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PRE-CONVERGENCE OF p-STACKS ON TOPOLOGICAL SPACES

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ABSTRACT. We introduce the notion of pre-convergence of p-stacks and characterize the pre-interior, pre-closure, separation axioms and pre-continuity on a topological space by using pre-convergence of p-stacks. We also introduce the notion of p-precompactness and investigate its properties in terms of pre-convergence of p-stacks.

1. INTRODUCTION

Mashhour *et. al.* [5] introduced the concepts of preopen sets and pre-continuity on a topological space and obtained many significant properties. Reilly and Vamanamurthy [12] introduced the concept of preirresolute function on a topological space and investigated some its properties. In [1, 2, 3, 10] the new separation axioms were defined by preopen sets. In [8], the author introduced the notion of pre-convergence of filters and characterized pre-continuity and pre-irresolute function in terms of pre-convergence of filters. In [4], Kent and the author introduced p-stacks which are more general than filters and showed some their properties.

In this paper, we introduce the notion of pre-convergence of p-stacks and characterize the pre-interior, pre-closure, separation axioms and pre-continuity on a topological space by using pre-convergence of p-stacks. We also introduce the notion of p-precompactness and investigate its properties in terms of pre-convergence of p-stacks.

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2. PRELEMINARIES

In the present paper we denote X and Y topological spaces. Let S be a subset of X. Then the closure (resp. interior) of S will be denoted by clS (resp. $intS$). A subset S of X is said to be *preopen* [5] if $S \subset int(cl(S))$. The complement of a preopen set is said to be *preclosed.* The family of all preopen sets in X will be denoted by $PO(X)$. A function $f : X \to Y$ is said to be *pre-continuous* [5] (resp. *preirresolute* [7, 12]) if $f^{-1}(V) \in PO(X)$ for each open (resp. preopen) set V of Y. A subset $P(x)$ of X is called a *pre-neighborhood* of a point $x \in X$ [9, 11] if there exists a preopen set S such that $x \in S \subset P(x)$.

Given a set X, a collection $\mathbf{C}$ of subsets of X is called a *stack* if $A \in \mathbf{C}$ whenever $B \in \mathbf{C}$ and $B \subset A$. A stack $\mathbf{H}$ on a set X is called a *p-stack* [1] if it satisfies the following condition:

(p) $A, B \in \mathbf{H}$ implies $A \cap B \neq \emptyset$.

Condition (p) is called the *pairwise intersection property* (PIP). A collection B of subsets of X with the PIP is called a *p-stack base.* For any collection $\mathbf{B}$, we denote by $\langle \mathbf{B} \rangle = \{A \subset X : \text{there exists } B \in \mathbf{B} \text{ such that } B \subset A\}$ the stack generated by $\mathbf{B}$, and if $\{B\}$ is a p-stack base, then $\langle\{B\}\rangle$ is a p-stack. We will denote simply $\langle\{B\}\rangle = \langle B\rangle$. In case $x \in X$ and $B = \{x\}$, $\langle B\rangle$ is usually denoted by $\dot{x}$. Let $pS(X)$ denote the collection of all p-stacks on X, partially ordered by inclusion. The maximal elements in $pS(X)$ are called *ultrapstacks* [4]. It is obvious that every ultrafilter is an ultrapstack, and that every p-stack is contained in an ultrapstack. For a function $f : X \to Y$ and $\mathbf{H} \in pS(X)$, the image p-stack $f(\mathbf{H})$ in $pS(Y)$ has p-stack base $\{f(H) : H \in \mathbf{H}\}$. Likewise, if $\mathbf{G} \in pS(Y)$, $f^{-1}(\mathbf{G})$ denotes the p-stack on X generated by $\{f^{-1}(G) : G \in \mathbf{G}\}$.

Definition 2.1 ([1, 2, 3, 10]). Let A be a subset in X.

