of orders to just over 2000 for *nmax=10800* (Fig. 1 and 2).

The numerical accuracy or error of the computed Legendre function can be evaluated from following

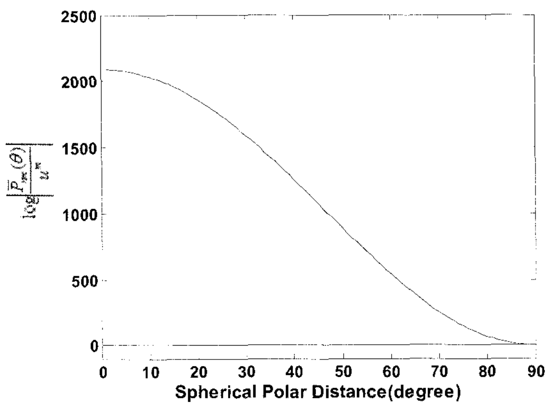

Fig. 1. Logarithm plot of maximum (upper line) and minimum (lower line) values of $\bar{P}_{nm}(\theta), \forall n, m \le 10000$.

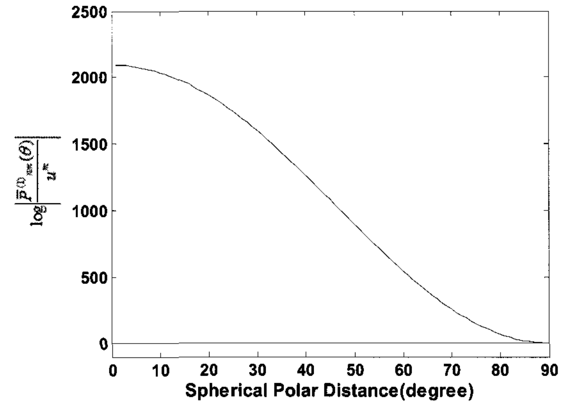

Fig. 2. Logarithm plot of maximum (upper line) and minimum (lower line) values of $\bar{P}_{nm}(\theta), \forall n,m \leq 10000$.

well-known equations.

$$accuracy = 2n+1-\left(\sum_{m=0}^{n}(\bar{P}nm(\theta))^2\right) \quad (7)$$

$$accuracy = (M+1)^2 - \sum_{n=0}^{M}\sum_{m=0}^{n}\left(\bar{P}_{nm}(\theta)\right)^2 \quad (8)$$

As can be seen in Fig. 3 and 4 corresponding to equations (7) and (8), respectively, the accuracy with respect to the spherical polar distance shows that the computation is stable except at the pole.

It should be noted that the numerical accuracy of the computation decreases as the computation point get closer to the pole and the maximum degree and order increases (Fig. 5).

Figure 6 shows the relationships between the accuracy and polar distance, and the accuracy and maximum degree and order of Legendre function in terms of RMS which is defined as:

$$RMS = \sqrt{\frac{\sum_{i=1}^{n} accuracy_i - mean\,of\,accuracy}{n}} . \quad (9)$$

Fig. 6. The RMS with respect to the polar distance with three different maximum degree and order.

Again, one can clearly notice the accuracy is decrease where close to the pole and when high degree and order is calculated.

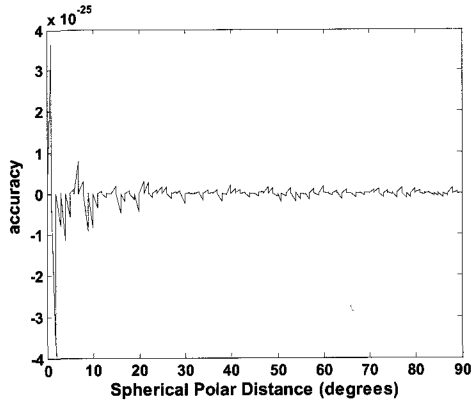

Fig. 3. The accuracy of the Legendre function with respect to the spherical polar distance with maximum degree and order of 10800: equation (7).

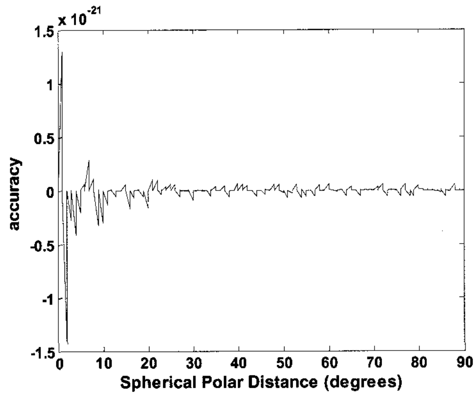

Fig. 4. The accuracy of the Legendre function with respect to the spherical polar distance with maximum degree and order of 10800: equation (8).

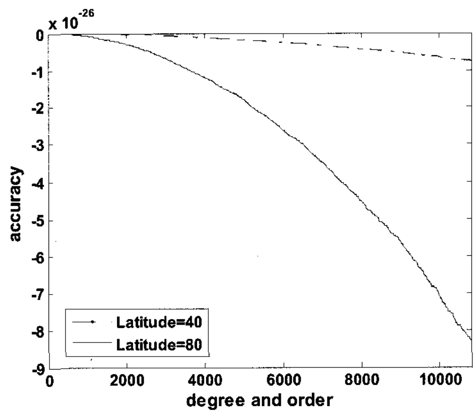

Fig. 5. The accuracy of the Legendre function according to the spherical polar distance and the maximum degree and order of Legendre function.