(1) $pint(A) = \cup\{U \in PO(X) : U \subset A\}$;
(2) $pcl(A) = \cap\{F \subset X : A \subset F \text{ and } X - F \in PO(X)\}$;
(3) X is *pre-T_1* if for every two distinct points x and y in X, there exist two preopen sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$;
(4) X is *pre-T_2* if for every two distinct points x and y in X, there exist two disjoint preopen sets U and V such that $x \in U$ and $y \in V$;
(5) X is *pre-regular* if for preclosed set H and $x \notin H$, there exist two disjoint preopen sets U and V such that $H \subset U$ and $x \in V$;

(6) X is *precompact* if each cover of X by preopen sets has a finite subcover.

Lemma 2.2 ([4]). *For $\mathbf{H} \in pS(X)$, the following are equivalent;*

(1) *$\mathbf{H}$ is an ultrapstack;*
(2) *If $A \cap H \neq \emptyset$ for all $H \in \mathbf{H}$, then $A \in \mathbf{H}$;*
(3) *$B \notin \mathbf{H}$ implies $X - B \in \mathbf{H}$.*

Theorem 2.3 ([4]). *Let $f : X \to Y$ be a function and $\mathbf{H} \in pS(X)$. If $\mathbf{H}$ is an ultrapstack, so is $f(\mathbf{H})$.*

3. Main Results

Definition 3.1. Let $\mathbf{P}_x = \{V \subset X : V \text{ is a pre-neighborhood of } x\}$ for $x \in X$. Then we call the family $\mathbf{P}_x$ the *pre-neighborhood stack* at x.

Definition 3.2. For $x \in X$ and $\mathbf{F} \in pS(X)$. A p-stack $\mathbf{F}$ on X *pre-converges* to x if $\mathbf{P}_x \subset \mathbf{F}$.

From Definition 3.2, we get the following theorem.

Theorem 3.3. *For $x \in X$, the following are valid:*

(1) *$\dot{x}$ pre-converges to x;*
(2) *For $\mathbf{F}$, $\mathbf{G} \in pS(X)$ if $\mathbf{F}$ pre-converges to x and $\mathbf{F} \subset \mathbf{G}$, then $\mathbf{G}$ pre-converges to x;*
(3) *If both $\mathbf{F}$ and $\mathbf{G}$ are p-stacks pre-converging to x, then $\mathbf{F} \cap \mathbf{G}$ pre-converges to x.*

Theorem 3.4. *Let A be a subset in X. Then $x \in pcl(A)$ for $x \in X$ if and only if there is $\mathbf{F} \in pS(X)$ such that $A \in \mathbf{F}$ and $\mathbf{F}$ pre-converges to x.*

Proof. Let x be an element in $pcl(A)$; then $\mathbf{P}_x \cup \langle A \rangle = \langle \{U \cap V : U \in \mathbf{P}_x, V \in \langle A \rangle\} \rangle$ is a p-stack which pre-converges to x and contains A.

For the converse, let $\mathbf{F}$ be a p-stack pre-converging to x and $A \in \mathbf{F}$; then $\mathbf{P}_x \subset \mathbf{F}$, so it follows that $U \cap A \neq \emptyset$ for all $U \in \mathbf{P}_x$. □

Theorem 3.5. *Let A be a subset X. Then $x \in pint(A)$ for $x \in X$ if and only if for every p-stack $\mathbf{F}$ pre-converging to x, $A \in \mathbf{F}$.*

Proof. Let x be an element in $pint(A)$ and let $\mathbf{F}$ be a p-stack pre-converging to x; then there is a preopen subset U such that $x \in U \subset A$, so by definition of pre-convergence of p-stacks, we can say $A \in \mathbf{F}$.

Conversely, suppose that for every p-stack $\mathbf{F}$ pre-converging to x, $A \in \mathbf{F}$. Since the pre-neighborhood stack $\mathbf{P}_x$ pre-converges to x, by hypothesis $A \in \mathbf{P}_x$, so $x \in pint(A)$. □

Now using pre-convergence of p-stacks, we characterize separation axioms defined by preopen sets on a topological space.

Theorem 3.6. *The following are equivalent:*

(1) (X, μ) *is pre-*T_1*;*

(2) $\cap \mathbf{P}_x = \{x\}$ *for* $x \in X$*;*

(3) *If* $\dot{x}$ *pre-converges to* y*, then* $x = y$*.*

Proof.