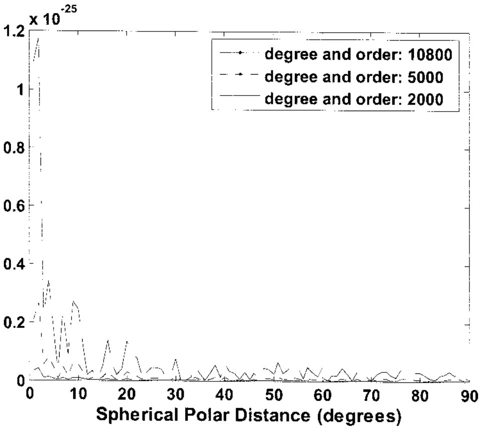

Fig. 6. The RMS with respect to the polar distance with three different maximum degree and order.

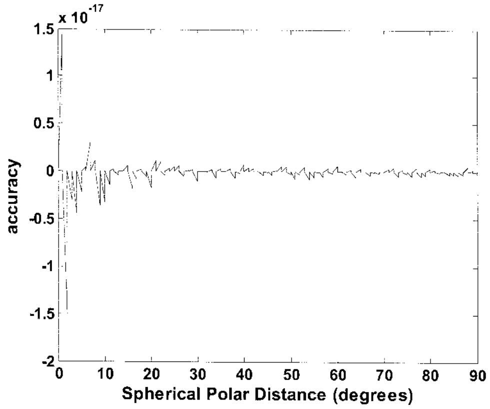

Fig. 7. The accuracy of the first derivatives of Legendre function with respect to the spherical polar distance with maximum degree and order of 10800:equation (10).

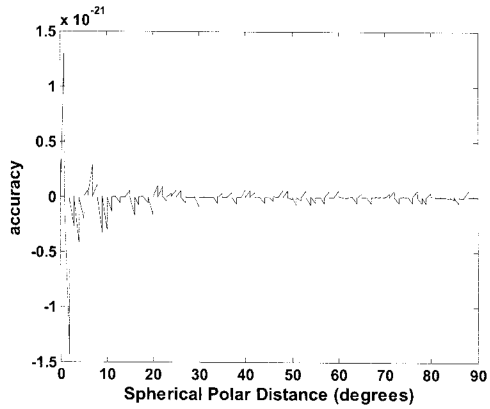

Fig. 8. The accuracy of the first derivatives of Legendre function with respect to the spherical polar distance with maximum degree and order of 10800: equation (11).

Similarly, the numerical accuracy of 1st derivative of Legendre function is computed by:

$$accuracy = \left(\frac{n(n+1)(2n+1)}{2}\right) - \sum_{m=0}^{n}\left(\bar{P}_{nm}^{(1)}(\theta)\right)^2 . \tag{10}$$

$$accuracy = \left(\frac{M(M+1)^2(M+2)}{4}\right) - \sum_{n=0}^{M}\sum_{m=0}^{n}\left(\bar{P}^{(1)}(\theta)\right)^2 \tag{11}$$

Figure 7 and 8 shows the evaluated accuracy of the first derivatives of Legendre functions according to equations (10) and (11). As one can notice immediately, the accuracy is several orders poorer than that of Legendre function.

4. Magnitudes of Orders vs Degree and Orders

Based on the 128 bit arithmetic, the trends of magnitudes of orders of Legendre function as degrees and orders of the expansions increase are investigated.

Figure 9 shows the maximum magnitudes of orders of Legendre function with respect to the degrees of orders in two latitudes of 40 and 80 degrees. Interestingly, one can see the nice linear relationship, and indeed, the slope for the latitude of 80 is steeper. With simple linear fitting, we found that the slopes are 0.0891 and 0.2031 for latitudes 40 and 80, respectively. In addition, it was found that the maximum degrees and orders for latitude

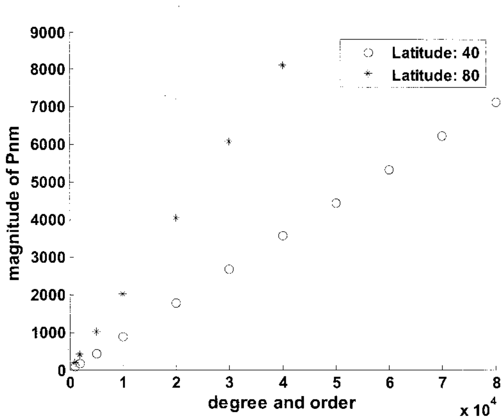

Fig. 9. The magnitude of orders of Legendre functions with respect to the degree and order of expansion.

40 and 80 are 103,240 and 45,294, respectively, using 128 bit arithmetic. The corresponding maximum spatial resolutions are 200 m and 450 m so that much more detailed analysis on the gravity field is possible with current technology. Indeed, the computation time is exponentially increases as maximum degrees and orders are getting higher.

5. Conclusion

In this study, associated Legendre functions and its first derivatives up to degree and order up to 10800 are computed. Since the orders of magnitude for the degree and order of 10800 ranges larger than 2000, 128 bit arithmetic should be performed and this is carried out using an Itanium cluster and Intel FORTRAN.

The analyzed numerical accuracy with respect to polar distance and maximum degree and order showed that the accuracy decreases as the computation point get close to the pole and the degree and order get high. In addition, the accuracy of the first derivatives of Legendre functions was on the order of 10^{-15} which is much poorer than that of Legendre function on the order of 10^{-25}.

In addition, it was found that the computable highest degree and order of Legendre function using 128 bit arithmetic are 103,240 and 45,294 at latitudes of 40 and 80, respectively. The relationship between the magnitude of orders and degrees and orders is linear and the slope of the linear trends increases as latitudes get close to the pole.

It is expected that the computed Legendre functions contribute to many geodetic and geophysical applications, for example, the gravity modeling with spatial accuracy of better than 2 km is possible and the numerical accuracy in theories such as upward continuation and gravity reductions can be verified.

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