(1) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (3) Let $\dot{x}$ pre-converge to y; then x is an element in $\cap \mathbf{P}_y$. Thus $x = y$.

(3) $\Rightarrow$ (1) Suppose that X is not pre-T_1. Then there are distinct elements x and y such that every preopen neighborhood of x contains y. Thus $\mathbf{P}_x \subset \dot{y}$ and $\dot{y}$ pre-converges to x. □

Theorem 3.7. *X is pre-T_2 if and only if every pre-convergent p-stack* $\mathbf{F}$ *on* X *pre-converges to exactly one point.*

Proof. Suppose that X is pre-T_2 and a p-stack $\mathbf{F}$ pre-converges to x. For any $y \neq x$, there are disjoint preopen sets $U(x)$ and $U(y)$ containing x and y, respectively. Since $\mathbf{F}$ is a p-stack pre-converging to x, both $U(x)$ and $X - U(y)$ are elements of $\mathbf{F}$. Thus $\mathbf{F}$ doesn't pre-converge to y.

Conversely suppose that X is not pre-T_2. Then there must exist x, y such that $U(x) \cap U(y) \neq \emptyset$ for every preopen sets $U(x)$ and $U(y)$ of x and y, respectively. Let $\mathbf{F} = \mathbf{P}_x \cup \mathbf{P}_y$ be a p-stack; then $\mathbf{F}$ is finer than $\mathbf{P}_x$ and $\mathbf{P}_y$, so the p-stack $\mathbf{F}$ pre-converges to both x and y.

Let $\mathbf{F} \in pS(X)$; then $\mathbf{B} = \{pcl(F) : F \in \mathbf{F}\}$ is a p-stack base on X. The p-stack generated by $\mathbf{B}$ is denoted by $pcl(\mathbf{F})$ and the p-stack $pcl(\mathbf{F})$ is called the *pre-closure p-stack* of $\mathbf{F}$. □

Theorem 3.8. *The following are equivalent:*

(1) X *is pre-regular;*

(2) *For every* x *in* X*,* $\mathbf{P}_x = pcl(\mathbf{P}_x)$*;*

(3) *If a p-stack* $\mathbf{F}$ *pre-converges to* x*, then the pre-closure p-stack* $pcl(\mathbf{F})$ *pre-converges to* x*.*

Proof. (1) $\Rightarrow$ (2) Let F be an element in $\mathbf{P}_x$; then there exists a preopen neighborhood $U(x)$ such that $U(x) \subset F$. By hypothesis, there is a preopen neighborhood $W(x)$ of x such that $W(x) \subset pcl(W(x)) \subset U(x) \subset F$ so $F \in pcl(\mathbf{P}_x)$.

(2) $\Rightarrow$ (3) It is obvioius.

(3) $\Rightarrow$ (1) Let U be a preopen set containing $x \in X$; then from (3), it follows $U \in pcl(\mathbf{P}_x)$. Thus there is a preopen neighborhood V of x such that $V \subset pcl(V) \subset U$. □

We know that a function $f : X \to Y$ is preirresolute if and only if for each x in X and each pre-neighborhood U of $f(x)$, there is a pre-neighborhood V of x such that $f(V) \subset U$.

Now we get another characterization of preirresolute functions on topological spaces by using p-stacks.

Theorem 3.9. *If $f : X \to Y$ is a function, then the following statements are equivalent:*

(1) *f is preirresolute;*
(2) *$\mathbf{P}_{f(x)} \subset f(\mathbf{P}_x)$, for all $x \in X$;*
(3) *If a p-stack $\mathbf{F}$ pre-converges to x, then the image p-stack $f(\mathbf{F})$ pre-converges to $f(x)$.*

Proof. (1) $\Rightarrow$ (2) Let $V \in \mathbf{P}_{f(x)}$ in Y; then there exists a preopen neighborhood $U \in \mathbf{P}_x$ such that $f(U) \subset W \subset V$, thus $V \in f(\mathbf{P}_x)$.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) If f is not preirresolute, then for some $x \in X$ there is a preopen neighborhood $V \in \mathbf{P}_{f(x)}$ such that for all preopen neighborhood $U \in \mathbf{P}_x$, $f(U)$ is not included in V. For all $U \in \mathbf{P}_x$, we get a p-stack $\mathbf{G} = f(\mathbf{P}_x) \cup \langle Y - V \rangle$ and also get a p-stack $\mathbf{F} = \mathbf{P}_x \cup \langle f^{-1}(Y - V) \rangle$ which pre-converges to x. But since $f(\mathbf{F})$ is a p-stack which is finer than $\mathbf{G}$ and $f^{-1}(Y - V) \in \mathbf{G}$, $f(\mathbf{F})$ can not pre-converge to $f(x)$. □

Now we introduce the concept of p-precompactness and investigate its properties by using p-stacks.

Definition 3.10. A subset A of X is said to be *p-precompact* if every ultrapstack containing A pre-converges to a point in A. A space X is *p-precompact* if X is p-precompact.

Example 3.11. Let $X = \{a, b, c\}$ and let (X, τ) be a topological space. In case τ is the discrete topology, $\tau = PO(X)$. Let $\mathbf{H}$ be an ultrapstack containing a p-stack

$\mathbf{F}$ generated by $\{\{a,b\},\{b,c\},\{a,c\}\}$; then it does not pre-converge to any point in X. Thus however X is finite, it is not p-precompact. In case $\tau = \{\emptyset, \{a\}, \{a,b\}, X\}$, $PO(X) = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, X\}$. The following pre-neighborhood stacks are obtained: $\mathbf{P}_\tau(a) = \{\{a\},\{a,b\},\{a,c\},X\}$, $\mathbf{P}_\tau(b) = \{\{a,b\},X\}$ and $\mathbf{P}_\tau(c) = \{\{a,c\},X\}$. Since every ultrapstack pre-converges to a point in X, X is p-precompact.

Theorem 3.12. *If X is p-precompact and $A \subset X$ is preclosed, then A is p-precompact.*

Proof. Let $\mathbf{F}$ be an ultrapstack containing A; then from Definition 3.10, there is $x \in X$ such that $\mathbf{F}$ pre-converges to x. From Theorem 3.4, it follows $x \in pcl(A)$. □

Theorem 3.13. *Let a function $f : X \to Y$ be preirresolute. If A is a p-precompact set in X, so is $f(A)$.*

Proof. Let $\mathbf{H}$ be an ultrapstack containing $f(A)$ and let $\mathbf{G}$ be an ultrapstack containing the p-stack base $\{f^{-1}(H) : H \in \mathbf{H}\} \cup \langle A \rangle$; then $\mathbf{G}$ pre-converges to x for some $x \in A$, and $\mathbf{H} = f(\mathbf{G})$ pre-converges to $f(x)$ by Theorem 2.3 and Theorem 3.9. Thus $f(A)$ is p-precompact. □

Theorem 3.14. *X is p-precompact if and only if each preopen cover of X has a two-element subcover.*

Proof. Suppose $\mathbf{H}$ is an ultrapstack in X such that it does not pre-converge to any point in X. Then for each $x \in X$, there is a preopen subset $U_x \in \mathbf{P}_x$ such that $U_x \notin \mathbf{H}$. By Lemma 2.2, $X - U_x \in \mathbf{H}$, for all $x \in X$. Thus $\mathbf{U} = \{U_x : x \in X\}$ is a preopen cover of X. But $\mathbf{U}$ has no a two-element subcover of X, for if $U, V \in \mathbf{U}$ and $X \subset U \cup V$, then $(X - U) \cap (X - V) = X - (U \cup V) = \emptyset$, contradicting the assumption that $\mathbf{H}$ is a p-stack.

Conversely, let $\mathbf{U}$ be a preopen cover of X with no two-element subcover of X. Then $\mathbf{B} = \{X - U : U \in \mathbf{U}\}$ is a p-stack base, and any ultrapstack containing $\mathbf{B}$ can not pre-converge to any point in X. □

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